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A KOOPMAN-VON NEUMANN TYPE THEOREM ON THE CONVERGENCE OF CESÀRO MEANS IN RIESZ SPACES

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Abstract. We extend the Koopman-von Neumann convergence condition on the Cesàro mean to the context of a Dedekind complete Riesz space with weak order unit. As a consequence, a characterisation of conditional weak mixing is given in the Riesz space setting. The results are applied to convergence in L^1.

1. Introduction

The Koopman-von Neumann Lemma, as referred to by Petersen [14, Section 2.6] (see also Krengel [7, Section 2.3], as well as Eisner, Farkas, Haase and Nagel [3, Section 9.2]), characterises the convergence to zero of the Cesàro mean of bounded, non-negative sequences of real numbers in terms of the existence of a convergent subsequence of the given sequence. Here, the subsequence is formed from the given sequence by the omission of a, so called, density zero set from $\mathbb{N}_0$, the index set of the given sequence.

In this paper, we consider the order convergence to zero of the Cesàro mean of an order bounded, non-negative sequence in a Dedekind complete Riesz space with weak order unit. This requires a more sophisticated density zero concept. In particular, we introduce a density zero sequence of band projections which forms the foundation for the Koopman-von Neumann condition in Riesz spaces. When the Riesz space is the real numbers the characterisation presented here gives the classical Koopman-von Neumann convergence condition.

As an application of the Koopman-von Neumann Lemma (Theorem 3.2), we give, in Section 4, a characterisation of conditional weak mixing in Riesz spaces. In Section 5, as an example, we apply Theorem 3.2 to characterise the order convergence of the Cesàro mean to zero of order bounded, non-negative sequences in L^1. We also refer the reader to the recent work of Gao, Troitsky and Xanthos on the UO-convergence and its application to Cesàro means in Banach lattices [5].

This work supplements the development of stochastic processes in Riesz spaces of Grobler [6], Stoica [15], Azouzi et al [2], Kuo, Labuschagne and Watson [9], and

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mixing processes in Riesz spaces as considered in Kuo, Rogans and Watson [11], and Kuo, Vardy and Watson [12].

2. Preliminaries

We refer the reader to Aliprantis and Border [1], Fremlin [4], Meyer-Nieberg [13], and Zaanen [18] and [19], for background in Riesz spaces and f-algebras.

We recall that, in a Riesz space, E, a sequence (f_n) in E converges to zero, in order, if and only if the sequence $(|f_n|)$ converges, in order, to zero in E. Further to this, in a Dedekind complete Riesz space, the absolute order convergence of a sum implies the order convergence of the sum, see below.

Lemma 2.1. *Let $(f_n)_{n\in\mathbb{N}_0}$ be a sequence in E, a Dedekind complete Riesz space, then order convergence of $\sum_{k=0}^{\infty}|f_k|$ implies the order convergence of $\sum_{k=0}^{\infty} f_k$.*

Proof. Suppose that $\sum_{k=0}^{n-1}|f_k|\to \ell$, in order, as $n\to\infty$. Then ℓ is an upper bound for the increasing sequences $\left(\sum_{k=0}^{n-1} f_k^{\pm}\right)$, which, from the Dedekind completeness of E, have order limits, say, $f^{\pm}$. Thus, $\sum_{k=0}^{n-1} f_k=\sum_{k=0}^{n-1} f_k^{+}-\sum_{k=0}^{n-1} f_k^{-}\to f^{+}-f^{-}$, in order, as $n\to\infty$. □

From [10, Lemma 2.1], we have the following theorem.

Theorem 2.2. *Let E be a Dedekind complete Riesz space and $(f_n)_{n\in\mathbb{N}_0}$ a sequence in E with $f_n\to 0$, in order, as $n\to\infty$, then $\frac{1}{n}\sum_{k=0}^{n-1}|f_k|\to 0$, in order, as $n\to\infty$.*

Corollary 2.3. *Let E be a Dedekind complete Riesz space and $(f_n)_{n\in\mathbb{N}_0}$ a sequence in E with order limit f, then we have $\frac{1}{n}\sum_{k=0}^{n-1} f_k\to f$ in order, as $n\to\infty$.*

Proof. Let $g_n := f_n - f$ for each $n\in\mathbb{N}_0$, then $(|g_n|)_{n\in\mathbb{N}_0}$ is order convergent to 0, by assumption. Thus, by Theorem 2.2,

$$0\le \left| f-\frac{1}{n}\sum_{k=0}^{n-1} f_k\right| \le \frac{1}{n}\sum_{k=0}^{n-1}|g_k|\to 0,$$

in order, as $n\to\infty$, and the result follows as E is Archimedean. □

3. The Koopman-von Neumann condition

In [14], a subset N of $\mathbb{N}_0$ is said to be of density zero if $\frac{1}{n}\sum_{k=0}^{n-1}\chi_N(k)\to 0$ as $n\to\infty$, where $\chi_N(k)=0$ if $k\in\mathbb{N}_0\setminus N$ and $\chi_N(k)=1$ if $k\in N$. The Koopman-von Neumann Lemma [14, Lemma 6.2] asserts that if a sequence $(a_n)_{n\in\mathbb{N}_0}$ of non-negative real numbers is bounded, then $\frac{1}{n}\sum_{k=0}^{n-1} a_k\to 0$ as $n\to\infty$ if and only if there is N, a subset of $\mathbb{N}_0$, of density zero, such that $a_n\to 0$ as $n\to\infty$, $n\in\mathbb{N}_0\setminus N$.

We recall that $E_e=\{f\in E \mid |f|\le ke \text{ for some } k\in\mathbb{R}_+\}$, the subspace of E consisting of the e bounded elements of E is an f-algebra, see [2, 16, 18]. For all band projections P and Q on E, we set $Pe\cdot Qe = PQe$. Here, $\cdot$ represents the f-algebra multiplication on E_e. The linear extension of this multiplication and use of order limits extends this multiplication to the f-algebra multiplication on E_e. Further, the weak order unit, e, is the multiplicative unit of the f-algebra.

In a Riesz space, E, for $u\in E_+$, $p\in E_+$ is called a component of u if $u\wedge(u-p)=0$, see [19, pg. 213]. Furthermore, if E has the principal projection property, then p is a component of u if and only if $p=P_f u$, for some principal projection P_f, [19, Theorem 32.7]. If E has a weak order unit, say e, then p is a component of e if and only if there is a band projection P on E with $Pe = p$. Further, $e - p = (I - P)e$ and $p \cdot f = Pf$ for each $f \in E$. We note here that E is an E_e module since the f-algebra multiplication in E is a restriction of that in E_u and if $f \in E_e$ with $|f| \leq ke$ and $g \in E$ then $|f \cdot g| \leq k\,|g|$ so $f \cdot g \in E$. Any $s \in E$ for which there exist pairwise disjoint components $p_1, ..., p_n$ of e and real numbers $\alpha_1, ..., \alpha_n$ such that $s = \sum_{k=1}^{n} \alpha_k p_k$ is called an e-step function. Notice that if E has the principal projection property, then there exist principal band projections $P_1, ..., P_n$ such that $s = \sum_{k=0}^{n} \alpha_k P_k e$, where the band projections are pairwise disjoint. Consequently, if E is a Dedekind complete Riesz space with weak order unit, say e, then $B_e = E \supseteq E_e$ (where B_e is the principal band generated by e), and any e-step function $s \in E$ can be represented by $s = \sum_{k=1}^{n} \alpha_k P_k e$.

In order to extend the Koopman-von Neumann Lemma to sequences in a Riesz space, we define a density zero sequence of band projections as follows.

Definition 3.1 (Density zero sequence of band projections)**.** A sequence $(P_n)_{n\in\mathbb{N}_0}$ of band projections in a Riesz space E with weak order unit e is said to be of density zero if $\frac{1}{n}\sum_{k=0}^{n-1} P_k e \to 0$, in order as $n \to \infty$.

The above definition could be rephrased as saying that a sequence of components of e is said to be of density zero if its Cesàro mean tends to zero in order.

With the above definition of density zero sequences of band projections, we can now give an analogue of the Koopman-von Neumann Lemma in Riesz spaces.

Theorem 3.2 (Koopman-von Neumann)**.** *Let E be a Dedekind complete Riesz space with weak order unit, say e, and let $(f_n)_{n\in\mathbb{N}_0}$ be an order bounded sequence in the positive cone, E_+, of E, then $\frac{1}{n}\sum_{k=0}^{n-1} f_k \to 0$, in order, as $n \to \infty$, if and only if there exists a density zero sequence of band projections $(P_n)_{n\in\mathbb{N}_0}$ on E such that $(I - P_n) f_n \to 0$, in order, as $n \to \infty$.*

Proof. Suppose that $(f_n)_{n\in\mathbb{N}_0} \subset E_+$ and there exists $g \in E_+$ with $f_n \leq g$, for all $n \in \mathbb{N}_0$.

If $(P_n)_{n\in\mathbb{N}_0}$ is a density zero sequence of band projections with $(I - P_n)f_n \to 0$, in order, as $n \to \infty$, then

$$0 \leq \frac{1}{n}\sum_{k=0}^{n-1} f_k = \frac{1}{n}\sum_{k=0}^{n-1} P_k e \cdot f_k + \frac{1}{n}\sum_{k=0}^{n-1} (I - P_k) f_k \tag{3.1a}$$

$$\leq \left(\frac{1}{n}\sum_{k=0}^{n-1} P_k e\right) \cdot g + \frac{1}{n}\sum_{k=0}^{n-1} (I - P_k) f_k. \tag{3.1b}$$

Since $(P_n)_{n\in\mathbb{N}_0}$ is of density zero, $\frac{1}{n}\sum_{k=0}^{n-1} P_k e \to 0$, in order, as $n \to \infty$, giving $\left(\frac{1}{n}\sum_{k=0}^{n-1} P_k e\right) \cdot g \to 0$, in order, as $n \to \infty$. Furthermore, by Theorem 2.2, as $(I - P_n)f_n \to 0$, in order, as $n \to \infty$ and $(I - P_n)f_n \geq 0$, we have $\frac{1}{n}\sum_{k=0}^{n-1} (I - P_k) f_k \to 0$, in order, as $n \to \infty$. Thus, by (3.1a)-(3.1b), as E is Archimedean, $\frac{1}{n}\sum_{k=0}^{n-1} f_k \to 0$, in order, as $n \to \infty$.

Conversely, suppose that $\frac{1}{n}\sum_{k=0}^{n-1} f_k \to 0$, in order, as $n \to \infty$. Let $P_{m,i}$ be the band projection onto the band generated by $\left(f_i - \frac{1}{m}e\right)^+$ and $p_{m,i} = P_{m,i}e$. Let $u_{m,j} := \sup_{k\geq j} \frac{1}{k}\sum_{i=0}^{k-1} p_{m,i}$. Let $R_{m,j}$ be the band projection onto the band generated by $\left(u_{m,j} - \frac{1}{m}e\right)^+$ and $r_{m,j} = R_{m,j}e$. As $0 \leq p_{m,i} \leq e$, we have that $0 \leq u_{m,j} \leq e$. Further, since $p_{m,i}$ is increasing in m for fixed i, it follows that $u_{m,j}$

is increasing in m for fixed j and hence $u_{m,j} - \frac{1}{m}e$ is increasing in m for fixed j, giving that $r_{m,j}$ is increasing in m for fixed j.

Since $\{k \in \mathbb{Z} \,|\, k \geq j+1\} \subset \{k \in \mathbb{Z} \,|\, k \geq j\}$, it follows that

$$u_{m,j+1} = \sup_{k \geq j+1} \frac{1}{k} \sum_{i=0}^{k-1} p_{m,i} \leq \sup_{k \geq j} \frac{1}{k} \sum_{i=0}^{k-1} p_{m,i} = u_{m,j},$$

giving that $u_{m,j}$ is decreasing in j. Hence, $r_{m,j}$ is decreasing in j, for fixed m. We now show that, for fixed m, $r_{m,j} \downarrow 0$, in order, as $j \to \infty$. Since $f_i \geq P_{m,i} f_i \geq \frac{1}{m} p_{m,i}$ (as $P_{m,i}$ is the projection onto the band generated by $\left(f_i - \frac{1}{m}e\right)^+$), we have

$$\sup_{k \geq j} \frac{1}{k} \sum_{i=0}^{k-1} f_i \geq \frac{1}{m} \sup_{k \geq j} \frac{1}{k} \sum_{i=0}^{k-1} p_{m,i} = \frac{1}{m} u_{m,j}. \tag{3.2}$$

However,

$$u_{m,j} \geq R_{m,j} u_{m,j} \geq \frac{1}{m} r_{m,j}, \tag{3.3}$$

(since $R_{m,j}$ is the band projection onto the band generated by $\left(u_{m,j} - \frac{1}{m}e\right)^+$), so, by (3.2) and (3.3), we have

$$0 \leq \frac{1}{m^2} r_{m,j} \leq \frac{1}{m} u_{m,j} \leq \sup_{k \geq j} \frac{1}{k} \sum_{i=0}^{k-1} f_i \to 0, \tag{3.4}$$

in order, as $j \to \infty$. Thus, for fixed m, $r_{m,j} \downarrow 0$, in order, as $j \to \infty$.

Observe that $R_{1,j} = 0$, since $R_{1,j}$ is the band projection onto the band generated by $(u_{1,j} - e)^+ = 0$. Let $H_j := \sup_{m \in \mathbb{N}} R_{m,j}$, for $j = 0, 1, \ldots$, then $H_j \downarrow H$, say, in order, as $j \to \infty$. Here, the band projections H_j and H can be explicitly determined from their components $h_j := H_j e = \sup_{m \in \mathbb{N}} r_{m,j}$, for $j = 0, 1, \ldots$, and $h_j \downarrow h := He$, in order, as $j \to \infty$. Further $r_{m+1,j} \cdot (e - r_{m,j})$, $m \in \mathbb{N}$, is a partition of h_j for each $j = 0, 1, \ldots$, that is,

$$(r_{m+1,j} \cdot (e - r_{m,j})) \wedge (r_{n+1,j} \cdot (e - r_{n,j})) = 0, \text{ for } m \neq n, \tag{3.5}$$

and

$$\sum_{m=1}^{M-1} r_{m+1,j} \cdot (e - r_{m,j}) = r_{M,j} \uparrow_M h_j, \tag{3.6}$$

in order, as $M \to \infty$, giving

$$\sum_{m \in \mathbb{N}} r_{m+1,j} \cdot (e - r_{m,j}) = h_j. \tag{3.7}$$

Let

$$Q_j := \bigvee_{m \in \mathbb{N}} P_{m,j} R_{m+1,j} \left(I - R_{m,j}\right) = \sum_{m \in \mathbb{N}} P_{m,j} R_{m+1,j} \left(I - R_{m,j}\right), \tag{3.8}$$

by (3.5), and in terms of components we have

$$q_j := Q_j e = \bigvee_{m \in \mathbb{N}} p_{m,j} \cdot r_{m+1,j} \cdot (e - r_{m,j}) = \sum_{m \in \mathbb{N}} p_{m,j} \cdot r_{m+1,j} \cdot (e - r_{m,j}). \tag{3.9}$$

Further, as $0 \leq p_{m,j} \leq e$, by (3.8),

$$q_j = \sum_{m \in \mathbb{N}} p_{m,j} \cdot r_{m+1,j} \cdot (e - r_{m,j}) \leq \sum_{m \in \mathbb{N}} r_{m+1,j} \cdot (e - r_{m,j}) = h_j. \tag{3.10}$$

Hence, subtracting the left-hand side of (3.10) from the right-hand side of (3.10), we obtain

$$(3.11)\quad (e-q_j)=(e-h_j)+(h_j-q_j)=(e-h_j)+\sum_{m\in\mathbb{N}}(e-p_{m,j})\cdot r_{m+1,j}\cdot(e-r_{m,j})\,.$$

For $m>j+1$, $j\in\mathbb{N}_0$, we have that $\frac{1}{j+1}>\frac{1}{m}$. Now, from the definition of $u_{m,j}$, we have

$$u_{m,j}=\sup_{k\geq j}\frac{1}{k}\sum_{i=0}^{k-1}p_{m,i}\geq\frac{1}{j+1}\sum_{i=0}^{j}p_{m,i}\geq\frac{1}{j+1}p_{m,j},$$

thus $P_{m,j}u_{m,j}\geq\frac{1}{j+1}p_{m,j}$, giving

$$P_{m,j}\left(u_{m,j}-\frac{1}{m}e\right)^+\geq P_{m,j}\left(u_{m,j}-\frac{1}{m}e\right)\geq\left(\frac{1}{j+1}-\frac{1}{m}\right)p_{m,j}.$$

Hence, $\left(u_{m,j}-\frac{1}{m}e\right)^+\geq\left(\frac{1}{j+1}-\frac{1}{m}\right)p_{m,j}$. Further, $\frac{1}{j+1}-\frac{1}{m}>0$, so, for $m>j+1$, $p_{m,j}$ is in the band generated by $\left(u_{m,j}-\frac{1}{m}e\right)^+$, giving $R_{m,j}\geq P_{m,j}$. Applying the above to f_j, we have

$$(3.12)\qquad r_{m,j}\cdot f_j\geq p_{m,j}\cdot f_j.$$

Taking the supremum over $m\in\mathbb{N}$ in (3.12) gives $h_j\cdot f_j\geq f_j$, since $\sup_{m\in\mathbb{N}}r_{m,j}=h_j$ and $\sup_{m\in\mathbb{N}}p_{m,j}\cdot f_j=f_j$. Further, $h_j\leq e$, hence, $(e-h_j)\cdot f_j=0$, so multiplying (3.11) by f_j gives

$$(3.13a)\qquad (e-q_j)\cdot f_j=\sum_{m\in\mathbb{N}}r_{m+1,j}\cdot(e-r_{m,j})\cdot(I-P_{m,j})\,f_j$$

$$(3.13b)\qquad \leq\sum_{m\in\mathbb{N}}\frac{1}{m}r_{m+1,j}\cdot(e-r_{m,j})\,,$$

where we have used that $(I-P_{m,j})\,f_j\leq\frac{1}{m}e$. Now, from (3.6), (3.7) and (3.13a)-(3.13b), for $k\leq j$,

$$\begin{aligned}(e-q_j)\cdot f_j&\leq\sum_{m=1}^{k-1}\frac{1}{m}r_{m+1,j}\cdot(e-r_{m,j})+\sum_{m=k}^{\infty}\frac{1}{m}r_{m+1,j}\cdot(e-r_{m,j})\\&\leq\sum_{m=1}^{k-1}r_{m+1,j}\cdot(e-r_{m,j})+\frac{1}{k}h_j\\&\leq r_{k,j}e+\frac{1}{k}e.\end{aligned}$$

So, taking the limit supremum as $j\to\infty$, we obtain

$$(3.15)\qquad 0\leq\limsup_{j\to\infty}(e-q_j)\cdot f_j\leq\frac{1}{k}e+\limsup_{j\to\infty}r_{k,j}=\frac{1}{k}e,$$

for each $k\in\mathbb{N}$. Thus, as E is Archimedean, $\limsup_{j\to\infty}(I-Q_j)\,f_j=0$ and $(I-Q_j)\,f_j\to 0$, in order, as $j\to\infty$.

It now remains to show that $\frac{1}{n}\sum_{j=0}^{n-1}q_j\to 0$, in order, as $n\to\infty$. For $j<n$ and $m\geq M+1$, we have $r_{M+1,n}\leq r_{m,j}$, giving $r_{M+1,n}\cdot(e-r_{m,j})=0$. Thus, for

$M, n \in \mathbb{N}$, we have

$$\begin{aligned}
&r_{M+1,n} \cdot (e - r_{M,n}) \cdot \left(\frac{1}{n} \sum_{j=0}^{n-1} q_j \right) \\
&= \frac{1}{n} \sum_{j=0}^{n-1} r_{M+1,n} \cdot (e - r_{M,n}) \cdot \left(\bigvee_{m\in\mathbb{N}} p_{m,j} \cdot r_{m+1,j} \cdot (e - r_{m,j}) \right) \\
&= \frac{1}{n} \sum_{j=0}^{n-1} \bigvee_{m\le M} r_{M+1,n} \cdot (e - r_{M,n}) \cdot p_{m,j} \cdot r_{m+1,j} \cdot (e - r_{m,j}) \\
&\le \frac{1}{n} \sum_{j=0}^{n-1} r_{M+1,n} \cdot (e - r_{M,n}) \cdot p_{M,j} \\
&= r_{M+1,n} \cdot (e - r_{M,n}) \cdot \left(\frac{1}{n} \sum_{j=0}^{n-1} p_{M,j} \right) \\
&\le r_{M+1,n} \cdot (I - R_{M,n}) u_{M,n} \\
&\le \frac{1}{M} r_{M+1,n} \cdot (I - R_{M,n}) e.
\end{aligned}$$

Summing the above over $M \ge K$ gives

$$(h_n - r_{K,n}) \cdot \left(\frac{1}{n} \sum_{j=0}^{n-1} q_j \right) \le \sum_{M\ge K} \frac{1}{M} r_{M+1,n} \cdot (e - r_{M,n}) \tag{3.16a}$$

$$\le \frac{1}{K}(h_n - r_{K,n}) \le \frac{1}{K} e. \tag{3.16b}$$

We recall, from (3.10), that $Q_j \le H_j$, so if $j \ge \lfloor \sqrt{n} \rfloor$ then $Q_j \le H_j \le H_{\lfloor \sqrt{n} \rfloor}$ and $Q_j H_{\lfloor \sqrt{n} \rfloor} = Q_j$, that is, $(e - h_{\lfloor \sqrt{n} \rfloor}) \cdot q_j = 0$. Hence,

$$(e - h_{\lfloor \sqrt{n} \rfloor}) \cdot \left(\frac{1}{n} \sum_{j=0}^{n-1} q_j \right) = \frac{1}{n} \sum_{j=0}^{\lfloor \sqrt{n} \rfloor - 1} q_j \le \frac{1}{\sqrt{n}} e. \tag{3.17}$$

Combining (3.16a)-(3.16b) and (3.17), we have, for each $K \in \mathbb{N}$,

(3.18a)

$$\frac{1}{n} \sum_{j=0}^{n-1} q_j = \left(\left(e - h_{\lfloor \sqrt{n} \rfloor} \right) + \left(h_{\lfloor \sqrt{n} \rfloor} - h_n \right) + (h_n - r_{K,n}) + r_{K,n} \right) \cdot \left(\frac{1}{n} \sum_{j=0}^{n-1} q_j \right)$$

$$\le \frac{1}{\sqrt{n}} e + (h_{\lfloor \sqrt{n} \rfloor} - h_n) + \frac{1}{K} e + r_{K,n}. \tag{3.18b}$$

Here, h_n and $h_{\lfloor \sqrt{n} \rfloor}$ both converge, in order, to h, so $h_{\lfloor \sqrt{n} \rfloor} - h_n$ converges to 0, in order, as $n \to \infty$, as does $r_{K,n}$. Thus, taking the limit supremum as $n \to \infty$ in (3.18b) gives

$$0 \le \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} q_j \le \frac{1}{K} e \tag{3.19}$$

for each $K \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} q_j = 0,$$

in order, as E is Archimedean. □

Note 3.3. If $(f_n)_{n\in\mathbb{N}_0}$ in Theorem 3.2 is not assumed to be order bounded, but $\frac{1}{n}\sum_{k=0}^{n-1} f_k \to 0$, in order, as $n \to \infty$, then it still follows that there is a density zero sequence of band projections, $(P_n)_{n\in\mathbb{N}_0}$, such that $(I - P_n)f_n \to 0$, in order, as $n \to \infty$, but one cannot conclude boundedness of $(P_n f_n)_{n\in\mathbb{N}_0}$. Further, the converse need not hold. In particular, if $(f_n)_{n\in\mathbb{N}_0}$ is not bounded and there is a density zero sequence of band projections $(P_n)_{n\in\mathbb{N}_0}$ such that $(I - P_n)f_n \to 0$, in order, as $n \to \infty$, then one cannot conclude that $\frac{1}{n}\sum_{k=0}^{n-1} f_k$ is order convergent to 0 as $n \to \infty$.

For example, working in the classical case of $E = \mathbb{R}$, for $1 < p < \infty$, we have

$$g_j^p = \begin{cases} n, & j = \lfloor n^p \rfloor, n \in \mathbb{N}_0, \\ 0, & \text{otherwise} \end{cases}$$

is unbounded and $N = \{\lfloor n^p \rfloor \mid n \in \mathbb{N}_0\}$ is a set of density zero. Here $g_j^p = 0 \to 0$ for $j \in \mathbb{N}_0 \backslash N$ as $j \to \infty$ but $\frac{1}{n}\sum_{k=0}^{n-1} g_k^p$ is convergent to 0 for $p > 2$, convergent to a non-zero value for $p = 2$ and divergent to ∞ for $1 < p < 2$.

4. Application to weak mixing

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, that is, Ω is a set, $\mathcal{A}$ is a σ-algebra of subsets of Ω and μ is a measure on $\mathcal{A}$ with $\mu(\Omega) = 1$. The mapping $\tau : \Omega \to \Omega$ is called a measure preserving transformation if $\mu\left(\tau^{-1}A\right) = \mu(A)$, for each $A \in \mathcal{A}$, in which case $(\Omega, \mathcal{A}, \mu, \tau)$ is called a measure preserving system. Further details may be found in [3, 7, 14].

The measure preserving system $(\Omega, \mathcal{A}, \mu, \tau)$ is said to be weakly mixing if

$$\frac{1}{n}\sum_{k=0}^{n-1} \left|\mu\left(\tau^{-1}(A) \cap B\right) - \mu(A)\mu(B)\right| \to 0 \tag{4.1}$$

as $n \to \infty$, for each $A, B \in \mathcal{A}$. To give a Riesz space analogue of a measure preserving system and weak mixing, we recall from [8] the definition of a conditional expectation operator on a Riesz space.

Definition 4.1. Let E be a Riesz space with weak order unit. A positive order continuous projection $T\colon E \to E$, with range, $\mathcal{R}(T)$, a Dedekind complete Riesz subspace of E, is called a conditional expectation operator if Te is a weak order unit of E for each weak order unit e of E.

If T is a conditional expectation operator on E with $Te = e$, then T is also a conditional expectation operator on E_e, since, if $f \in E_e$, then $|f| \le ke$, giving $|Tf| \le T|f| \le Tke = ke$.

The Riesz space analogue of a measure preserving system is introduced in the following definition.

Definition 4.2. Let E be a Dedekind complete Riesz space with weak order unit, say, e, and T be a conditional expectation operator on E with $Te = e$. If S is an order continuous Riesz homomorphism on E with $Se = e$ and $TSPe = TPe$ for each band projection P on E, then (E,T,S,e) is called a conditional expectation preserving system.

By Freudenthal's Spectral Theorem, see [19, Theorem 33.2], the condition $TSPe = TPe$ for each band projection P on E in the above definition is equivalent to $TSf = Tf$ for all $f \in E$. In [9, Theorems 3.7 and 3.9] various generalisations of Birkhoff's ergodic theorem to conditional expectation preserving systems (E,T,S,e) were given, resulting in convergence conditions for $\left(\frac{1}{n}\sum_{k=0}^{n-1} S^k f\right)$, $f \in E$.

We are now in a position to define conditional weak mixing on a Riesz space with a conditional expectation operator and weak order unit.

Definition 4.3 (Conditional weak mixing)**.** The conditional expectation preserving system (E,T,S,e) is said to be conditionally weak mixing if, for all band projections P and Q on E,

$$\frac{1}{n}\sum_{k=0}^{n-1}\left|T\left(\left(S^k Pe\right)\cdot Qe\right) - TPe\cdot TQe\right| \to 0, \tag{4.2}$$

in order, as $n \to \infty$.

We note that for $E = L^1(\Omega,\mathcal{A},\mu)$, the band projections on E are of the form $P_A f = \chi_A f$, for $f \in E$ and $A \in \mathcal{A}$. Definition 4.3 now gives a conditional weak mixing condition on E, conditioned by $T = \mathbb{E}[\cdot \mid \mathcal{B}]$, for $\mathcal{B}$ a sub-σ-algebra of $\mathcal{A}$. If $\mathcal{B} = \{\Omega\setminus C \mid C \in \mathcal{A}, \mu(C) = 0\} \cup \{C \in \mathcal{A} \mid \mu(C) = 0\}$, then conditional weak mixing coincides with the weak mixing on $(\Omega,\mathcal{A},\mu)$.

We recall, [19, pg. 49], that if E is a Riesz space and $(x_n)_{n\in\mathbb{N}} \subset E$ converges to $x \in E$, we say that x_n converges to x u-uniformly for given $0 < u \in E$ if for each $0 < \epsilon \in \mathbb{R}$, there is some $N_\epsilon \in \mathbb{N}$ such that $|x_n - x| \le \epsilon u$ whenever $n \ge N_\epsilon$.

Theorem 4.4. *Given the conditional expectation preserving system (E,T,S,e), then the following statements are equivalent.*

1. *(E,T,S,e) is conditionally weak mixing.*
2. *For all $f, g \in E_e$, we have that*
$$\frac{1}{n}\sum_{k=0}^{n-1}\left|T\left(\left(S^k f\right)\cdot g\right) - Tf\cdot Tg\right| \to 0,$$
in order, as $n \to \infty$.
3. *For each pair of band projections P and Q on E, there is a sequence of density zero band projections, $(R_n)_{n\in\mathbb{N}_0}$, in E such that*
$$(I - R_n)\left|T\left(\left(S^n Pe\right)\cdot Qe\right) - T(Pe)\cdot T(Qe)\right| \to 0,$$
in order, as $n \to \infty$.

Proof.

(1)⇒(2): Suppose that (E,T,S,e) is conditionally weak mixing. Let $s, t \in E$ be e-step functions with $s = \sum_{i=1}^m \alpha_i P_i e$ and $t = \sum_{j=1}^r \beta_j Q_j e$, where P_i and Q_j are band projections on E and α_i and β_j are real numbers, $i = 1,\dots,m$, $j = 1,\dots,r$, then

$$\text{(4.3a)} \quad \frac{1}{n}\sum_{k=0}^{n-1}\left|T\left(\left(S^{k}(s)\right)\cdot t\right)-T(s)\cdot T(t)\right|$$

$$\text{(4.3b)} \quad \leq \sum_{i=1}^{m}\sum_{j=1}^{r}|\alpha_i\beta_j|\frac{1}{n}\sum_{k=0}^{n-1}\left|T\left(\left(S^{k}(P_ie)\right)\cdot Q_je\right)-T(P_ie)\cdot T(Q_je)\right|\to 0,$$

in order, as $n\to\infty$.

By Freudenthal's Spectral Theorem, [19, Theorem 33.2], $f,g\in E_e$ can be expressed as e-uniform order limits of sequences, say $(s_i)_{i\in\mathbb{N}},(t_j)_{j\in\mathbb{N}}$, of e-step functions in E_e and there is $K>0$ so that $|s_i|,|t_j|,|f|,|g|\leq Ke$, for all $i,j\in\mathbb{N}$. This implies that $T|s_i|,T|t_j|,T|f|,T|g|\leq Ke$, for each $i,j\in\mathbb{N}$. For each $\epsilon>0$ there is $N_\epsilon\in\mathbb{N}$ so that $|s_i-f|\leq\epsilon e$ and $|t_j-g|\leq\epsilon e$ for each $i,j\geq N_\epsilon\in\mathbb{N}$. Hence, $T|s_i-f|\leq\epsilon e$ and $T|t_j-g|\leq\epsilon e$ for each $i,j\geq N_\epsilon$. Let $b_k:=\left|T\left(S^{k}(f)\cdot g\right)-Tf\cdot Tg\right|$ and $b_{k,i,j}:=\left|T\left(S^{k}(s_i)\cdot t_j\right)-Ts_i\cdot Tt_j\right|$, then

$$\begin{aligned}
&|b_{k,i,j}-b_k|\\
&\quad=\left|\left|T\left(S^{k}(s_i)\cdot t_j\right)-Ts_i\cdot Tt_j\right|-\left|T\left(S^{k}(f)\cdot g\right)-Tf\cdot Tg\right|\right|\\
&\quad\leq\left|T\left(S^{k}(s_i)\cdot t_j\right)-Ts_i\cdot Tt_j-T\left(S^{k}(f)\cdot g\right)+Tf\cdot Tg\right|\\
&\quad\leq\left|T\left(S^{k}(f)\cdot g-S^{k}(s_i)\cdot t_j\right)\right|+|Tf\cdot Tg-Ts_i\cdot Tt_j|\\
&\quad\leq T\left|\left(S^{k}(f-s_i)\right)\cdot g\right|+T\left|S^{k}(s_i)\cdot(g-t_j)\right|+T|f-s_i|\cdot T|g|+T|s_i|\cdot T|g-t_j|\\
&\quad\leq Ke\cdot\left(T\left|S^{k}(f-s_i)\right|+T|g-t_j|+T|f-s_i|+T|g-t_j|\right)\\
&\quad\leq 4K\epsilon e.
\end{aligned}$$

Hence, $b_k\leq b_{k,i,j}+4K\epsilon e$, for all $i,j\in\mathbb{N}$ with $i,j\geq N_\epsilon$, so

$$0\leq\frac{1}{n}\sum_{k=0}^{n-1}b_k\leq\frac{1}{n}\sum_{k=0}^{n-1}b_{k,N_\epsilon,N_\epsilon}+4K\epsilon e,$$

for all $n\in\mathbb{N}$. By (4.3a)-(4.3b), $\frac{1}{n}\sum_{k=0}^{n-1}b_{k,N_\epsilon,N_\epsilon}\to 0$, in order, as $n\to\infty$, so

$$0\leq\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}b_k\leq 4K\epsilon e,$$

for all $\epsilon>0$, implying that $\frac{1}{n}\sum_{k=0}^{n-1}b_k\to 0$, in order.

(2) ⇒ (1): Choosing $f=Pe$ and $g=Qe$, the result follows directly.

(1)⇔(3): Taking $f_n=|T((S^nPe)\cdot Qe)-TPe\cdot TQe|$, the result follows from Theorem 3.2. □

5. Application to measurable processes

If we consider the Riesz space E of equivalence classes of almost everywhere identical functions in $L^1(\Omega,\mathcal{A},\mu)$, where μ is a finite measure (the case of μ σ-finite is an easy extension of this case), then E is a Dedekind complete Riesz space under a.e. pointwise ordering and $\mathbf{1}$, the a.e. equivalence class of the constant function with value 1, is a weak order for E. Here the band projections, P, on E are multiplication by the characteristic functions of measurable sets, i.e., P is of the form $Pf=\chi_A f, f\in E$, for $A\in\mathcal{A}$. We recall from [1, Lemma 8.17] and [13, pg. 9, Example (ii)] that a sequence $(g_n)_{n\in\mathbb{N}}\subset L^1(\Omega,\mathcal{A},\mu)$ is order convergent in $L^1(\Omega, \mathcal{A}, \mu)$ if and only if $g_n \to g$ a.e. pointwise and there exists $h \in L^1(\Omega, \mathcal{A}, \mu)$ for which $|g_n| \le h$ a.e. for all $n \in \mathbb{N}$.

If $(f_n)_{n\in\mathbb{N}_0}$ is a non-negative order bounded sequence in E, i.e., $f_n \ge 0$ a.e. and there exists $g \in L^1(\Omega, \mathcal{A}, \mu)$ so that $f_n \le g$ a.e. for all n, then, by Theorem 3.2,

$$s_n := \frac{1}{n}\sum_{j=0}^{n-1} f_j \to 0, \quad \text{in order, as} \quad n \to \infty, \tag{5.1}$$

if and only if there is a density zero sequence of band projections $(P_n)_{n\in\mathbb{N}_0}$ such that $(I - P_n)f_n \to 0$, in order, as $n \to \infty$, i.e., there is a sequence of measurable sets $(A_n)_{n\in\mathbb{N}_0}$ with

$$c_n := \frac{1}{n}\sum_{j=0}^{n-1} \chi_{A_j} \to 0 \quad \text{in order as} \quad n \to \infty, \tag{5.2}$$

with $\chi_{\Omega\setminus A_n} f_n \to 0$ in order as $n \to \infty$.

In the above, $0 \le s_n \le g$ and $0 \le c_n \le \mathbf{1}$, for each n, so $(s_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are order bounded and thus order convergent if and only if they are a.e. pointwise convergent. Further, Lebesgue's Dominated Convergence Theorem is applicable. Thus we have the following.

Corollary 5.1. *If $(f_n)_{n\in\mathbb{N}_0}$ is a non-negative sequence in $L^1(\Omega, \mathcal{A}, \mu)$, where μ is a finite measure, and there exists $g \in L^1(\Omega, \mathcal{A}, \mu)$ so that $f_n \le g$ a.e. for all $n \in \mathbb{N}_0$, then $\frac{1}{n}\sum_{j=0}^{n-1} f_j \to 0$ as $n \to \infty$, if and only if there is a sequence of measurable sets $(A_n)_{n\in\mathbb{N}_0}$ with $\frac{1}{n}\sum_{j=0}^{n-1} \chi_{A_j} \to 0$ and $\chi_{\Omega\setminus A_n} f_n \to 0$ as $n \to \infty$. Here the limits can be taken as either a.e. pointwise or in norm.*

Proceeding as in [11, Section 5], the conditional weak mixing of Section 4 can be carried over to $L^1(\Omega, \mathcal{A}, \mu)$ to give a characterisation of conditional weak mixing in measure spaces.

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