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# A CHARACTERIZATION OF $X$ FOR WHICH SPACES $C_p(X)$ ARE DISTINGUISHED AND ITS APPLICATIONS

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Abstract. We prove that the locally convex space $C_p(X)$ of continuous real-valued functions on a Tychonoff space $X$ equipped with the topology of pointwise convergence is distinguished if and only if $X$ is a $\Delta$-space in the sense of Knight in [Trans. Amer. Math. Soc. 339 (1993), pp. 45–60]. As an application of this characterization theorem we obtain the following results:

1) If $X$ is a Čech-complete (in particular, compact) space such that $C_p(X)$ is distinguished, then $X$ is scattered.
2) For every separable compact space of the Isbell–Mrówka type $X$, the space $C_p(X)$ is distinguished.
3) If $X$ is the compact space of ordinals $[0, \omega_1]$, then $C_p(X)$ is not distinguished.

We observe that the existence of an uncountable separable metrizable space $X$ such that $C_p(X)$ is distinguished, is independent of ZFC. We also explore the question to which extent the class of $\Delta$-spaces is invariant under basic topological operations.

## 1. Introduction

Following J. Dieudonné and L. Schwartz [9] a locally convex space (lcs) $E$ is called *distinguished* if every bounded subset of the bidual of $E$ in the weak*-topology is contained in the closure of the weak*-topology of some bounded subset of $E$. Equivalently, a lcs $E$ is distinguished if and only if the strong dual of $E$ (i.e. the topological dual of $E$ endowed with the strong topology) is *barrelled*, (see [19, 8.7.1]). A. Grothendieck [17] proved that a metrizable lcs $E$ is distinguished if and only if its strong dual is *bornological*. We refer the reader to survey articles [6] and [7] which present several more modern results about distinguished metrizable and Fréchet lcs.

Throughout the article, all topological spaces are assumed to be Tychonoff and infinite. By $C_p(X)$ and $C_k(X)$ we mean the spaces of all real-valued continuous

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functions on a Tychonoff space $X$ endowed with the topology of pointwise convergence and the compact-open topology, respectively. By a *bounded set* in a topological vector space (in particular, $C_p(X)$) we understand any set which is absorbed by every 0-neighbourhood.

For spaces $C_p(X)$ we proved in [14] the following theorem (the equivalence (1) $\Leftrightarrow$ (4) has been obtained in [12]).

**Theorem 1.1.** *For a Tychonoff space $X$, the following conditions are equivalent*:

(1) $C_p(X)$ *is distinguished.*
(2) $C_p(X)$ *is a large subspace of* $\mathbb{R}^X$, *i.e. for every bounded set $A$ in $\mathbb{R}^X$ there exists a bounded set $B$ in $C_p(X)$ such that $A \subset cl_{\mathbb{R}^X}(B)$.*
(3) *For every $f \in \mathbb{R}^X$ there is a bounded set $B \subset C_p(X)$ such that $f \in cl_{\mathbb{R}^X}(B)$.*
(4) *The strong dual of the space $C_p(X)$ carries the finest locally convex topology.*

Several examples of $C_p(X)$ with(out) distinguished property have been provided in papers [12], [13] and [14]. The aim of this research is to continue our initial work on distinguished spaces $C_p(X)$.

The following concept plays a key role in our paper. We show its applicability for the studying of distinguished spaces $C_p(X)$.

**Definition 1.2** ([21])**.** A topological space $X$ is said to be a $\Delta$-space if for every decreasing sequence $\{D_n : n \in \omega\}$ of subsets of $X$ with empty intersection, there is a decreasing sequence $\{V_n : n \in \omega\}$ consisting of open subsets of $X$, also with empty intersection, and such that $D_n \subset V_n$ for every $n \in \omega$.

We should mention that R. W. Knight [21] called all topological spaces $X$ satisfying the above Definition 1.2 by $\Delta$-sets. The original definition of a $\Delta$-set of the real line $\mathbb{R}$ is due to G. M. Reed and E. K. van Douwen (see [28]). In this paper, for general topological spaces satisfying Definition 1.2 we reserve the term $\Delta$-*space.* The class of all $\Delta$-spaces is denoted by $\Delta$.

One of the main results of our paper, Theorem 2.1 says that $X$ is a $\Delta$-space if and only if $C_p(X)$ is a distinguished space. This characterization theorem has been applied systematically for obtaining a range of results from our paper.

Our main result in Section 3 states that a Čech-complete (in particular, compact) $X \in \Delta$ must be scattered. A very natural question arises about what those scattered compact spaces $X \in \Delta$ are. In view of Theorem 2.1, it is known that a Corson compact $X$ belongs to the class $\Delta$ if and only if $X$ is a scattered Eberlein compact space [14]. With the help of Theorems 2.1 and 3.8 we show that the class $\Delta$ contains also all separable compact spaces of the Isbell–Mrówka type. Nevertheless, as we demonstrate in Section 3, there are compact scattered spaces $X \notin \Delta$ (for example, the compact space $[0, \omega_1]$).

Section 4 deals with the questions about metrizable spaces $X \in \Delta$. We notice that every $\sigma$-scattered metrizable space $X$ belongs to the class $\Delta$. For separable metrizable spaces $X$, our analysis reveals a tight connection between distinguished $C_p(X)$ and well-known set-theoretic problems about special subsets of the real line $\mathbb{R}$. We observe that the existence of an uncountable separable metrizable space $X$ such that $C_p(X)$ is distinguished is independent of ZFC and it is equivalent to the existence of a separable countably paracompact nonnormal Moore space. We refer readers to [24] for the history of the normal Moore problem.

In Section 5 we study whether the class $\Delta$ is invariant under the basic topological operations: subspaces, (quotient) continuous images, finite/countable unions and finite products. We pose several new open problems.

## 2. Characterization theorem

In this section we provide a characterization of distinguished spaces $C_p(X)$ in terms of topological properties of the space $X$. For the reader's convenience we recall some relevant terminology.

(a) A disjoint cover $\{X_\gamma : \gamma \in \Gamma\}$ of $X$ is called a *partition* of $X$.
(b) A collection of sets $\{U_\gamma : \gamma \in \Gamma\}$ is called an *expansion* of a collection of sets $\{X_\gamma : \gamma \in \Gamma\}$ in $X$ if $X_\gamma \subseteq U_\gamma \subseteq X$ for every index $\gamma \in \Gamma$.
(c) A collection of sets $\{U_\gamma : \gamma \in \Gamma\}$ is called *point-finite* if no point belongs to infinitely many $U_\gamma$ -s.

**Theorem 2.1.** *For a Tychonoff space $X$, the following conditions are equivalent*:
(1) *$C_p(X)$ is distinguished.*
(2) *Any countable partition of $X$ admits a point-finite open expansion in $X$.*
(3) *Any countable disjoint collection of subsets of $X$ admits a point-finite open expansion in $X$.*
(4) *$X$ is a $\Delta$-space.*

*Proof.* Observe that every collection of pairwise disjoint subsets of $X$, $\{X_\gamma : \gamma \in \Gamma\}$ can be extended to a partition by adding a single set $X_* = X \setminus \bigcup\{X_\gamma : \gamma \in \Gamma\}$. If the obtained partition admits a point-finite open expansion in $X$, then removing one open set we get a point-finite open expansion of the original disjoint collection. This shows evidently the equivalence (2) $\Leftrightarrow$ (3).

Assume now that (3) holds. Let $\{D_n : n \in \omega\}$ be a decreasing sequence subsets of $X$ with empty intersection. Define $X_n = D_n \setminus D_{n+1}$ for each $n \in \omega$. By assumption, a disjoint collection $\{X_n : n \in \omega\}$ admits a point-finite open expansion $\{U_n : n \in \omega\}$ in $X$. Then $\{V_n = \bigcup\{U_i : i \geq n\} : n \in \omega\}$ is an open decreasing expansion in $X$ with empty intersection. This proves the implication (3) $\Rightarrow$ (4).

Next we show (4) $\Rightarrow$ (2). Let $\{X_n : n \in \omega\}$ be any countable partition of $X$. Define $D_0 = X$ and $D_n = X \setminus \bigcup\{X_i : i < n\}$. Then $X_n \subset D_n$ for every $n$, the sequence $\{D_n : n \in \omega\}$ is decreasing and its intersection is empty. Assuming (4), we find an open decreasing expansion $\{U_n : n \in \omega\}$ of $\{D_n : n \in \omega\}$ in $X$ such that $\bigcap\{U_n : n \in \omega\} = \emptyset$. For every $x \in X$ there is $n$ such that $x \notin U_m$ for each $m > n$, it means that $\{U_n : n \in \omega\}$ is a point-finite expansion of $\{X_n : n \in \omega\}$ in $X$. This finishes the proof (3) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

Now we prove the implication (1) $\Rightarrow$ (2). Let $\{X_n : n \in \omega\}$ be any countable partition of $X$. Fix any function $f \in \mathbb{R}^X$ which satisfies the following conditions: for each $n \in \omega$ and every $x \in X_n$ the value of $f(x)$ is greater than $n$. By assumption, there is a bounded subset $B$ of $C_p(X)$ such that $f \in cl_{\mathbb{R}^X}(B)$. Hence, for every $n \in \omega$ and every point $x \in X_n$, there exists $f_x \in B$ such that $f_x(x) > n$. But $f_x$ is a continuous function, therefore there is an open neighbourhood $U_x \subset X$ of $x$ such that $f_x(y) > n$ for every $y \in U_x$. We define an open set $U_n \subset X$ as follows: $U_n = \bigcup\{U_x : x \in X_n\}$. Evidently, $X_n \subseteq U_n$ for each $n \in \omega$. If we assume that the open expansion $\{U_n : n \in \omega\}$ is not point-finite, then there exists a point $y \in X$ such that there are infinitely many numbers $n$ with $y \in U_{x_n}$ for some $x_n \in X_n$. This means that $\sup\{g(y) : g \in B\} = \infty$, which contradicts the boundedness of $B$.

It remains to prove (2) ⇒ (1). By Theorem 1.1, we need to show that for every mapping $f \in \mathbb{R}^X$ there is a bounded set $B \subset C_p(X)$ such that $f \in cl_{\mathbb{R}^X}(B)$. If there exists a constant $r > 0$ such that $\sup\{|f(x)| : x \in X\} < r$, then we take $B = \{h \in C(X) : \sup\{|h(x)| : x \in X\} < r\}$. It is easy to see that $B$ is as required.

Let $f \in \mathbb{R}^X$ be unbounded. Denote by $Y_0 = \emptyset$ and $Y_n = \{x \in X : n-1 \leq |f(x)| < n\}$ for each non-zero $n \in \omega$. Define $\varphi : X \to \omega$ by the rule: if $Y_n \neq \emptyset$ then $\varphi(x) = n$ for every $x \in Y_n$. So, $|f| < \varphi$. Put $X_n = \varphi^{-1}(n)$ for each $n \in \omega$. Note that some sets $X_n$ might happen to be empty, but the collection $\{X_n : n \in \omega\}$ is a partition of $X$ with countably many nonempty $X_n$ -s. By our assumption, there exists a point-finite open expansion $\{U_n : n \in \omega\}$ of the partition $\{X_n : n \in \omega\}$. Define $F : X \to \omega$ by $F(x) = \max\{n : x \in U_n\}$. Obviously, $f < F$. Finally, we define $B = \{h \in C_p(X) : |h| \leq F\}$. Then $f \in cl_{\mathbb{R}^X}(B)$, because for every finite subset $K \subset X$ there is a function $h \in B$ such that $f \restriction_K = h \restriction_K$. Indeed, given a finite subset $K \subset X$, let $\{V_x : x \in K\}$ be the family of pairwise disjoint open sets such that $x \in V_x \subset U_{\varphi(x)}$ for every $x \in K$. For each $x \in K$, fix a continuous function $h_x : X \to [-\varphi(x), \varphi(x)]$ such that $h_x(x) = f(x)$ and $h_x$ is equal to the constant value 0 on the closed set $X \setminus V_x$. One can verify that $h = \Sigma_{x\in K} h_x \in B$ is as required. □

Below we present a straightforward application of Theorem 2.1.

**Corollary 2.2** ([14]). *Let $Z$ be any subspace of $X$. If $X$ belongs to the class $\Delta$, then $Z$ also belongs to the class $\Delta$.*

*Proof.* If $\{Z_\gamma : \gamma \in \Gamma\}$ is any collection of pairwise disjoint subsets of $Z$ and there exists a point-finite open expansion $\{U_\gamma : \gamma \in \Gamma\}$ in $X$, then obviously $\{U_\gamma \cap Z : \gamma \in \Gamma\}$ is a point-finite expansion consisting of the sets relatively open in $Z$. It remains to apply Theorem 2.1. □

The last result can be reversed, assuming that $X \setminus Z$ is finite.

**Proposition 2.3.** *Let $Z$ be a subspace of $X$ such that $Y = X \setminus Z$ is finite. If $Z$ belongs to the class $\Delta$, then $X$ belongs to $\Delta$ as well.*

*Proof.* Let $\{X_n : n \in \omega\}$ be any countable collection of pairwise disjoint subsets of $X$. Denote by $F$ the set of those $n \in \omega$ such that $X_n \bigcap Y \neq \emptyset$. There might be only finitely many $X_n$ -s which intersect the finite set $Y$, hence $F \subset \omega$ is finite. If $n \in F$, then we simply declare that $U_n$ is equal to $X$. Consider the subcollection $\{X_n : n \in \omega \setminus F\}$. It is a countable collection of pairwise disjoint subsets of $Z$. Since $Z \in \Delta$, by Theorem 2.1, there is a point-finite open expansion $\{U_n : n \in \omega \setminus F\}$ in $Z$. Observe that $Z$ is open in $X$, therefore all those $U_n$ -s remain open in $X$. Bringing all $U_n$ -s of both sorts together we obtain a point-finite open expansion $\{U_n : n \in \omega\}$ in $X$. Finally, $X \in \Delta$, by Theorem 2.1. □

*Remark* 2.4. The following applicable concept has been re-introduced in [14]. A family $\{\mathcal{N}_x : x \in X\}$ of subsets of a Tychonoff space $X$ is called a *scant cover* for $X$ if each $\mathcal{N}_x$ is an open neighbourhood of $x$ and for each $u \in X$ the set $X_u = \{x \in X : u \in \mathcal{N}_x\}$ is finite.[1]

Our Theorem 2.1 generalizes one of the results obtained in [14] stating that if $X$ admits a scant cover $\{\mathcal{N}_x : x \in X\}$ then $C_p(X)$ is distinguished. Indeed, let

[1] The referee kindly informed the authors that this notion also is known in the literature under the name *the point-finite neighbourhood assignment.*

$\{X_\gamma : \gamma \in \Gamma\}$ be any collection of pairwise disjoint subsets of $X$. Define $U_\gamma = \bigcup\{\mathcal{N}_x : x \in X_\gamma\}$. It is easily seen that $\{U_\gamma : \gamma \in \Gamma\}$ is a point-finite open expansion in $X$, by definition of a scant cover. Applying Theorem 2.1, we conclude that $C_p(X)$ is distinguished.

## 3. Applications to compact spaces $X \in \Delta$

First we recall a few definitions and facts (probably well-known) which will be used in the sequel. A space $X$ is said to be *scattered* if every nonempty subset $A$ of $X$ has an isolated point in $A$. Denote by $A^{(1)}$ the set of all non-isolated (in $A$) points of $A \subset X$. For ordinal numbers $\alpha$, the $\alpha$-th derivative of a topological space $X$ is defined by transfinite induction as follows.

$X^{(0)} = X$; $X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}$; $X^{(\gamma)} = \bigcap_{\alpha<\gamma} X^{(\alpha)}$ for limit ordinals $\gamma$.

For a scattered space $X$, the smallest ordinal $\alpha$ such that $X^{(\alpha)} = \emptyset$ is called the *scattered height* of $X$ and is denoted by $ht(X)$. For instance, $X$ is discrete if and only if $ht(X) = 1$.

The following classical theorem is due to A. Pełczyński and Z. Semadeni.

**Theorem 3.1** ([29, Theorem 8.5.4])**.** *A compact space $X$ is scattered if and only if there is no continuous mapping of $X$ onto the segment* $[0, 1]$.

A continuous surjection $\pi : X \to Y$ is called *irreducible* (see [29, Definition 7.1.11]) if for every closed subset $F$ of $X$ the condition $\pi(F) = Y$ implies $F = X$.

**Proposition 3.2** ([29, Proposition 7.1.13])**.** *Let $X$ be a compact space and let $\pi : X \to Y$ be a continuous surjection. Then there exists a closed subset $F$ of $X$ such that $\pi(F) = Y$ and the restriction $\pi\restriction_F : F \to Y$ is irreducible.*

**Proposition 3.3** ([29, Proposition 25.2.1])**.** *Let $X$ be a compact space and let $\pi : X \to Y$ be a continuous surjection. Then $\pi$ is irreducible if and only if whenever $E \subset X$ and $\pi(E)$ is dense in $Y$, then $E$ is dense in $X$.*

Recall that a Tychonoff space $X$ is *Čech-complete* if $X$ is a $G_\delta$-set in some (equivalently, any) compactification of $X$, (see [10, 3.9.1]). It is well known that every locally compact space and every completely metrizable space is Čech-complete. Next statement resolves an open question posed in [14].

**Theorem 3.4.** *Every Čech-complete (in particular, compact) $\Delta$-space is scattered.*

*Proof.*

*Step* 1 ($X$ is compact). On the contrary, assume that $X$ is not scattered. First, by Theorem 3.1, there is a continuous mapping $\pi$ from $X$ onto the segment $[0, 1]$. Second, by Proposition 3.2, there exists a closed subset $F$ of $X$ such that $\pi(F) = [0, 1]$ and the restriction $\pi\restriction_F : F \to [0, 1]$ is irreducible. Since $X \in \Delta$ the compact space $F$ also belongs to $\Delta$, by Corollary 2.2. For simplicity, without loss of generality we may assume that $F$ is $X$ itself and $\pi : X \to [0, 1]$ is irreducible.

Let $\{X_n : n \in \omega\}$ be a partition of $[0, 1]$ into dense sets. Put $Y_n = \bigcup_{k\geq n} X_k$, and $Z_n = \pi^{-1}(Y_n)$ for all $n \in \omega$. Then all sets $Z_n$ are dense in $X$ by Proposition 3.3 and the intersection $\bigcap_{n\in\omega} Z_n$ is empty. Every compact space $X$ is a Baire space, i.e. the Baire category theorem holds in $X$, hence if $\{U_n : n \in \omega\}$ is any open expansion of $\{Z_n : n \in \omega\}$, then the intersection $\bigcap_{n\in\omega} U_n$ is dense in $X$. In view of our Theorem 2.1 this conclusion contradicts the assumption $X \in \Delta$, and the proof follows.

*Step* 2 ($X$ is any Čech-complete space). By the first step we deduce that every compact subset of $X$ is scattered. But any Čech-complete space $X$ is scattered if and only if every compact subset of $X$ is scattered. A detailed proof of this probably folklore statement can be found in [30]. □

**Proposition 3.5.** *If $X$ is a first-countable compact space, then $X \in \Delta$ if and only if $X$ is countable.*

*Proof.* If $X \in \Delta$, then $X$ is scattered, by Theorem 3.4. By the classical theorem of S. Mazurkiewicz and W. Sierpiński [29, Theorem 8.6.10], a first-countable compact space is scattered if and only if it is countable. This proves (i) $\Rightarrow$ (ii). The converse is known [14] and follows from the fact that any countable space $X = \{x_n : n \in \omega\}$ admits a scant cover. Indeed, define $X_n = \{x_i : i \geq n\}$. Then the family $\{X_n : n \in \omega\}$ is a scant cover of $X$. Now it suffices to mention Remark 2.4. □

*Remark* 3.6. Theorem 3.4 extends also a well-known result of B. Knaster and K. Urbanik stating that every countable Čech-complete space is scattered [20]. It is easy to see that a countable Baire space contains a dense subset of isolated points, but in general does not have to be scattered. We don't know whether every Baire $\Delta$-space must have isolated points.

Recall that an Eberlein compact is a compact space homeomorphic to a subset of a Banach space with the weak topology. A compact space is said to be a Corson compact space if it can be embedded in a $\Sigma$-product of the real lines. Every Eberlein compact is Corson, but not vice versa. However, every scattered Corson compact space is a scattered Eberlein compact space [1].

**Theorem 3.7** ([14])**.** *A Corson compact space $X$ belongs to the class $\Delta$ if and only if $X$ is a scattered Eberlein compact space.*

Bearing in mind Theorem 3.4, to show Theorem 3.7 it suffices to use the fact that every scattered Eberlein compact space admits a scant cover (the latter follows from the proof of [5, Lemma 1.1]) and then apply Remark 2.4.

Being motivated by the previous results one can ask if there exist scattered compact spaces $X \in \Delta$ which are not Eberlein compact. The next question is also crucial: Does a compact scattered space $X \notin \Delta$ exist? Below we answer both questions positively.

We need the following somewhat technical

**Theorem 3.8.** *Let $Z = C_0 \cup C_1$ be a space such that*

(1) $C_0 \cap C_1 = \emptyset$.
(2) *$C_0$ is an open $F_\sigma$ subset of $Z$.*
(3) *both $C_0$ and $C_1$ belong to the class $\Delta$.*

*Then $Z$ also belongs to the class $\Delta$.*

*Proof.* By assumption, $C_0 = \bigcup\{F_n : n \in \omega\}$, where each $F_n$ is closed in $Z$. Let $\{X_n : n \in \omega\}$ be any countable collection of pairwise disjoint subsets of $Z$. Our target is to define open sets $U_n \supseteq X_n$, $n \in \omega$ in such a way that the collection $\{U_n : n \in \omega\}$ is point-finite. We decompose the sets $X_n = X_n^0 \cup X_n^1$, where $X_n^0 = X_n \cap C_0$ and $X_n^1 = X_n \cap C_1$. By Theorem 2.1, the collection $\{X_n^0 : n \in \omega\}$ expands to a point-finite open collection $\{U_n^0 : n \in \omega\}$ in $C_0$. The set $C_0$ is open in $Z$, therefore $U_n^0$ are open in $Z$ as well.

Now we consider the disjoint collection $\{X_n^1 : n \in \omega\}$ in $C_1$. By assumption, $C_1 \in \Delta$, therefore applying Theorem 2.1 once more, we find a point-finite expansion $\{V_n^1 : n \in \omega\}$ in $C_1$ consisting of sets which are open in $C_1$. Every set $V_n^1$ is a trace of some set $W_n^1$, which is open in $Z$, i.e. $V_n^1 = W_n^1 \cap C_1$, and every $W_n^1$ is open in $Z$. We refine the sets $W_n^1$ by the formula $U_n^1 = W_n^1 \setminus \bigcup\{F_i : i \leq n\}$. Since all sets $F_i$ are closed in $Z$, the sets $U_n^1$ remain open in $Z$. Since all sets $F_i$ are disjoint with $C_1$, the collection $\{U_n^1 : n \in \omega\}$ remains to be an expansion of $\{X_n^1 : n \in \omega\}$. Furthermore, the collection $\{U_n^1 : n \in \omega\}$ is point-finite, because $\{V_n^1 : n \in \omega\}$ is point-finite, and every point $z \in C_0$ belongs to some $F_n$, hence $z \notin U_m^1$ for every $m \geq n$. Finally, we define $U_n = U_n^0 \cup U_n^1$. The collection $\{U_n : n \in \omega\}$ is a point-finite open expansion of $\{X_n : n \in \omega\}$, and the proof is complete. □

This yields the following

**Corollary 3.9.** *Let $Z$ be any separable scattered space such that its scattered height $ht(Z)$ is equal to 2. Then $Z \in \Delta$.*

*Proof.* The structure of $Z$ is the following. $Z = C_0 \cup C_1$, where $C_0$ is a countable dense in $Z$ set consisting of isolated in $Z$ points and $C_1$ consists of all accumulation points. Moreover, the space $C_1$ with the topology induced from $Z$ is discrete. All conditions of Theorem 3.8 are satisfied, and the result follows. □

Our first example will be the one-point compactification of an Isbell–Mrówka space $\Psi(\mathcal{A})$. We recall the construction and basic properties of $\Psi(\mathcal{A})$. Let $\mathcal{A}$ be an almost disjoint family of subsets of the set of natural numbers $\mathbb{N}$ and let $\Psi(\mathcal{A})$ be the set $\mathbb{N} \cup \mathcal{A}$ equipped with the topology defined as follows. For each $n \in \mathbb{N}$, the singleton $\{n\}$ is open, and for each $A \in \mathcal{A}$, a base of neighbourhoods of $A$ is the collection of all sets of the form $\{A\} \cup B$, where $B \subset A$ and $|A \setminus B| < \omega$. The space $\Psi(\mathcal{A})$ is then a first-countable separable locally compact Tychonoff space. If $\mathcal{A}$ is a maximal almost disjoint (MAD) family, then the corresponding Isbell–Mrówka space $\Psi(\mathcal{A})$ would be in addition pseudocompact. (Readers are advised to consult [18] which surveys various topological properties of these spaces).

**Theorem 3.10.** *There exists a separable scattered compact space $X$ with the following properties:*

(a) *The scattered height of $X$ is equal to 3.*
(b) *$X \in \Delta$.*
(c) *$X$ is not an Eberlein compact space.*

*Proof.* Let $\mathcal{A}$ be any uncountable almost disjoint (in particular, MAD) family of subsets of $\mathbb{N}$ and let $Z$ be the corresponding first-countable separable locally compact Isbell–Mrówka space $\Psi(\mathcal{A})$. It is easy to see that $Z = \Psi(\mathcal{A})$ satisfies the assumptions of Corollary 3.9. Hence, $Z \in \Delta$. Now, denote by $X$ the one-point compactification of the separable locally compact space $Z$. Then the scattered height of $X$ is equal to 3. Note that $X \in \Delta$ by Proposition 2.3. Moreover, $X$ is not an Eberlein compact space, since every separable Eberlein compact space is metrizable, while $\Psi(\mathcal{A})$ is metrizable if and only if $\mathcal{A}$ is countable. □

Now we show that there do exist scattered compact spaces which are not in the class $\Delta$. We will use the classical Pressing Down Lemma. Let $[0, \omega_1)$ be the set of all countable ordinals equipped with the order topology. For simplicity, we identify $[0, \omega_1)$ with $\omega_1$. A subset $S$ of $\omega_1$ is called a *stationary* subset if $S$ has nonempty intersection with every closed and unbounded set in $\omega_1$. A mapping $\varphi : S \to \omega_1$ is called *regressive* if $\varphi(\alpha) < \alpha$ for each $\alpha \in S$. The proof of the following fundamental statement can be found for instance in [22].

**Theorem 3.11** (*Pressing Down Lemma*)**.** *Let $\varphi : S \to \omega_1$ be a regressive mapping, where $S$ is a stationary subset of $\omega_1$. Then for some $\gamma < \omega_1$, $\varphi^{-1}\{\gamma\}$ is a stationary subset of $\omega_1$.*

It is known that there are plenty of stationary subsets of $\omega_1$. In particular, every stationary set can be partitioned into countably many pairwise disjoint stationary sets [22]. Note that $\omega_1$ is a scattered locally compact and first-countable space. Next statement resolves an open question posed in [14].

**Theorem 3.12.** *The compact scattered space $[0, \omega_1]$ is not in the class $\Delta$.*

*Proof.* It suffices to show that $\omega_1$ does not belong to the class $\Delta$. Assume, on the contrary, that $\omega_1 \in \Delta$. Denote by $L$ the set of all countable limit ordinals. Evidently, $L$ is a closed unbounded set in $\omega_1$. Take any representation of $L$ as the union of countably many pairwise disjoint stationary sets $\{S_n : n \in \omega\}$. By Theorem 2.1, there exists a point-finite open expansion $\{U_n : n \in \omega\}$ in $\omega_1$.

For every $\alpha \in U_n$ there is an ordinal $\beta(\alpha) < \alpha$ such that $[\beta(\alpha), \alpha] \subset U_n$. In fact, for every $n \in \omega$ we can define a regressive mapping $\varphi_n : S_n \to \omega_1$ by the formula: $\varphi_n(\alpha) = \beta(\alpha)$. Since $S_n$ is a stationary set for every $n$, we can apply to $\varphi_n$ the Pressing Down Lemma. Hence, for each $n$ there are a countable ordinal $\gamma_n$ and an uncountable subset $T_n \subset S_n$ with the following property: $[\gamma_n, \alpha] \subset U_n$ for every $\alpha \in T_n$. Denote $\gamma = \sup\{\gamma_n : n \in \omega\} \in \omega_1$. Because all $T_n$ are unbounded, for all natural $n$ we have an ordinal $\alpha_n \in T_n$ such that $\gamma < \alpha_n$ and $[\gamma_n, \alpha_n] \subset U_n$. This implies that $\gamma \in U_n$ for every $n \in \omega$. However, a collection $\{U_n : n \in \omega\}$ is point-finite. The obtained contradiction finishes the proof. □

The function space $C_k(X)$ is called *Asplund* if every separable vector subspace of $C_k(X)$ isomorphic to a Banach space, has the separable dual.

**Proposition 3.13.** *If $X \in \Delta$, then the space $C_k(X)$ is Asplund. The converse conclusion fails in general.*

*Proof.* Let $\mathcal{K}(X)$ be the family of all compact subset of $X$. By the assumption and Corollary 2.2, each $K \in \mathcal{K}(X)$ belongs to the class $\Delta$. Clearly, $C_k(X)$ is isomorphic to a linear subspace of the product $\Pi = \prod_{K \in \mathcal{K}(X)} C_k(K)$ of Banach spaces $C_k(K)$. Assume that $E$ is a separable vector subspace of $C_k(X)$ isomorphic to a Banach space. Observe that $E$ is isomorphic to a subspace of the finite product $\prod_{j \in F} C_k(K_j)$ for $K_j \in \mathcal{K}(X)$ and $j \in F$. Indeed, let $B$ be the unit (bounded) ball of the normed space $E$. Then there exists a finite set $F$ such that $\bigcap_{j \in F} \pi_j^{-1}(U_j) \cap E \subset B$, where $U_j$ are balls in spaces $C_k(K_j)$, $j \in F$, and $\pi_j$ are natural projections from $E$ onto $C_k(K_j)$. Let $\pi_F$ be the (continuous) projection from $\Pi$ onto $\prod_{j \in F} C_k(K_j)$. Then $\pi_F \restriction_E$ is an injective continuous and open map from $E$ onto $(\pi_F \restriction_E)(E) \subset \prod_{j \in F} C_k(K_j)$. The injectivity of $\pi_F \restriction_E$ follows from the fact that $B$ is a bounded neighbourhood of zero in $E$. It is easy to see that the image $(\pi_F \restriction_E)(B)$ is an open neighbourhood of zero in $\prod_{j \in F} C_k(K_j)$. On the other hand, $\prod_{j \in F} C_k(K_j)$ is isomorphic to the space $C_k(\bigoplus_{j \in F} K_j)$ and the compact space $\bigoplus_{j \in F} K_j$ is scattered. By the classical [11, Theorem 12.29] $E$ must have the separable dual $E^*$. Hence, $C_k(X)$ is Asplund. The converse fails, as Theorem 3.12 shows for $X = [0, \omega_1]$. □

Since every infinite compact scattered space $X$ contains a nontrivial converging sequence, for such $X$ the Banach space $C(X)$ is not a Grothendieck space, (see [8]).

**Corollary 3.14.** *If $X$ is an infinite compact and $X \in \Delta$, then the Banach space $C(X)$ is not a Grothendieck space. The converse fails, as $X = [0, \omega_1]$ applies.*

For non-scattered spaces $X$ Theorem 3.4 implies immediately the following

**Corollary 3.15.** *If $X$ is a non-scattered space, the Stone-Čech compactification $\beta X$ is not in the class $\Delta$.*

**Proposition 3.16.** *Let $X = \beta Z \setminus Z$, where $Z$ is any infinite discrete space. Then $X$ is not in the class $\Delta$.*

*Proof.* $\beta Z \setminus Z$ does not have isolated points for any infinite discrete space $Z$. □

It is known that $X = [0, \omega_1]$ is the Stone-Čech compactification of $[0, \omega_1)$. We showed that $X \notin \Delta$. Also, $\beta Z \notin \Delta$ for any infinite discrete space $Z$. Every scattered Eberlein compact space belongs to the class $\Delta$ by Theorem 3.7; however, no Eberlein compact $X$ can be the Stone-Čech compactification $\beta Z$ for any proper subset $Z$ of $X$ by the Preiss–Simon theorem (see [2, Theorem IV.5.8]). All of these facts provide a motivation for the following result.

**Example 3.17.** There exists an Isbell–Mrówka space $Z$ which is *almost compact* in the sense that the one-point compactification of $Z$ coincides with $\beta Z$ (see [18, Theorem 8.6.1]). Define $X = \beta Z$. Then $X \in \Delta$, by Theorem 3.10.

## 4. Metrizable spaces $X \in \Delta$

In this section we try to describe constructively the structure of nontrivial metrizable spaces $X \in \Delta$. Note first that every scattered metrizable $X$ is in the class $\Delta$ since every such space $X$ homeomorphically embeds into a scattered Eberlein compact [3], and then Theorem 3.7 and Corollary 2.2 apply. We extend this result as follows.

A topological space $X$ is said to be *$\sigma$-scattered* if $X$ can be represented as a countable union of scattered subspaces and $X$ is called *strongly $\sigma$-discrete* if it is a union of countably many of its closed discrete subspaces. Strongly $\sigma$-discreteness of $X$ implies that $X$ is $\sigma$-scattered, for any topological space. For metrizable $X$, by the classical result of A. H. Stone [31], these two properties are equivalent.

**Proposition 4.1.** *Any $\sigma$-scattered metrizable space belongs to the class $\Delta$.*

*Proof.* In view of aforementioned equivalence, every subset of $X$ is $F_\sigma$. If every subset of $X$ is $F_\sigma$, then $X \in \Delta$. This fact apparently is well-known (see also a comment after Claim 4.2). For the sake of completeness we include a direct argument. We show that $X$ satisfies the condition (2) of Theorem 2.1. Let $\{X_n : n \in \omega\}$ be any countable disjoint partition of $X$. Denote $X_n = \bigcup\{F_{n,m} : m \in \omega\}$, where each $F_{n,m}$ is closed in $X$. Define open sets $U_n$ as follows: $U_0 = X$ and $U_n = X \setminus \bigcup\{F_{k,m} : k < n, m < n\}$ for $n \geq 1$. Then $\{U_n : n \in \omega\}$ is a point-finite open expansion of $\{X_n : n \in \omega\}$ in $X$. □

A metrizable space $A$ is called an *absolutely analytic* if $A$ is homeomorphic to a Souslin subspace of a complete metric space $X$ (of an arbitrary weight), i.e. $A$ is expressible as $A = \bigcup_{\sigma \in \mathbb{N}^\mathbb{N}} \bigcap_{n \in \mathbb{N}} A_{\sigma|n}$, where each $A_{\sigma|n}$ is a closed subset of $X$. It is known that every absolutely analytic metrizable space $X$ (in particular, every Borel subspace of a complete metric space) either contains a homeomorphic copy of the Cantor set or it is strongly $\sigma$-discrete. Therefore, for absolutely analytic metrizable space $X$ the converse is true: $X \in \Delta$ implies that $X$ is strongly $\sigma$-discrete [14].

However, the last structural result can not be proved in general for all (separable) metrizable spaces without extra set-theoretic assumptions. Let us recall several definitions of special subsets of the real line $\mathbb{R}$ (see [23], [28]).

(a) A $Q$-set $X$ is a subset of $\mathbb{R}$ such that each subset of $X$ is $F_\sigma$, or, equivalently, each subset of $X$ is $G_\delta$ in $X$.
(b) A $\lambda$-set $X$ is a subset of $\mathbb{R}$ such that each countable $A \subset X$ is $G_\delta$ in $X$.
(c) A $\Delta$-set $X$ is a subset of $\mathbb{R}$ such that for every decreasing sequence $\{D_n : n \in \omega\}$ subsets of $X$ with empty intersection there is a decreasing expansion $\{V_n : n \in \omega\}$ consisting of open subsets of $X$ with empty intersection.

*Claim* 4.2. The existence of an uncountable separable metrizable $\Delta$-space is equivalent to the existence of an uncountable $\Delta$-set.

*Proof.* Note that every separable metrizable space homeomorphically embeds into a Polish space $\mathbb{R}^\omega$ and the latter space is a one-to-one continuous image of the set of irrationals $\mathbb{P}$. Therefore, if $M$ is an uncountable separable metrizable space, then there exist an uncountable set $X \subset \mathbb{R}$ and a one-to-one continuous mapping from $X$ onto $M$. It is easy to see that $X$ is a $\Delta$-set provided $M$ is a $\Delta$-space. □

Note that in the original definition of a $\Delta$-set, G. M. Reed used $G_\delta$-sets instead of open sets and E. van Douwen observed that these two versions are equivalent [28]. From the original definition it is obvious that each $Q$-set must be a $\Delta$-set. The fact that every $\Delta$-set is a $\lambda$-set is known as well. K. Kuratowski showed that in ZFC there exist uncountable $\lambda$-sets. The existence of an uncountable $Q$-set is one of the fundamental set-theoretical problems considered by many authors. F. Hausdorff showed that the cardinality of an uncountable $Q$-set $X$ has to be strictly smaller than the continuum $\mathfrak{c} = 2^{\aleph_0}$, so in models of ZFC plus the Continuum Hypothesis (CH) there are no uncountable $Q$-sets. Let us outline several of the most relevant known facts.

(1) Martin's Axiom plus the negation of the Continuum Hypothesis (MA $+\neg$CH) implies that every subset $X \subset \mathbb{R}$ of cardinality less than $\mathfrak{c}$ is a $Q$-set (see [16]).

(2) It is consistent that there is a $Q$-set $X$ such that its square $X^2$ is not a $Q$-set [15].

(3) The existence of an uncountable $Q$-set is equivalent to the existence of an uncountable strong $Q$-set, i.e. a $Q$-set all finite powers of which are $Q$-sets [26].

(4) No $\Delta$-set $X$ can have cardinality $\mathfrak{c}$ [27]. Hence, under MA, every subset of $\mathbb{R}$ that is a $\Delta$-set is also a $Q$-set. Recently we proved the following claim: If $X$ has a countable network and $|X| = \mathfrak{c}$, then $C_p(X)$ is not distinguished [14]. In view of our Theorem 2.1 this fact means that no $\Delta$-space $X$ with a countable network can have cardinality $\mathfrak{c}$.[2]

(5) It is consistent that there exists a $\Delta$-set $X$ that is not a $Q$-set [21]. Of course, there are plenty of nonmetrizable $\Delta$-spaces with non-$G_\delta$ subsets, in ZFC.

(6) An uncountable $\Delta$-set exists if and only if there exists a separable countably paracompact nonnormal Moore space (see [33] and [27]).

[2]The referee kindly informed the authors that the last result can be derived easily from the actual argument of [27].

Summarizing, the following conclusion is an immediate consequence of our Theorem 2.1 and the known facts about $\Delta$-sets listed above.

**Corollary 4.3.**

(1) *The existence of an uncountable separable metrizable space such that* $C_p(X)$ *is distinguished, is independent of ZFC.*

(2) *There exists an uncountable separable metrizable space* $X$ *such that* $C_p(X)$ *is distinguished, if and only if there exists a separable countably paracompact nonnormal Moore space.*

## 5. Basic operations in $\Delta$ and open problems

In this section we consider the question whether the class $\Delta$ is invariant under the following basic topological operations: subspaces, continuous images, quotient continuous images, finite/countable unions, finite products.

*1. Subspaces.* Trivial because of Corollary 2.2.

*2. (Quotient) continuous images.* Evidently, every topological space is a continuous image of a discrete one. The following assertion in fact has been remarked for the first time in [25].

**Proposition 5.1** ([25])**.** *There exists in ZFC a MAD family* $\mathcal{A}$ *on* $\mathbb{N}$ *such that the corresponding Isbell–Mrówka space* $\Psi(\mathcal{A})$ *admits a continuous mapping onto the closed interval* $[0,1]$*.*

Detailed constructions of such MAD families $\mathcal{M}$ can be found in [4], [32].

Thus, the class $\Delta$ is not invariant under continuous images even for first-countable separable locally compact spaces. However, a continuous mapping in Proposition 5.1 cannot be quotient.

**Proposition 5.2.** *Every quotient continuous image of any Isbell–Mrówka space is a* $\Delta$*-space.*

*Proof.* We observe that by construction any Isbell–Mrówka space $\Psi(\mathcal{A}) = \mathbb{N} \cup \mathcal{A}$ satisfies the following property: every subset of $\mathcal{A}$ is closed in $\Psi(\mathcal{A})$ and $\mathbb{N}$ is obviously countable. Let $Y$ be the image of $\Psi(\mathcal{A})$ under a quotient continuous mapping $f$. We show that $Y$ enjoys the same property. Indeed, denote by $M$ the image $f(\mathbb{N})$ and put $F = Y \setminus M$. Evidently, $M$ is at most countable and for every subset $B$ of $F$ the preimage $f^{-1}(B)$ is closed in $\Psi(\mathcal{A})$ as a subset of $\mathcal{A}$, therefore $B$ is closed in $Y$. It follows that $Y$ is $Q$-set space, i.e. every subset of $Y$ is $F_\sigma$. We noticed in the proof of Proposition 4.1 that the latter property implies that $Y \in \Delta$. □

Note also that the class of scattered Eberlein compact spaces preserves continuous images. We were unable to resolve the following major open problem.

**Problem 5.3.** Let $X$ be any compact $\Delta$-space and $Y$ be a continuous image of $X$. Is $Y$ a $\Delta$-space?

Even a more general question is open.

**Problem 5.4.** Let $X$ be any $\Delta$-space and $Y$ be a quotient continuous image of $X$. Is $Y$ a $\Delta$-space?

Towards a solution of these problems we obtained several partial positive results.

**Proposition 5.5.** *Let $X$ be any $\Delta$-space and $\varphi : X \to Y$ be a quotient continuous surjection with only finitely many nontrivial fibers. Then $Y$ is also a $\Delta$-space.*

*Proof.* By assumption, there exists a closed subset $K \subset X$ such that $\varphi(K)$ is finite and $\varphi \restriction_{X\setminus K}: X \setminus K \to Y \setminus \varphi(K)$ is a one-to-one mapping. Both sets $X \setminus K$ and $Y \setminus \varphi(K)$ are open in $X$ and $Y$, respectively. Since $\varphi$ is a quotient continuous mapping, it is easy to see that $\varphi \restriction_{X\setminus K}$ is a homeomorphism. $X \setminus K$ is a $\Delta$-space, hence $Y \setminus \varphi(K)$ is also a $\Delta$-space. Finally, $Y$ is a $\Delta$-space, by Proposition 2.3. □

**Proposition 5.6.** *Let $X$ be any $\Delta$-space and $\varphi : X \to Y$ be a closed continuous surjection with finite fibers. Then $Y$ is also a $\Delta$-space.*

*Proof.* Let $\{Y_n : n \in \omega\}$ be a partition of $Y$. By assumption, the partition $\{\varphi^{-1}(Y_n) : n \in \omega\}$ admits a point-finite open expansion $\{U_n : n \in \omega\}$ in $X$. Clearly, $\varphi(X \setminus U_n)$ are closed sets in $Y$. Define $V_n = Y \setminus \varphi(X \setminus U_n)$ for each $n \in \omega$. We have that $\{V_n : n \in \omega\}$ is an open expansion of $\{Y_n : n \in \omega\}$ in $Y$. It remains to verify that the family $\{V_n : n \in \omega\}$ is point-finite. Indeed, let $y \in Y$ be any point. Each point in the fiber $\varphi^{-1}(y)$ belongs to a finite number of sets $U_n$. Since the fiber $\varphi^{-1}(y)$ is finite, $y$ is contained only in a finite number of sets $V_n$ which finishes the proof. □

*3. Finite/countable unions.*

**Proposition 5.7.** *Assume that $X$ is a finite union of closed subsets $X_i$, where each $X_i$ belongs to the class $\Delta$. Then $X$ also belongs to $\Delta$. In particular, a finite union of compact $\Delta$-spaces is also a $\Delta$-space.*

*Proof.* Denote by $Z$ the discrete finite union of $\Delta$-spaces $X_i$. Obviously, $Z$ is a $\Delta$-space which admits a natural closed continuous mapping onto $X$. Since all fibers of this mapping are finite, the result follows from Proposition 5.6. □

We recall a definition of the Michael line. The Michael line $X$ is the refinement of the real line $\mathbb{R}$ obtained by isolating all irrational points. So, $X$ can be represented as a countable disjoint union of singletons (rationals) and an open discrete set. Nevertheless, the Michael line $X$ is not in $\Delta$ [14]. This example and Proposition 5.7 justify the following

**Problem 5.8.** Let $X$ be a countable union of compact subspaces $X_i$ such that each $X_i$ belongs to the class $\Delta$. Does $X$ belong to the class $\Delta$?

*4. Finite products.* We already mentioned earlier that the existence of a $Q$-set $X \subset \mathbb{R}$ such that its square $X^2$ is not a $Q$-set, is consistent with ZFC.

**Problem 5.9.** Is the existence of a $\Delta$-set $X \subset \mathbb{R}$ such that its square $X^2$ is not a $\Delta$-set, consistent with ZFC?

It is known that the finite product of scattered Eberlein compact spaces is a scattered Eberlein compact.

**Problem 5.10.** Let $X$ be the product of two compact spaces $X_1$ and $X_2$ such that each $X_i$ belongs to the class $\Delta$. Does $X$ belong to the class $\Delta$?

Our last problem is inspired by Theorem 3.10.

**Problem 5.11.** Let $X$ be any scattered compact space with a finite scattered height. Does $X$ belong to the class $\Delta$?

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