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A NOTE ON FINITENESS PROPERTIES OF GRAPHS OF GROUPS

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ABSTRACT. We show that if G is of type $\mathcal{F}_n$, and G splits as a finite graph of groups, then the vertex groups are of type $\mathcal{F}_n$ if the edge groups are of type $\mathcal{F}_n$.

1. INTRODUCTION

Definition 1.1. A group G is $\mathcal{F}_n$ if G is π_1 of an aspherical complex X whose n-skeleton is compact. Equivalently, G is $\mathcal{F}_n$ if it acts freely and cocompactly on an $(n-1)$-connected n-complex. See [Geo08, Sec 7.2].

Every group is $\mathcal{F}_0$ since (-1)-connected just means nonempty. $\mathcal{F}_1$ means finitely generated, and $\mathcal{F}_2$ means finitely presented.

The purpose of this note is to explain the following which is proven in Theorem 5.1:

Theorem 1.2. *Let G split as a finite graph of groups with $\mathcal{F}_n$ edge groups. If G is $\mathcal{F}_n$ then each vertex group is $\mathcal{F}_n$.*

For $n = 1$, Theorem 1.2 is the following. It is obtained in [DD89] but the idea goes back to Stallings' binding ties [Sta65], and the theorem is surely older.

Theorem 1.3. *Let G be a finitely generated group that splits as a graph Γ of groups. If each edge group is finitely generated then each vertex group is finitely generated.*

For $n = 2$, Theorem 1.2 is the following:

Theorem 1.4. *Let G be a finitely presented group that splits as a graph of groups. If each edge group is finitely presented then each vertex group is finitely presented.*

Theorem 1.4 appears to be a "folk theorem". Dunwoody suggested to us that it could be obtained by applying [DD89, Thm VI.4.4] followed by a folding sequence [BF91]. There is a proof of it by Guirardel-Levitt who obtained a more powerful version relating to relative properties [GL17, Prop 4.9].

Theorem 1.2 is the converse to the following classical statement, which holds since a graph of $K(\pi, 1)$ spaces with π_1-injective attaching maps is a $K(\pi, 1)$. See Theorem 2.3.

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Theorem 1.5. *Let G split as a finite graph of groups with $\mathcal{F}_n$ edge groups. If each G_v is $\mathcal{F}_n$ then G is $\mathcal{F}_n$.*

Remark 1.6. Theorem 1.2 holds with the word "finite" removed. Indeed, if a finitely generated group G splits as a graph Γ of groups then for each vertex v of Γ there is a finite subgraph Γ' containing v such that map $\Gamma' \to \Gamma$ induces an isomorphism between the fundamental groups of graphs of groups.

In contrast, Theorem 1.5 fails to hold with "finite" removed. For instance, a free group of infinite rank splits as an infinite graph of trivial groups.

2. Examples and a problem

There are many examples illustrating the failure of $\mathcal{F}_n$ for the vertex or edge groups of an $\mathcal{F}_n$ group that splits as a graph of groups. The most highly studied examples arise in the course of studying finiteness properties of the subgroup N arising from a short exact sequence:

$$1 \to N \to G \to \mathbb{Z} \to 1$$

In this case, $G \cong N \rtimes \mathbb{Z}$ can be thought of as an HNN extension where the edge and vertex groups are copies of N.

There are many examples where G is $\mathcal{F}_n$ but N fails to be $\mathcal{F}_n$. Stallings and then Bieri [Sta63, Bie81] understood the motivating case where $G = (F_2)^n$ and the homomorphism sends the generators of each F_2 factor to the generators of $\mathbb{Z}$. Remarkably, while G is $\mathcal{F}_n$, the subgroup N is $\mathcal{F}_{n-1}$ but not $\mathcal{F}_n$. This led to the Morse theory of Bestina-Brady providing a plethora of similar examples [BB97].

In fact, in this context, it is difficult for N to be $\mathcal{F}_n$ without $\mathsf{cd}(N) < \mathsf{cd}(G)$, as explained by Bieri [Bie81].

Example 2.1. The groups $G = N \rtimes \mathbb{Z}$ above provide examples of $\mathcal{F}_n$ groups that split as an HNN extension with an $\mathcal{F}_{n-1}$ edge group but where the vertex group is not $\mathcal{F}_n$. There are likewise $\mathcal{F}_n$ amalgamated free products $K = V *_E V'$ such that E is $\mathcal{F}_{n-1}$ but V and V' are not $\mathcal{F}_n$. Indeed $K = G * \mathbb{Z}$ has this property. For, we may express $G * \mathbb{Z}$ as $(N * \mathbb{Z}) *_{N*N} (N * \mathbb{Z})$. Note that $N * \mathbb{Z}$ is $\mathcal{F}_{n-1}$ (a trivial instance of Theorem 1.5) but not $\mathcal{F}_n$ by Theorem 1.2 since N isn't. To verify the amalgamated product, consider the splitting of $K = G * \mathbb{Z}$ as a graph of groups whose underlying graph has edges a, b, c that are each joined to vertices u, v. Let $G_u = N$ and $G_v = N$, and let $G_a = N$ and $G_b = N$ but $G_c = 1$. We can choose the inclusions of G_a and G_b into G_u, G_v so that the subgraph of groups over $\Theta - c$ yields G. The subgraphs over $\Theta - a$ and $\Theta - b$ yield the groups $N * \mathbb{Z}$, and the subgraph over c yields $N * N$. Thus the splitting of Θ as $(\Theta - a) \cup_c (\Theta - b)$ yields $G * \mathbb{Z} = (N * \mathbb{Z}) *_{N*N} (N * \mathbb{Z})$ as claimed.

Example 2.2. Let $E \subset V$ be a subgroup (of a free group) that is not finitely generated. Let $G = V *_E V$ be the *double* of V along E. Then G is finitely generated and splits as an amalgamated product where each vertex group is finitely generated but the edge group is not finitely generated. Note that G is not f.p. since $\mathsf{H}_2(G)$ is not finitely generated (see Theorem 1.5). One can likewise produce doubles of the same type where G and V are $\mathcal{F}_n$ but E is not $\mathcal{F}_n$.

The following is a weak form of [Geo08, Thm 7.3.1]:

Theorem 2.3. *Let G act cocompactly on an $(n-1)$-connected complex. Suppose that for each $g \in G$, if g stabilizes a cell then g fixes it pointwise. If the stabilizer of each cell is $\mathcal{F}_n$ then G is $\mathcal{F}_n$.*

In parallel with Theorem 1.2 but generalizing from trees to CAT(0) cube complexes, we propose two formulations of a converse which we believe are equivalent:

Conjecture 2.4. *Let G be $\mathcal{F}_n$ and suppose G acts cocompactly on a CAT(0) cube complex. Then each vertex stabilizer is $\mathcal{F}_n$ provided the stabilizer of each k-cube is $\mathcal{F}_n$ for $k > 0$.*

Conjecture 2.5. *Let G be $\mathcal{F}_n$ and suppose G acts cocompactly on a CAT(0) cube complex. Then each vertex stabilizer is $\mathcal{F}_n$ provided the stabilizers of hyperplanes of each codimension are $\mathcal{F}_n$.*

Conjecture 2.5 relates to results about quasiconvexity of the vertex groups obtaining stronger conclusions with geometric hypotheses [BW13, HR17, GM18].

The following shows that assuming all codimension-1 hyperplane stabilizers are $\mathcal{F}_n$ does not ensure that vertex stabilizers are $\mathcal{F}_n$.

Example 2.6. Let $G = F_2 \times \mathbb{Z} = \langle a, b\rangle \times \langle t\rangle$. Let $\phi_1 : G \to \mathbb{Z}$ be the homomorphism induced by $\phi_1(a) = \phi_1(b) = 0$ and $\phi_1(t) = 1$. Let $\phi_2 : G \to \mathbb{Z}$ be the homomorphism induced by $\phi_2(a) = \phi_2(b) = -1$ and $\phi_2(t) = 1$. Let $\phi : G \to \mathbb{Z} \times \mathbb{Z}$ be the product homomorphism $\phi(g) = (\phi_1(g), \phi_2(g))$. Composing with the standard action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ we obtain an action of G on $\mathbb{R}^2$ which we view as a CAT(0) square complex.

The stabilizer of any point (and hence of 0-cubes and squares) equals $\ker(\phi)$. The stabilizers of the hyperplanes in the two directions are equal to $\ker(\phi_1)$ and $\ker(\phi_2)$.

We claim that $\ker(\phi_1)$ and $\ker(\phi_2)$ are finitely generated but $\ker(\phi) = \ker(\phi_1) \cap \ker(\phi_2)$ is not finitely generated. Indeed, $\ker(\phi_1) = \langle a, b\rangle$ and $\ker(\phi_2) = \langle at, bt\rangle$. However, $\ker(\phi)$ is the kernel of the homomorphism $\langle a, b\rangle \to \mathbb{Z}$ sending a and b to the generator 1, and thus not finitely generated [Mol68].

3. Background

Choose a generator α of $\mathsf{H}_n(S^n)$. The *Hurewicz homomorphism* $h : \pi_n(X, x) \to \mathsf{H}_n(X)$ is defined by viewing any based n-sphere $f : (S^n, s) \to (X, x)$ as an n-cycle via $h(f) = [f_*(\alpha)]$.

We use the following form of the Hurewicz Theorem [Hat02, Thm 4.32]:

Theorem 3.1. *If X is $(n-1)$-connected and $n \geq 2$ then $\widetilde{\mathsf{H}}_k(X) = 0$ for $k < n$ and $h : \pi_n X \to \mathsf{H}_n(X)$ is an isomorphism.*

Let $D^n \subset S^n$ be a hemisphere containing the basepoint s, and let $[\alpha]$ represent a generator of $\mathsf{H}_n(S^n, D^n)$. The *relative Hurewicz homomorphism* $h : \pi_n(X, A, a) \to \mathsf{H}_n(X, A)$ is defined by viewing any relative based n-sphere $f : (S^n, D^n, s) \to (X, A, a)$ as an n-cycle via $h(f) = [f_*(\alpha)]$. We use the following relative form of the Hurewicz Theorem [Hat02, Thm 4.37] adapted to the simpler case where A is simply-connected (to ensure injectivity of h).

Theorem 3.2. *For $n \geq 2$, if (X, A) is $(n-1)$-connected and A is simply-connected and nonempty, then $\mathsf{H}_i(X, A) = 0$ for $i < n$ and $h : \pi_n(X, A, a) \to \mathsf{H}_n(X, A)$ is an isomorphism.*

Remark 3.3. The $(n-1)$-connectivity of X holds precisely when $\mathsf{H}_m(X)=0$ for $m<n$ and $\pi_1 X=1$ if $n\geq 2$. Note that (X,A) is $(n-1)$-connected when both X and A are $(n-1)$-connected. For details on k-connectivity, see [Hat02, pp.346].

For low dimensions we use that path connectivity is detected by $\widetilde{\mathsf{H}}_0=0$, as well as the following well-known statement [Hat02, Thm 2A.1]:

Theorem 3.4. *If X is path connected then $\pi_1 X\to \mathsf{H}_1(X)$ is a surjection.*

The following statement will also be crucial [Geo08, Thm 8.2.1]:

Theorem 3.5. *Let H be $\mathcal{F}_m$ with $m\geq 1$. If H acts freely and cocompactly on an $(m-1)$-complex Z that is $(m-2)$-connected then we can add finitely many H-orbits of m-cells to obtain an H-cocompact free action on an $(m-1)$-connected complex.*

4. Useless tree definitions and useful subtree lemmas

Definition 4.1 (Trees)**.** Let T be a tree. We let T' denote its barycentric subdivision. The original vertices of T are called *T-vertices* of T', and we sometimes refer to edges of T' as *half-edges* of T. The barycenter of an edge e of T is denoted by $\dot{e}$. For each T-vertex v of T', let $S(v)$ be the union of v and the closed half-edges adjacent to v. When $v\neq v'$ the intersection $S(v)\cap S(v')$ is either empty or consists of the barycenter of an edge e joining v,v'. Thus T is isomorphic to the nerve of the covering $\{S(v)\}_{v\in T^0}$ of T'.

Definition 4.2 (Trees of complexes)**.** As we will be working with G-equivariant maps $X\to T$ from complexes to trees, we delineate the framework that we work in. A *tree of complexes* is a complex X and a map $\phi: X\to T$ such that the resulting map $\phi: X\to T'$ is cellular and surjects onto the vertices of T'.

For each T-vertex v, let $X_v=\phi^{-1}(S(v))$. For each T-edge e, let $X_e=\phi^{-1}(\{\dot{e}\})$. The subcomplexes $\{X_v\}$ and $\{X_e\}$ are the *vertex spaces* and *edge spaces* of X.

With this viewpoint, letting $X=T$, the vertex spaces of T are the stars $(S(v))$ and the edge spaces of T are the barycenters $\dot{e}$. Hence the map $X\to T$ maps vertex spaces to vertex spaces and edge spaces to edge spaces (possibly not surjectively when X is not connected).

Our seemingly artificial requirement that $X\to T'$ is surjective on vertices allows us to naturally recover T from X as the nerve of the covering by vertex spaces.

Finally, as $\phi: X\to T'$ is G-equivariant and surjective on vertices we have Stabilizer$(X_v)=G_v$ and Stabilizer$(X_e)=G_e$ for each vertex v and edge e of T.

Definition 4.3 (Footprint)**.** Let $X\to T'$ be a cellular map. Let c be a nontrivial n-chain in X. The *footprint* of c is the smallest subtree of T' containing all images of n-cells of c. (We use the n-cells of c with a nonzero coefficient and ignore orientations.)

We likewise define the footprint of a combinatorial path in X.

A footprint F is finite. Its *complexity* is the number of T-vertices in F. A *T-leaf* is a T-vertex of F that is incident with exactly one T-edge in F.

Lemma 4.4 (H-arboricide)**.** *Let X split as a tree of complexes. Suppose each edge space X_e is $(m-1)$-connected and each vertex space X_v is $(m-1)$-connected.*

Let $c\in\widetilde{\mathsf{H}}_m(X)$. Then we can add finitely many $(m+1)$-balls to the vertex and edge spaces to obtain X' such that c maps to 0 under $\widetilde{\mathsf{H}}_m(X)\to\widetilde{\mathsf{H}}_m(X')$.

Proof. We will prove the result by induction on the complexity of the footprint of the cycle c. We focus on the cases $m = 1$ and $m \geq 2$ together. We turn to the case $m = 0$ at the end. That proof is essentially the same but is stripped of the interesting algebraic topology, and the reader may wish to consider that case first.

When the complexity is 0, the footprint is the midpoint of an edge e. By Theorem 3.1, as X_e is $(m-1)$-connected $[c] \in \mathsf{H}_m(X_e)$ is represented by an m-sphere, and we attach an $(m+1)$-ball to fill it. The analogous statement holds for $m = 1$ using Theorem 3.4.

When the complexity is 1 the footprint consists of a vertex v and possibly some half edges, the argument is similar: By Theorem 3.1 or Theorem 3.4, as X_v is $(m-1)$-connected $[c] \in \mathsf{H}_m(X_v)$ is represented by an m-sphere, and we attach an $(m+1)$-ball to fill it.

Otherwise, F has a vertex v that is incident with a single edge e. Then $c = c_v + c'$ where c_v is the part of the m-chain in X_v and c' is an m-chain consisting of a sum of oriented m-cells outside of X_v.

Note that (X_v, X_e) is $(m-1)$-connected (for $m \geq 1$) since X_v and X_e are. By Theorem 3.2, the element c_v is the image of a relative ball $b_v \in \pi_m(X_v, X_e)$. By $(m-1)$-connectedness of X_e, the $(m-1)$-sphere ∂b_v in X_e bounds an m-ball c_e in X_e. We attach an $(m+1)$-ball a_v whose boundary is $b_v \cup_{\partial b_v} c_e$. Now c is homologous to c' in the space with the added balls. Finally, the footprint of c' has fewer T-vertices than the footprint of c does, and so either $c' = 0$ or its complexity is smaller.

We now consider the case where $m = 0$. When the footprint of c is the midpoint of an edge e, we can add 1-balls to X_e whose endpoints agree with the cancelling oriented points of c (here is where we use reduced homology). And we can likewise do the same when the footprint of c contains a single vertex v. Otherwise, the footprint contains a vertex v with a single edge e, we let $c = c_v + c'$ where $c_v = \sum \pm p_i$ consists of the oriented 0-cells of c that lie in X_v. As X_e is (-1)-connected (i.e. nonempty), we let $c_e \in X_e$ be a 0-cell. We then attach 1-balls joining p_i and c_e. Then $[c'] = [c]$ in $\widetilde{\mathsf{H}}_0$ of the space obtained by adding these 1-balls as before. But either $c' = 0$ or the complexity of the footprint of c' is strictly smaller. ◻

Lemma 4.5 (π_1-arboricide)**.** *Let X split as a tree of complexes. Suppose each edge space X_e is connected and each vertex space X_v is connected.*

Let $c \to X$ be a map from a circle to X. Then we can add finitely many 2-balls to the vertex and edge spaces to obtain X' such that c is null-homotopic in X'.

Proof. By homotoping, we may assume that $c \to X$ is a combinatorial path to X^1.

We will prove the result by induction on the complexity of the footprint F of c. Suppose the complexity is at most 1. If F consists of the the midpoint of an edge e then we attach a 2-cell d to X_e along $\partial d = c$. Likewise, if F contains a single T-vertex v then we attach a 2-cell d to X_v along $\partial d = c$.

When the complexity is ≥ 2, the path $c \to T$ has one or more "backtracks" which shall organize a decrease of complexity. A *backtrack* of $c \to T$ consists of a subpath $k'PQk'' \subset c$ where $PQ \to T$ maps to a single vertex space X_v but k', k'' do not map to X_v, and the initial and terminal points p, q of PQ map to vertices in an edge space X_e. By connectivity of X_e, there is a combinatorial path $S \to X_e$ from q to p. This enables us to *push* as follows: We attach a disk D to X_v with ∂_D attached along the cycle PQS. Letting $c = c'PQ$, in the presence of D, the cycle c

is homotopic to $c'S^{-1}$ and c' has fewer backtracks. Repeating this process finitely many times, we arrive at a cycle c'' with a smaller footprint in T. $\square$

5. Main result

In this section we prove our main result expressed in terms of actions on trees instead of graphs of groups.

Theorem 5.1. *Let G act cocompactly and without inversions on a tree T. Suppose G is $\mathcal{F}_n$ and each edge group is $\mathcal{F}_n$. Then there is a free action of G on an n-dimensional complex X and a G-equivariant cellular map $X \to T'$ such that:*

1. *X is G-cocompact.*
2. *X is $(n-1)$-connected.*
3. *each X_e is $(n-1)$-connected.*
4. *Consequently: each X_v is $(n-1)$-connected.*

Corollary 5.2. *G_v is $\mathcal{F}_n$ for each vertex v.*

Proof. The free action of G_v on X_v is cocompact by Conclusion (1). Hence the result follows by Conclusion (4) as $X_v \neq \emptyset$. $\square$

Before proceeding to the main part of the proof, we explain the final consequence:

Proof that (2)+(3) $\Rightarrow$ (4). The m-acyclicity of X_v holds for $0 \leq m < n$ as follows: Let $X = X_v \cup \bar{X}_v$ where $\bar{X}_v = X - \text{Int}(X_v)$. Note that $X_v \cap \bar{X}_v = \cup_e X_e$ where e varies over the edges at v. Exactness of

$$\mathsf{H}_m(X_v \cap \bar{X}_v) \to \mathsf{H}_m(X_v) \oplus \mathsf{H}_m(\bar{X}_v) \to \mathsf{H}_m(X)$$

shows that since $\mathsf{H}_m(X_v \cap \bar{X}_v) = 0$ for $0 < m \leq n-1$ we have an injection $\mathsf{H}_m(X_v) \to \mathsf{H}_m(X_v \cup \bar{X}_v) = 0$. When $m = 0$, the image of $\mathsf{H}_0(X_v \cap \bar{X}_v)$ in $\mathsf{H}_0(X_v) \oplus \mathsf{H}_0(\bar{X}_v)$ intersects $\mathsf{H}_0(X_v)$ trivially so $\mathsf{H}_0(X_v) \to \mathsf{H}_0(X)$ is injective. Indeed, $\mathsf{H}_0(X_v \cap \bar{X}_v) = \mathsf{H}_0(\sqcup_e X_e) \to \mathsf{H}_0(\bar{X}_v)$ where the final homomorphism is an isomorphism since each X_e maps to a distinct component of $\bar{X}_v$ as X is a tree of spaces.

π_1-injectivity of $G_v \backslash X_v \to G \backslash X$ is a standard consequence of π_1-injectivity of each $G_e \backslash X_e \to G_v \backslash X_v$, that is, the vertex groups in a graph of groups embed since the edge groups embed. Indeed, consider a closed combinatorial path $P \to X_v$. Since X is 1-connected, there is a disk diagram $D \to X^2$, which we can assume to be combinatorial. The preimage of each X_e provides a subdiagram that can be replaced by a diagram in X_e since X_e is 1-connected. We thus obtain a disk diagram for P lying entirely in X_v.

Finally, $(n-1)$-connectivity of X_v holds since $\mathsf{H}_m(X_v) = 0$ for $m < n$ and $\pi_1 X_v = 1$ if $n \geq 2$ as in Remark 3.3. $\square$

Main proof of Theorem 5.1. We prove the asserted statement $\mathcal{S}_n$ by induction on n.

The base case where $n = 0$ holds as follows: Let V and $\dot{E}$ be representatives of G-orbits of the vertices and barycenters of edges of T. Let $X = G \times (V \sqcup \dot{E})$ where G acts by $g(a, b) = (ga, b)$. The map $X \to T'$ is given by $(g, k) \mapsto gk$ which is G-equivariant. Observe that $X \to T'$ is surjective on the vertices of T'. The G-cocompactness and nonemptyness properties are immediate.

Suppose $\mathcal{S}_{n-1}$ holds. Note that if G and each G_e is $\mathcal{F}_n$ then G and each G_e is $\mathcal{F}_{n-1}$. Thus there exists a free cocompact action of G on an $(n-1)$-complex X

and a G-equivariant map $X \to T'$ such that X is $(n-2)$-connected and each X_e and X_v is $(n-2)$-connected and in particular, nonempty.

By Theorem 3.5 we can add finitely many G-orbits of n-cells to the edge spaces so that each edge space is now $(n-1)$-connected. Let Y denote the resulting n-complex with G-equivariant map $Y \to T'$. Note that Y remains $(n-2)$-connected since we have only added n-balls. Note that $X = Y^{n-1}$.

By Theorem 3.5, there are finitely many G-orbits of n-balls $\{b_i^n\}_{i\in I}$ to add to Y to obtain an $(n-1)$-connected complex.

A key point here is that if we attach them we might not obtain a G-equivariant map to T'. We shall therefore kill each ∂b_i^n using a collection of balls that are added within vertex spaces as follows:

For $n=2$, Lemma 4.5 provides a finite collection $\{\bar{b}_{ij}^2\}_{j\in J_i}$ of 2-balls such that ∂b_i^n is nullhomotopic in $(Y \cup \bigcup \bar{b}_{ij}^2)$. For $n \neq 2$, regard ∂b_i^n as a $(n-1)$-cycle (which is reduced if $n=1$). Lemma 4.4 now provides a finite collection $\{\bar{b}_{ij}^n\}_{j\in J_i}$ of n-balls such that ∂b_i^n maps to 0 in $\widetilde{\mathsf{H}}_{n-1}(Y \cup \bigcup \bar{b}_{ij}^n)$.

Let $X' = Y \cup \bigcup_{i\in I} \bigcup_{j\in J_i} \bigcup_{g\in G} g\bar{b}_{ij}^n$. Then $g\partial b_i^n = 0$ in $\widetilde{\mathsf{H}}_{n-1}(X')$. Thus $\widetilde{\mathsf{H}}_{n-1}(X') = 0$. For $n=1$ it follows that X' is connected. For $n>2$, Theorem 3.2 implies that X' is $(n-1)$-connected. For $n=2$ it follows that X' is 1-connected as above.

A map $X' \to T'$ exists since the n-balls are attached along boundaries that lie within vertex spaces. The G-cocompactness of X' holds since only finitely many G-orbits of balls where added. Each edge space $X'_e = Y_e$ is unchanged and hence $(n-1)$-connected. □

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