PERIODIC SOLUTIONS OF PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we study the existence of periodic solutions to the partial functional differential equation

$$\begin{cases} \frac{dy(t)}{dt} = By(t) + \hat{L}(y_t) + f(t, y_t), \ \forall t \geq 0, \\ y_0 = \varphi \in C_B. \end{cases}$$

where $B : Y \to Y$ is a Hille-Yosida operator on a Banach space Y. For $C_B := \{\varphi \in C([-r, 0]; Y) : \varphi(0) \in \overline{D(B)}\}$, $y_t \in C_B$ is defined by $y_t(\theta) = y(t + \theta)$, $\theta \in [-r, 0]$, $\hat{L} : C_B \to Y$ is a bounded linear operator, and $f : \mathbb{R} \times C_B \to Y$ is a continuous map and is T-periodic in the time variable t. Sufficient conditions on B, $\hat{L}$ and $f(t, y_t)$ are given to ensure the existence of T-periodic solutions. The results then are applied to establish the existence of periodic solutions in a reaction-diffusion equation with time delay and the diffusive Nicholson's blowflies equation.

1. INTRODUCTION

The aim of this paper is to study the existence of periodic solutions for the following partial functional differential equation (PFDE):

$$(1.1) \qquad \begin{cases} \frac{dy(t)}{dt} = By(t) + \hat{L}(y_t) + f(t, y_t), \ \forall t \geq 0, \\ y_0 = \varphi \in C_B. \end{cases}$$

where $B : Y \to Y$ is a Hille-Yosida operator on a Banach space Y. Denote $C_B := \{\varphi \in C([-r, 0]; Y) : \varphi(0) \in \overline{D(B)}\}$. $y_t \in C_B$ is defined by $y_t(\theta) = y(t + \theta)$, $\theta \in [-r, 0]$, $\hat{L} : C_B \to Y$ is a bounded linear operator, and $f : \mathbb{R} \times C_B \to Y$ is a continuous map and is T-periodic in the time variable t.

The existence of periodic solutions in abstract evolution equations has been widely studied in the literature. By applying Horn's fixed point theorem to the Poincaré map, Liu [12] and Ezzinbi and Liu [7] established the existence of bounded and ultimate bounded solutions of evolution equations with or without delay, which contains partial functional differential equation, implying the existence of periodic solutions. Benkhalti and Ezzinbi [2] and Kpoumiè et al. [9] proved that under some conditions, the existence of a bounded solution for some non-densely defined

nonautonomous partial functional differential equations implies the existence of periodic solutions. The approach was to construct a map on the space of T-periodic functions from the corresponding nonhomogeneous linear equation and use a fixed-point theorem concerning set-valued maps. Li [10] used analytic semigroup theory to discuss the existence and stability of periodic solutions in evolution equations with multiple delays. Li et al. [11] proved several Massera-type criteria for linear periodic evolution equations with delay and applied the results to nonlinear evolution equations, functional, and partial differential equations. For fundamental theories on partial functional differential equations, we refer to the monograph of Wu [17].

In this paper, we study the existence of periodic solutions of the partial functional differential equation (1.1). In section 2, we recall some preliminary results on existence of mild periodic solutions of abstract semilinear equations. In section 3, using the existence theorem of mild periodic solutions presented in section 2, we show the existence of periodic mild solutions of partial functional differential equations. In section 4, we apply the main results of this paper to a reaction-diffusion equation with time delay and the diffusive Nicholson's blowflies equation.

2. Preliminary results

In this section, we first recall an existence theorem of classical solutions of partial functional differential equations. Then we review a theorem obtained in Su and Ruan [16], which will be applied to prove our main theorem in the next section.

Consider an abstract semilinear functional differential equation on a Banach space X given by

$$(2.1) \qquad \begin{cases} \frac{du(t)}{dt} = A_0 u(t) + F(t, u_t), \ \forall t \geq 0, \\ u_0 = \varphi \in C_X, \end{cases}$$

where $A_0 : D(A_0) \subseteq X \to X$ is a linear Hille-Yosida operator, $C_X = C([-r, 0], X)$ denotes the space of continuous functions from $[-r, 0]$ to X with the uniform convergence topology. $u_t(\theta) = u(t+\theta)$ for $\theta \in [-r, 0]$, F is a function from $[0, \infty) \times C_X$ into X, and $\varphi \in C_X$ is the given initial value.

The following theorem gives the existence of classical solutions of problem (2.1).

Theorem 2.1 (Ezzinbi and Adimy [6, Theorem 13]). *Assume that $F(t, \varphi)$ is continuous differentiable and satisfies the following locally Lipschitz conditions: for each $\alpha > 0$ there exists a constant $C_1(\alpha) > 0$ such that*

$$|F(t, \varphi_1) - F(t, \varphi_2)| \leq C_1(\alpha) \|\varphi_1 - \varphi_2\|,$$

$$|D_t F(t, \varphi_1) - D_t F(t, \varphi_2)| \leq C_1(\alpha) \|\varphi_1 - \varphi_2\|,$$

$$|D_\varphi F(t, \varphi_1) - D_\varphi F(t, \varphi_2)| \leq C_1(\alpha) \|\varphi_1 - \varphi_2\|$$

for all $t \in [0, T_\varphi]$ and $\varphi_1, \varphi_2 \in C_X$ with $\|\varphi_1\|, \|\varphi_2\| \leq \alpha$, where $D_t F$ and $D_\varphi F$ denote the derivatives of $F(t, \varphi)$ with respect to t and φ, respectively. For given $\varphi \in C_X^1 := C^1([-r, 0], X)$ such that

$$\varphi(0) \in D(A_0), \varphi'(0) \in \overline{D(A_0)} \ \text{and} \ \varphi'(0) = A_0 \varphi(0) + F(0, \varphi),$$

let $u(., \varphi) : [-r, T_\varphi) \to X$ be the unique integral solution of equation (2.1). Then, $u(., \varphi)$ is continuously differentiable on $[-r, T_\varphi)$ and satisfies equation (2.1).

Now consider the abstract semilinear equation

$$(2.2) \qquad \frac{du}{dt} = Au(t) + F(t, u(t)), \ \ t \geq 0$$

in a Banach space X, where A is a linear operator on X (not necessarily densely defined) satisfying the Hille-Yoshida condition (see the following) and $F : [0, \infty) \times \overline{D(A)} \to X$ is continuous and T-periodic in t.

Assumption 2.2.

(H1) There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $\|(\lambda I - A)^{-n}\|_{L(X)} \leq \frac{M}{(\lambda - \omega)^n}$ for $\lambda > \omega$, $n \geq 1$;

(H2) $F : [0, \infty) \times \overline{D(A)} \to X$ is continuous and Lipschitz on bounded sets; i.e., for each $C > 0$ there exists $K_F(C) \geq 0$ such that $\|F(t, u) - F(t, v)\| \leq K_F(C) \|u - v\|$ for $t \in [0, \infty)$ and $\|u\| \leq C$ and $\|v\| \leq C$;

(H3) $F : [0, \infty) \times \overline{D(A)} \to X$ is continuous and bounded on bounded sets; i.e., there exists $L_F(T, \rho) \geq 0$ such that $\|F(t, u)\| \leq L_F(T, \rho)$ for $t \leq T$ and $\|u\| \leq \rho$.

Definition 2.3. A linear operator $A : D(A) \subset X \to X$ satisfying Assumption 2.2 (H1) is called *a Hille-Yosida operator*.

With these assumptions, we have the following result for equation (2.2).

Theorem 2.4 (Su and Ruan [16, Theorem 3.3])**.** *Let Assumption* 2.2 *hold with* $\omega < 0$, $M = 1$ *and* F *being* T-periodic in t. *Suppose that there exists* $\rho > 0$ *such that* $(N + T)K_F(\rho) < 1$ *and* $(N + T)L_F(T, \rho) \leq \rho$, *where* $N = \frac{T}{1 - e^{\omega T}}$. *Then the abstract semilinear equation* (2.2) *has a mild* T-periodic solution.

3. Existence of periodic solutions

In this section, we rewrite the partial functional differential equation as an abstract semilinear equation and present an existence theorem of periodic solutions.

Let $B : D(B) \subset Y \to Y$ be a linear operator on a Banach space $(Y, \|\cdot\|_Y)$. Assume that B is a Hille-Yosida operator; that is, there exist $\omega_B \in \mathbb{R}$ and $M_B > 0$ such that $(\omega_B, +\infty) \subset \rho(B)$ and

$$\left\|(\lambda I - B)^{-n}\right\| \leq \frac{M_B}{(\lambda - \omega_B)^n}, \ \forall \lambda > \omega_B, \ n \geq 1.$$

Set $Y_0 := \overline{D(B)}$. Consider the part of B in Y_0, denoted B_0, which is defined by

$$B_0 y = By, \ \forall y \in D(B_0)$$

with

$$D(B_0) := \{y \in D(B) : By \in Y_0\}.$$

For $r \geq 0$, set $C := C([-r, 0]; Y)$ endowed with the supremum norm

$$\|\varphi\|_\infty = \sup_{\theta \in [-r, 0]} \|\varphi(\theta)\|_Y .$$

Consider the PFDE

$$(3.1) \qquad \begin{cases} \frac{dy(t)}{dt} = By(t) + \hat{L}(y_t) + f(t, y_t), \ \forall t \geq 0, \\ y_0 = \varphi \in C_B, \end{cases}$$

where $C_B := \{\varphi \in C([-r,0];Y) : \varphi(0) \in \overline{D(B)}\}$, $y_t \in C_B$ is defined by $y_t(\theta) = y(t+\theta)$, $\theta \in [-r,0]$, $\hat{L} : C_B \to Y$ is a bounded linear operator, and $f : \mathbb{R} \times C_B \to Y$ is a continuous map.

Now we rewrite the PFDE (3.1) as an abstract non-densely defined Cauchy problem such that our theorems can be applied. Firstly, following Ducrot et al. [5] we regard the PFDE (3.1) as a PDE. Define $u \in C([0,+\infty) \times [-r,0], Y)$ by

$$u(t,\theta) = y(t+\theta), \ \forall t \geq 0, \ \forall \theta \in [-r,0].$$

If $y \in C^1([-r,+\infty), Y)$, then

$$\frac{\partial u(t,\theta)}{\partial t} = y'(t+\theta) = \frac{\partial u(t,\theta)}{\partial \theta}.$$

Moreover, for $\theta = 0$, we obtain

$$\frac{\partial u(t,0)}{\partial \theta} = y'(t) = By(t) + \hat{L}(y_t) + f(t,y_t) = Bu(t,0) + \hat{L}(u(t,.)) + f(t,u(t,.)), \ \forall t \geq 0.$$

Therefore, we deduce that u satisfy a PDE

$$(3.2) \qquad \begin{cases} \frac{\partial u(t,\theta)}{\partial t} - \frac{\partial u(t,\theta)}{\partial \theta} = 0, \\ \frac{\partial u(t,0)}{\partial \theta} = Bu(t,0) + \hat{L}(u(t,.)) + f(t,u(t,.)), \ \forall t \geq 0, \\ u(0,.) = \varphi \in C_B. \end{cases}$$

In order to write the PDE (3.2) as an abstract non-densely defined Cauchy problem, we extend the state space to take into account the boundary conditions. Let $X = Y \times C$ with the usual product norm

$$\left\| \begin{pmatrix} y \\ \varphi \end{pmatrix} \right\| = \|y\|_Y + \|\varphi\|_\infty.$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$(3.3) \qquad A \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix}, \ \forall \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in D(A)$$

with

$$D(A) = \{0_Y\} \times \{\varphi \in C^1([-r,0],Y), \ \varphi(0) \in D(B)\}.$$

Note that A is non-densely defined because

$$X_0 := \overline{D(A)} = 0_Y \times C_B \neq X.$$

Now define $L : X_0 \to X$ by

$$L \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} := \begin{pmatrix} \hat{L}(\varphi) \\ 0_C \end{pmatrix}$$

and $F : \mathbb{R} \times X_0 \to X$ by

$$F(t, \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}) := \begin{pmatrix} f(t,\varphi) \\ 0_C \end{pmatrix}.$$

Let

$$v(t) := \begin{pmatrix} 0_Y \\ u(t) \end{pmatrix}.$$

Then we can rewrite the PDE (3.2) as the following non-densely defined Cauchy problem

$$(3.4) \qquad \frac{dv(t)}{dt} = Av(t) + L(v(t)) + F(t,v(t)), \ t \geq 0; \ v(0) = \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in X_0.$$

To state an existence theorem of periodic solutions for equation (3.1), we make the following assumptions.

Assumption 3.1.
  (C1) $f : \mathbb{R} \times C_B \to Y$ is Lipschitz on bounded sets; i.e., for each $C > 0$ there exists $K_f(C) \geq 0$ such that $\|f(t, u) - f(t, v)\| \leq K_f(C) \|u - v\|$ for $t \in [0, \infty)$ and $\|u\| \leq C$ and $\|v\| \leq C$;
  (C2) $f : \mathbb{R} \times C_B \to Y$ is bounded on bounded sets; i.e., there exists $L_f(T, \rho) \geq 0$ such that $\|f(t, u)\| \leq L_f(T, \rho)$ for $t \leq T$ and $\|u\| \leq \rho$.

With these assumptions, we have the following result for equation (3.1).

Theorem 3.2. *Let Assumption* 3.1 *hold with* $\omega_B < 0$, $M_B = 1$ *and* f *being T-periodic in* t. *Suppose that there exists* $\rho > 0$ *such that* $(N + T)(K_f(\rho) + \left\|\hat{L}\right\|_\rho) < 1$ *and* $(N + T)(L_f(T, \rho) + \left\|\hat{L}\right\|_\rho \rho) \leq \rho$, *where* $N = \frac{T}{1 - e^{\omega_B T}}$, *then equation* (3.1) *has a T-periodic mild solution.*

Proof. Since (3.1) can be written as (3.4), denote $G(t, v(t)) = L(v(t)) + F(t, v(t))$, it suffices to prove that
  (a) A satisfies Assumption 2.2 (H1) with $\omega = \omega_B < 0$ and $M = 1$;
  (b) $G : [0, \infty) \times \{0_Y\} \times C_B \to Y \times C$ satisfies Assumption 2.2 (H1) (H2);
  (c) There exists $\rho > 0$ such that $(N + T)K_G(\rho) < 1$ and $(N + T)L_G(T, \rho) \leq \rho$, where $N = \frac{T}{1 - e^{\omega T}} = \frac{T}{1 - e^{\omega_B T}}$.

It follows from Theorem 2.4 that equation (3.4) has a T-periodic mild solution, which implies that equation (3.1) has a T-periodic mild solution with initial value $u(0, .) = \varphi \in C_B$.

From Lemma 3.6 and its proof in Ducrot et al. [5], we know that A as defined in (3.3) is a Hille-Yoshida operator with $\omega = \omega_B < 0$ and $M = M_B = 1$, which proves (a).

For $\varphi_1, \varphi_2 \in C_B$ such that $\|\varphi_1\| \leq C$ and $\|\varphi_2\| \leq C$, we have
$$\left(\begin{array}{c} 0_Y \\ \varphi_1 \end{array} \right), \left(\begin{array}{c} 0_Y \\ \varphi_2 \end{array} \right) \in 0_Y \times C_B = \overline{D(A)}$$
and
$$\left\| \left(\begin{array}{c} 0_Y \\ \varphi_1 \end{array} \right) \right\| = \|\varphi_1\| \leq C, \left\| \left(\begin{array}{c} 0_Y \\ \varphi_2 \end{array} \right) \right\| = \|\varphi_2\| \leq C.$$
Then
$$\left\| G(t, \left(\begin{array}{c} 0_Y \\ \varphi_1 \end{array} \right)) - G(t, \left(\begin{array}{c} 0_Y \\ \varphi_2 \end{array} \right)) \right\|$$
$$= \left\| L(\left(\begin{array}{c} 0_Y \\ \varphi_1 \end{array} \right)) - L(\left(\begin{array}{c} 0_Y \\ \varphi_2 \end{array} \right)) + F(t, \left(\begin{array}{c} 0_Y \\ \varphi_1 \end{array} \right)) - F(t, \left(\begin{array}{c} 0_Y \\ \varphi_2 \end{array} \right)) \right\|$$
$$\leq \left\| L(\left(\begin{array}{c} 0_Y \\ \varphi_1 \end{array} \right)) - L(\left(\begin{array}{c} 0_Y \\ \varphi_2 \end{array} \right)) \right\| + \left\| F(t, \left(\begin{array}{c} 0_Y \\ \varphi_1 \end{array} \right)) - F(t, \left(\begin{array}{c} 0_Y \\ \varphi_2 \end{array} \right)) \right\|$$
$$= \left\| \left(\begin{array}{c} \hat{L}(\varphi_1 - \varphi_2) \\ 0_C \end{array} \right) \right\| + \left\| \left(\begin{array}{c} f(t, \varphi_1) - f(t, \varphi_2) \\ 0_C \end{array} \right) \right\|$$
$$= \left\| \hat{L}(\varphi_1 - \varphi_2) \right\|_Y + \|f(t, \varphi_1) - f(t, \varphi_2)\|_Y$$

$$\leq K_f(C) \|\varphi_1 - \varphi_2\| + \left\|\hat{L}\right\|_C \|\varphi_1 - \varphi_2\|$$

$$= (K_f(C) + \left\|\hat{L}\right\|_C) \|\varphi_1 - \varphi_2\|$$

$$= (K_f(C) + \left\|\hat{L}\right\|_C) \left\| \begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix} - \begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix} \right\|.$$

So there exists $K_G(C) = K_f(C) + \left\|\hat{L}\right\|_C$ such that

$$\left\| G(t, \begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix}) - G(t, \begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix}) \right\| \leq K_G(C) \left\| \begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix} - \begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix} \right\|.$$

Furthermore, for $t \leq T$ and $\left\| \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \right\| \leq \rho$, we have

$$\left\| G(t, \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}) \right\| = \left\| L \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} + F(t, \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}) \right\|$$

$$\leq \left\| L \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \right\| + \left\| F(t, \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}) \right\|$$

$$= \left\| \begin{pmatrix} \hat{L}(\varphi) \\ 0_C \end{pmatrix} \right\| + \left\| \begin{pmatrix} f(t, \varphi) \\ 0_C \end{pmatrix} \right\|$$

$$= \left\| \hat{L}(\varphi) \right\|_Y + \|f(t, \varphi)\|_Y$$

$$\leq \left\|\hat{L}\right\|_\rho \rho + L_f(T, \rho).$$

So there exists $L_G(T, \rho) = \left\|\hat{L}\right\|_\rho \rho + L_f(T, \rho)$ such that $\left\| G(t, \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}) \right\| \leq L_G(T, \rho)$, which completes the proof of (b).

With $K_G(C)$ and $L_G(T, \rho)$ given as above, (c) follows directly from the assumptions. $\qquad\square$

4. APPLICATIONS

In this section, we apply the results in last section to a reaction-diffusion equation with time delay and the diffusive Nicholson's blowflies equation.

4.1. A reaction-diffusion equation with time delay.
Let us consider the following periodic reaction-diffusion equation with time delay:

(4.1)
$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - au(x,t) - b(t)u(x, t-r), \ 0 \leq x \leq 1, \ t \geq 0 \\ u(0,t) = u(1,t) = k, \ t \geq 0 \\ u(x,t) = \phi(t)(x), \ 0 \leq x \leq 1, \ -r \leq t \leq 0, \end{cases}$$

where k is a constant and $a \geq 0$, $b \in C([0, \infty), \mathbb{R}_+)$ is T-periodic. We will study the existence of T-periodic solution of problem (4.1).

Let $v(x,t) = u(x,t) - k$, then we have the following equation:
(4.2)
$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = \frac{\partial^2 v(t,x)}{\partial x^2} - av(x,t) - b(t)v(x, t-r) - ka - kb(t), \ 0 \leq x \leq 1, \ t \geq 0 \\ v(0,t) = v(1,t) = 0, \ t \geq 0 \\ v(x,t) = \phi(t)(x) - k, \ 0 \leq x \leq 1, \ -r \leq t \leq 0. \end{cases}$$

We know that the existence of T-periodic solutions of equation (4.2) is equivalent to the existence of T-periodic solutions of equation (4.1).

Let $X = C(0,1)$ and $B : X \to X$ be defined by

$$B\phi = \phi'' - a\phi$$

with

$$D(B) = \{\phi \in C^2([0,1], \mathbb{R}), \phi(0) = \phi(1) = 0\}.$$

Let $C_B := \{\phi \in C([-r,0], X) : \phi(0) \in \overline{D(B)}\}$ and define $f : [0, \infty) \times C_B \to X$ by

$$f(t, \phi) = -b(t)\phi(-r) - ka - kb(t).$$

Then equation (4.2) can be written as

(4.3) $$\begin{cases} \frac{dy(t)}{dt} = By(t) + f(t, y_t), \ t \geq 0 \\ y_0 = \varphi \in C_B \end{cases}$$

Proposition 4.1. *Assume that*

(i) $a > 0$, $0 \leq b(t) \leq b_+$ *and* $b(t + T) = b(t)$ *for* $t \geq 0$;

(ii) $(\frac{T}{1 - e^{-aT}} + T)b_+ < 1$;

(iii) *There exists* $\rho > 0$ *such that* $(\frac{T}{1 - e^{-aT}} + T)(ka + kb_+ + b_+\rho) \leq \rho$.

Then equation (4.2) thus (4.1) has a T-periodic solution.

Proof. Since equation (4.2) can be written as (4.3), it suffices to check the assumptions of Theorem 3.2. Let $\psi \in X$ and let $\lambda > -a$. Then

$$(\lambda I - B)\varphi = \psi \Leftrightarrow (\lambda + a)\varphi - \varphi'' = \psi.$$

Set $\hat{\varphi} = \varphi'$. Then

$$(\lambda I - B)\varphi = \psi \Leftrightarrow \begin{cases} \varphi' = \hat{\varphi} \\ \hat{\varphi}' = (\lambda + a)\varphi - \psi \end{cases}$$

$$\Leftrightarrow \begin{cases} \sqrt{\lambda + a}\varphi' + \hat{\varphi}' = \sqrt{\lambda + a}(\sqrt{\lambda + a}\varphi + \hat{\varphi}) - \psi \\ \sqrt{\lambda + a}\varphi' - \hat{\varphi}' = -\sqrt{\lambda + a}(\sqrt{\lambda + a}\varphi - \hat{\varphi}) + \psi. \end{cases}$$

Define

$$w = (\sqrt{\lambda + a}\varphi + \hat{\varphi}),$$
$$\hat{w} = (\sqrt{\lambda + a}\varphi - \hat{\varphi}).$$

Then we have

(4.4) $$(\lambda I - B)\varphi = \psi \Leftrightarrow \begin{cases} w' = \sqrt{\lambda + a}w - \psi, \\ \hat{w}' = -\sqrt{\lambda + a}\hat{w} + \psi. \end{cases}$$

The first equation of (4.4) is equivalent to

(4.5) $$e^{-\sqrt{\lambda + a}x}w(x) = e^{-\sqrt{\lambda + a}y}w(y) - \int_y^x e^{-\sqrt{\lambda + a}l}\psi(l)dl, \ \forall x \geq y.$$

In (4.5) let $y = 0$, then we obtain

(4.6) $$w(x) = e^{\sqrt{\lambda + a}x}w(0) - e^{\sqrt{\lambda + a}x}\int_0^x e^{-\sqrt{\lambda + a}l}\psi(l)dl,$$

where $w(0) = \sqrt{\lambda + a}\varphi(0) + \hat{\varphi}(0) = \hat{\varphi}(0)$. In (4.5) let $x = 1$, we have

(4.7) $$w(y) = e^{\sqrt{\lambda + a}y - \sqrt{\lambda + a}}w(1) + e^{\sqrt{\lambda + a}y}\int_y^1 e^{-\sqrt{\lambda + a}l}\psi(l)dl,$$

where $w(1) = \sqrt{\lambda + a}\varphi(1) + \hat{\varphi}(1) = \hat{\varphi}(1)$.

The second equation of (4.4) is equivalent to

$$(4.8) \qquad e^{\sqrt{\lambda+a}x}\hat{w}(x) = e^{\sqrt{\lambda+a}y}\hat{w}(y) + \int_y^x e^{\sqrt{\lambda+a}l}\psi(l)dl, \ \forall x \geq y.$$

In (4.8) let $y = 0$, then we have

$$(4.9) \qquad \hat{w}(x) = e^{-\sqrt{\lambda+a}x}\hat{w}(0) + e^{-\sqrt{\lambda+a}x}\int_0^x e^{\sqrt{\lambda+a}l}\psi(l)dl,$$

where $\hat{w}(0) = \sqrt{\lambda+a}\varphi(0) - \hat{\varphi}(0) = -\hat{\varphi}(0)$. In (4.8) let $x = 1$, we have

$$(4.10) \qquad \hat{w}(y) = e^{\sqrt{\lambda+a}-\sqrt{\lambda+a}y}\hat{w}(1) - e^{-\sqrt{\lambda+a}y}\int_y^1 e^{\sqrt{\lambda+a}l}\psi(l)dl,$$

where $\hat{w}(1) = \sqrt{\lambda+a}\varphi(1) - \hat{\varphi}(1) = -\hat{\varphi}(1)$.

From (4.6) and (4.9), we have

$$(4.11) \qquad e^{2\sqrt{\lambda+a}x}\hat{w}(x) + w(x) = \int_0^x e^{\sqrt{\lambda+a}x}(e^{\sqrt{\lambda+a}l} - e^{-\sqrt{\lambda+a}l})\psi(l)dl,$$

where $x \in [0,1]$. Combining (4.7) and (4.10), we obtain

$$(4.12) \quad e^{2\sqrt{\lambda+a}(1-x)}w(x) + \hat{w}(x) = \int_x^1 e^{-\sqrt{\lambda+a}x}(e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}l} - e^{\sqrt{\lambda+a}l})\psi(l)dl.$$

Since $\hat{w} = \sqrt{\lambda+a}\varphi - \hat{\varphi}$ and $w = \sqrt{\lambda+a}\varphi + \hat{\varphi}$, (4.11) and (4.12) can be written as

$$(4.13)$$
$$\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}x} + 1)\varphi + (1 - e^{2\sqrt{\lambda+a}x})\hat{\varphi} = \int_0^x e^{\sqrt{\lambda+a}x}(e^{\sqrt{\lambda+a}l} - e^{-\sqrt{\lambda+a}l})\psi(l)dl$$

and

$$(4.14) \qquad (e^{2\sqrt{\lambda+a}(1-x)} + 1)\sqrt{\lambda+a}\varphi + (e^{2\sqrt{\lambda+a}(1-x)} - 1)\hat{\varphi}$$
$$= \int_x^1 e^{-\sqrt{\lambda+a}x}(e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}l} - e^{\sqrt{\lambda+a}l})\psi(l)dl.$$

Combining (4.13) and (4.14), we have the following

$$\varphi(x) = \frac{(e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}x} - e^{\sqrt{\lambda+a}x})\int_0^x (e^{\sqrt{\lambda+a}l} - e^{-\sqrt{\lambda+a}l})\psi(l)dl}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}} - 1)}$$

$$- \frac{(e^{-\sqrt{\lambda+a}x} - e^{\sqrt{\lambda+a}x})\int_x^1 (e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}l} - e^{\sqrt{\lambda+a}l})\psi(l)dl}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}} - 1)}$$

$$= \frac{\int_0^x (e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}(x-l)} - e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}(x+l)} - e^{\sqrt{\lambda+a}(x+l)} + e^{\sqrt{\lambda+a}(x-l)})\psi(l)dl}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}} - 1)}$$

$$- \frac{\int_x^1 (e^{2\sqrt{\lambda+a}- !\sqrt{\lambda+a}(x+l)} - e^{\sqrt{\lambda+a}(l-x)} - e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}(l-x)} + e^{\sqrt{\lambda+a}(x+l)})\psi(l)dl}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}} - 1)}$$

$$= \frac{\int_0^x (e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}|x-l|} - e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}(x+l)} - e^{\sqrt{\lambda+a}(x+l)} + e^{\sqrt{\lambda+a}|x-l|})\psi(l)dl}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}} - 1)}$$

$$+ \frac{\int_x^1 (e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}|l-x|} - e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}(x+l)} - e^{\sqrt{\lambda+a}(x+l)} + e^{\sqrt{\lambda+a}|l-x|})\psi(l)dl}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}} - 1)}$$

$$= \frac{\int_0^1 (e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}|x-l|} - e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}(x+l)} - e^{\sqrt{\lambda+a}(x+l)} + e^{\sqrt{\lambda+a}|x-l|})\psi(l)dl}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}}-1)}.$$

Since $\varphi \in \overline{D(A)}$, it follows that

$$\|\varphi\|$$

$$= \sup_{x\in[0,1]} |\varphi(x)|$$

$$= \sup_{x\in[0,1]} \left| \frac{\int_0^1 (e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}|x-l|} - e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}(x+l)} - e^{\sqrt{\lambda+a}(x+l)} + e^{\sqrt{\lambda+a}|x-l|})\psi(l)dl}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}}-1)} \right|.$$

Since $e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}|x-l|} - e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}(x+l)} - e^{\sqrt{\lambda+a}(x+l)} + e^{\sqrt{\lambda+a}|x-l|} \geq 0$ for $x \in [0,1]$ and $l \in [0,1]$, we have

$$\|\varphi\|$$

$$\leq \sup_{x\in[0,1]} |\psi(x)| \sup_{x\in[0,1]} \frac{\int_0^1 \left| e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}|x-l|} - e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}(x+l)} - e^{\sqrt{\lambda+a}(x+l)} + e^{\sqrt{\lambda+a}|x-l|} \right| dl}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}}-1)}$$

$$= \sup_{x\in[0,1]} |\psi(x)| \sup_{x\in[0,1]} \frac{\int_0^1 (e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}|x-l|} - e^{2\sqrt{\lambda+a}-\sqrt{\lambda+a}(x+l)} - e^{\sqrt{\lambda+a}(x+l)} + e^{\sqrt{\lambda+a}|x-l|})dl}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}}-1)}$$

$$= \sup_{x\in[0,1]} |\psi(x)| \sup_{x\in[0,1]} \frac{1}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}}-1)} \left[e^{2\sqrt{\lambda+a}} \left(\frac{2}{\sqrt{\lambda+a}} - \frac{1}{\sqrt{\lambda+a}} e^{\sqrt{\lambda+a}(x-1)} \right. \right.$$

$$- \frac{1}{\sqrt{\lambda+a}} e^{-\sqrt{\lambda+a}x} + \frac{1}{\sqrt{\lambda+a}} e^{-\sqrt{\lambda+a}(x+1)} - \frac{1}{\sqrt{\lambda+a}} e^{-\sqrt{\lambda+a}x} \right)$$

$$\left. - \frac{2}{\sqrt{\lambda+a}} + \frac{1}{\sqrt{\lambda+a}} e^{-\sqrt{\lambda+a}(x-1)} + \frac{1}{\sqrt{\lambda+a}} e^{\sqrt{\lambda+a}x} - \frac{1}{\sqrt{\lambda+a}} e^{\sqrt{\lambda+a}(x+1)} + \frac{1}{\sqrt{\lambda+a}} e^{\sqrt{\lambda+a}x} \right]$$

$$\leq \sup_{x\in[0,1]} |\psi(x)| \sup_{x\in[0,1]} \frac{e^{2\sqrt{\lambda+a}}-1}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}}-1)} \left(\frac{2}{\sqrt{\lambda+a}} - \frac{1}{\sqrt{\lambda+a}} e^{\sqrt{\lambda+a}(x-1)} - \frac{1}{\sqrt{\lambda+a}} e^{-\sqrt{\lambda+a}x} \right.$$

$$\left. + \frac{1}{\sqrt{\lambda+a}} e^{-\sqrt{\lambda+a}(x+1)} - \frac{1}{\sqrt{\lambda+a}} e^{-\sqrt{\lambda+a}x} \right)$$

$$\leq \sup_{x\in[0,1]} |\psi(x)| \sup_{x\in[0,1]} \frac{(e^{2\sqrt{\lambda+a}}-1)\frac{2}{\sqrt{\lambda+a}}}{2\sqrt{\lambda+a}(e^{2\sqrt{\lambda+a}}-1)}$$

$$= \frac{1}{\lambda+a} \sup_{x\in[0,1]} |\psi(x)|$$

$$= \frac{1}{\lambda+a} \|\psi\|.$$

Now we have $\|(\lambda I - B)^{-1}\psi\| \leq \frac{1}{\lambda+a}\|\psi\|$, which implies that $\|(\lambda I - B)^{-1}\| \leq \frac{1}{\lambda+a}$. So B is a Hille-Yoshida operator with $M = 1$ and $\omega_B = -a < 0$. We conclude that

$$\|(\lambda I - B)^{-1}\| \leq \frac{1}{\lambda+a}, \ \forall \lambda > -a,$$

For $\varphi_1, \varphi_2 \in C_B$ and $\|\varphi_1\| \leq C$, $\|\varphi_2\| \leq C$, we have

$$\|f(t,\varphi_1) - f(t,\varphi_2)\| = \|-b(t)\varphi_1(-r) - ka - kb(t) + b(t)\varphi_2(-r) + ka + kb(t)\|$$

$$= \|b(t)(\varphi_1(-r) - \varphi_2(-r))\|$$

$$\leq \|b(t)\| \|\varphi_1(-r) - \varphi_2(-r)\|$$

$$\leq b_+ \|\varphi_1 - \varphi_2\|.$$

So $K_f(\rho) = b_+$ for $\forall \rho > 0$. Moreover, for $\varphi \in C_B$ with $\|\varphi\| \leq \rho$ and $0 \leq t \leq T$,

$$\|f(t,\varphi)\| = \|-b(t)\varphi(-r) - ka - kb(t)\|$$

FIGURE 1. A T-periodic solution of the delayed reaction-diffusion equation (4.2) with initial condition $\varphi(x,t) = 0.5$ for $t \in [-1,0]$, $x \in [0,1]$, where $b(t) = 0.15 + 0.15\sin(2\pi t)$, $T = 1$, $r = 1$, $k = 0.5$ and $a = 1$.

$$\leq b_+ \|\varphi\| + ka + kb_+$$
$$\leq b_+\rho + ka + kb_+.$$

So we have $L_f(T,\rho) = b_+\rho + ka + kb_+$. Therefore, assumptions (ii) and (iii) imply $(N+T)(K_f(\rho) + \left\|\hat{L}\right\|) < 1$ and $(N+T)(L_f(T,\rho) + \left\|\hat{L}\right\|\rho) \leq \rho$ in Theorem 3.2, respectively. The conclusion follows from Theorem 3.2. $\square$

Now we choose parameters for equation (4.2) such that assumptions in Proposition 4.1 are satisfied. We will perform some numerical simulations to demonstrate the existence of T-periodic solutions.

Let $T = 1$, $k = 0.5$, $r = 1$, $a = 1$ and $b(t) = 0.15 + 0.15\sin(2\pi t)$. We can verify that conditions in Proposition 4.1 are satisfied, so there exists a T-periodic solution, which can be seen from Figure 1.

Now we change the parameters so that the conditions in Proposition 4.1 do not hold. Let $T = 1$, $a = 1$, $k = 0.5$, $r = 1$ and $b(t) = 1.5 + 10\sin(2\pi t)$. Figure 2 gives a solution in this scenario.

4.2. **The diffusive Nicholson's blowflies equation.** We consider the diffusive Nicholson's blowflies equation (So and Yang [15], Yang and So [18], So et al. [14]) (4.15)
$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - \tau u(t,x) + \beta(t)\tau u(t-1,x)e^{u(t-1,x)}, \ t \geq 0, \ x \in [0,1] \\ u(t,0) = u(t,1) = k, \ t \geq 0, \end{cases}$$

where k is a constant and $\beta(t)$ is T-periodic. To study existence of T-periodic solutions of equation (4.15), let $v(t,x) = u(t,x) - k$. Then we have

(4.16)
$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = \frac{\partial^2 v(t,x)}{\partial x^2} - \tau v(t,x) + \beta(t)\tau v(t-1,x)e^{-[v(t-1,x)+k]} \\ \qquad +k\beta(t)\tau e^{-[v(t-1,x)+k]} - k\tau \\ v(t,0) = v(t,1) = 0. \end{cases}$$

We know that existence of T-periodic solutions of equation (4.16) is equivalent to the existence of T-periodic solutions of equation (4.15).

Let $X = C[0,1]$ and let $B : X \to X$ be defined by
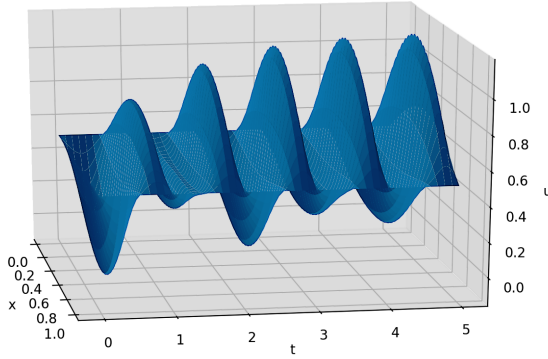
$$B\phi = \phi'' - \tau\phi$$

FIGURE 2. A solution of the delayed reaction-diffusion equation
(4.2) with initial condition $\varphi(x, t) = 0.5$ for $t \in [-1, 0]$, $x \in [0, 1]$,
where $b(t) = 1.5 + 10\sin(2\pi t)$, $T = 1$, $k = 0.5$, $r = 1$ and $a = 1$.

with $D(B) = \{\phi \in C^2([0, 1], \mathbb{R}), \phi(0) = \phi(1) = 0\}$. Let $C_B := \{\phi \in C([-1, 0], X) :$
$\phi(0) \in \overline{D(B)}\}$ and define $f : [0, \infty) \times C_B \to X$ by

$$f(t, \phi) = \beta(t)\tau\phi(-1)e^{-[\phi(-1)+k]} + k\beta(t)\tau e^{-[\phi(-1)+k]} - k\tau.$$

Then equation (4.16) can be written as

(4.17) $$\frac{dy(t)}{dt} = By(t) + f(t, y_t), \ t \geq 0$$

Proposition 4.2. *Assume that*
  (i) *$\tau > 0$, $0 \leq \beta(t) \leq \beta_+$ and $\beta(t) = \beta(t + T)$ for $\forall t \geq 0$;*
  (ii) *There exists $\rho > 0$ such that $(\frac{T}{1-e^{-\tau T}} + T)\beta_+\tau e^{-k}(\rho + k + 1)e^\rho < 1$ and*
  $(\frac{T}{1-e^{-\tau T}} + T)\tau(k + \beta_+\rho e^{\rho-k} + k\beta_+ e^{\rho-k}) \leq \rho$.
Then equation (4.16) thus (4.15) has a T-periodic solution.

Proof. Since equation (4.16) can be written as (4.17), it suffices to check assumptions of Theorem 3.2. Let $\psi \in X$ and let $\lambda > -\tau$. Then

$$(\lambda I - B)\varphi = \psi \Leftrightarrow (\lambda + \tau)\varphi - \varphi'' = \psi.$$

By following exactly the same way as in the proof of Proposition 4.1, we obtain that

$$\left\|(\lambda I - B)^{-1}\right\| \leq \frac{1}{\lambda + \tau}, \ \forall \lambda > -\tau,$$

which implies that $\omega_B = -\tau < 0$. For $\varphi_1, \varphi_2 \in C_B$ and $\|\varphi_1\| \leq \rho$, $\|\varphi_2\| \leq \rho$, we have

$$f(t, \varphi_1) - f(t, \varphi_2) = \beta(t)\tau\varphi_1(-1)e^{-[\varphi_1(-1)+k]} + k\beta(t)\tau e^{-[\varphi_1(-1)+k]} - k\tau$$
$$- \beta(t)\tau\varphi_2(-1)e^{-[\varphi_2(-1)+k]} - k\beta(t)\tau e^{-[\varphi_2(-1)+k]} + k\tau$$

and

$$\|f(t, \varphi_1) - f(t, \varphi_2)\| \leq \left\|\beta(t)\tau\varphi_1(-1)e^{-[\varphi_1(-1)+k]} - \beta(t)\tau\varphi_2(-1)e^{-[\varphi_2(-1)+k]}\right\|$$
$$+ \left\|k\beta(t)\tau e^{-[\varphi_1(-1)+k]} - k\beta(t)\tau e^{-[\varphi_2(-1)+k]}\right\|$$
$$\leq \left\|\beta(t)\tau e^{-k}(\varphi_1(-1)e^{-\varphi_1(-1)} - \varphi_1(-1)e^{-\varphi_2(-1)})\right\|$$

$$+ \left\| \beta(t)\tau e^{-k}(\varphi_1(-1)e^{-\varphi_2(-1)} - \varphi_2(-1)e^{-\varphi_2(-1)}) \right\|$$

$$+ \left\| k\beta(t)\tau e^{-k}(e^{-\varphi_1(-1)} - e^{-\varphi_2(-1)}) \right\|$$

$$\leq \beta_+\tau e^{-k}(\rho+1)e^\rho \|\varphi_1 - \varphi_2\| + k\beta_+\tau e^{-k}e^\rho \|\varphi_1 - \varphi_2\|$$

$$= \beta_+\tau e^{-k}(\rho+k+1)e^\rho \|\varphi_1 - \varphi_2\|.$$

So we have $K_f(\rho) = \beta_+\tau e^{-k}(\rho+k+1)e^\rho$ for $\rho > 0$. Moreover, for $\varphi \in C_B$ with $\|\varphi\| \leq \rho$ and $0 \leq t \leq T$,

$$\|f(t,\varphi)\| = \left\| \beta(t)\tau\varphi(-1)e^{-[\varphi(-1)+k]} + k\beta(t)\tau e^{-[\varphi(-1)+k]} - k\tau \right\|$$

$$\leq \beta_+\tau e^{-k} \left\| \varphi(-1)e^{-\varphi(-1)} \right\| + k\beta_+\tau e^{-k} \left\| e^{-\varphi(-1)} \right\| + k\tau$$

$$\leq \tau(k + \beta_+\rho e^{\rho-k} + k\beta_+ e^{\rho-k}).$$

Hence, we have $L_f(T,\rho) = \tau(k + \beta_+\rho e^{\rho-k} + k\beta_+ e^{\rho-k})$. Therefore, assumption (ii) implies $(N+T)(K_f(\rho) + \left\| \hat{L} \right\|) < 1$ and $(N+T)(L_f(T,\rho) + \left\| \hat{L} \right\| \rho) \leq \rho$ in Theorem 3.2. The conclusion follows from Theorem 3.2. $\qquad \square$

Now we choose parameters for equation (4.15) such that assumptions in Proposition 4.2 are satisfied. Let $T = 1$, $\tau = 1$, $k = 0.1$ and $\beta(t) = 0.025 + 0.015\cos 2\pi t$ in equation (4.15), then it is easy to check that assumptions of Proposition 4.2 are satisfied. So there exists a T-periodic solution, which can be seen from Figure 3.
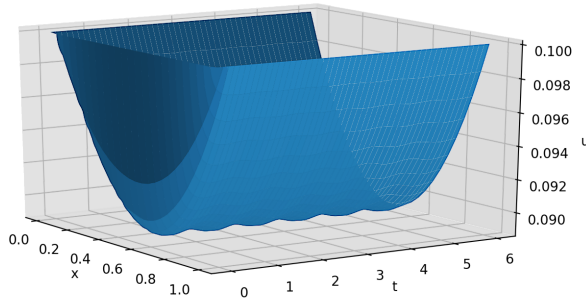

FIGURE 3. A T-periodic solution of the diffusive Nicholson's blowflies equation (4.15) with initial condition $\varphi(x,t) = 0.1$ for $t \in [-1,0]$, $x \in [0,1]$, where $\beta(t) = 0.025 + 0.015\cos(2\pi t)$, $T = 1$, $k = 0.1$ and $\tau = 1$.

Remark 4.3. When $u(t)$ does not depend on the spatial variable x in equations (4.1) and (4.15), the conclusions in Propositions 4.1 and 4.2 still hold. We then obtain conditions for the existence of periodic solutions in delayed periodic logistic equation (Chen [3]) and delayed periodic Nicholson's blowflies equation (Chen [4]), respectively.

Remark 4.4. The techniques and arguments used in this paper can be modified to study the existence of periodic solutions in partial functional differential equations with infinite delay (Benkhalti et al. [1]).

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