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GENERATOR FUNCTIONS AND THEIR APPLICATIONS

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ABSTRACT. We had introduced so called *generators* functions to precisely follow the regularity of analytic solutions of Navier-Stokes equations earlier (see Grenier and Nguyen [Ann. PDE 5 (2019)]. In this short note, we give a short presentation of these generator functions and use them to construct analytic solutions to classical evolution equations, which provides an alternative way to the use of the classical abstract Cauchy-Kovalevskaya theorem (see Asano [Proc. Japan Acad. Ser. A Math. Sci. 64 (1988), pp. 102–105], Baouendi and Goulaouic [Comm. Partial Differential Equations 2 (1977), pp. 1151–1162], Caflisch [Bull. Amer. Math. Soc. (N.S.) 23 (1990), pp. 495–500], Nirenberg [J. Differential Geom. 6 (1972), pp. 561–576], Safonov [Comm. Pure Appl. Math. 48 (1995), pp. 629–637]).

1. INTRODUCTION

The general abstract Cauchy-Kovalevskaya theorem has been intensively used to construct analytic solutions to various evolution partial differential equations, including first order hyperbolic and parabolic equations, or Euler and Navier-Stokes equations. We refer for instance to Asano [1], Baouendi and Goulaouic [2], Caflisch [6], Nirenberg [16], Safonov [17], among others. In this short note, we use generators functions as introduced in [8] to obtain existence results for these equations. The results in this paper are not new, but show the versatility and simplicity of use of these generator functions. We believe that the approach could be used on many other equations and provide easy ways to obtain analytic solutions, including those at the large time, and to investigate instabilities.

Let us first introduce generator functions in the particular case of a periodic function $f(t,x)$ on $t \geq 0$ and $x \in \mathbb{T}^d$, $d \geq 1$. For $z \in \mathbb{R}$, we introduce the *generator function* $Gen[f]$ defined by

$$Gen[f](t,z) = \sum_{\alpha \in \mathbb{Z}^d} e^{z|\alpha|} |f_\alpha(t)| \tag{1.1}$$

in which f_α denotes the Fourier transform of $f(t,x)$ with respect to $x \in \mathbb{T}^d$. If f is analytic in x, $Gen[f]$ is only defined for small enough $|z|$, up to the analyticity radius of $f(t,.)$. The results in this note also apply to the case when $x \in \mathbb{R}^d$, with which the above summation is replaced by the integral over $\mathbb{R}^d$. In applications, we may also introduce generator functions depending on multi-variables $z = (z_1, \cdots, z_d)$

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that correspond to the analyticity radius of $f(x)$ in $(x_1, \cdots, x_d)$, respectively; see, for instance, [8] for the case of boundary layers on the half space $\mathbb{T} \times \mathbb{R}_+$.

First note that generator functions are non negative, and that all their derivatives are non negative and non decreasing in z. Moreover generator functions have very nice properties with respect to algebraic operations and differentiation. Namely they "commute" with the product, the sum and the differentiation, making their use very versatile.

Lemma 1.1. *For any f, g, there hold the following properties*

$$Gen[f+g] \le Gen[f] + Gen[g] \tag{1.2}$$

$$Gen[fg] \le Gen[f]Gen[g] \tag{1.3}$$

$$Gen[\nabla_x f] = \partial_z Gen[f], \qquad \partial_t Gen[f] \le Gen[\partial_t f], \tag{1.4}$$

for all $z \ge 0$.

Proof. Let f_α and g_α be the Fourier transform of f and g, respectively. It follows that

$$(fg)_\alpha = f_\alpha \star_\alpha g_\alpha$$

for $\alpha \in \mathbb{Z}^d$. For $z \ge 0$, we compute

$$\begin{aligned} Gen[fg](z) = \sum_{\alpha \in \mathbb{Z}^d} e^{z|\alpha|} |f_\alpha \star_\alpha g_\alpha| &\le \sum_{\alpha \in \mathbb{Z}^d} \sum_{\beta \in \mathbb{Z}^d} e^{z|\beta|} e^{z|\alpha-\beta|} |f_\beta||g_{\alpha-\beta}| \\ &\le Gen[f]Gen[g] \end{aligned}$$

which is the second inequality. The first and third identities follow directly from the definition. The last inequality is a direct consequence of $\partial_t |f_\alpha| \le |\partial_t f_\alpha|$. □

The use of generator functions are not limited to polynomial operations. Namely, we have

Lemma 1.2. *Let $F, \widetilde{F}$ be analytic functions*

$$F(z) = \sum_{n \ge 0} a_n z^n, \qquad \widetilde{F}(z) = \sum_{n \ge 0} |a_n| z^n.$$

with some convergence radius ρ. Then, for any function $f(x)$, provided $\|f\|_{L^\infty} < \rho$, there holds

$$Gen[F(f)] \le \widetilde{F}(Gen[f]). \tag{1.5}$$

Proof. First, using the algebra (1.3), we have

$$Gen[a_n f^n] \le |a_n| Gen[f]^n.$$

Multiplying the inequality by z^n and summing over n give the lemma. □

2. An analyticity framework

In this section, we present our analyticity framework to construct analytic solutions via generator functions defined as in (1.1). The main point of the approach is that if $u(t,x)$ satisfies a non linear partial differential equation then $Gen[u](t,z)$ satisfies a transport differential inequality; see (2.7). This inequality describes in an acute way how the radius of analyticity shrinks as time goes on, and allows to get analytic bounds on u, and in particular to bound all its derivatives at the same time.

To precise the framework, we consider the following general system of evolution equations

$$\partial_t u = A(u, \nabla_x u) \tag{2.1}$$

for a vector valued function $u = u(x,t)$, with $x \in \mathbb{T}^d$, $d \geq 1$, and $t \geq 0$. We assume that the function $A(\cdot,\cdot)$ satisfies

$$Gen[A(u, \nabla_x u)] \leq C_0 F(Gen[u])\Big(1 + \partial_z Gen[u]\Big) \tag{2.2}$$

for some constant C_0 and some analytic functions F. For instance, A may be a quadratic polynomial in u and ∇u, in which case $F = Id$, or more generally, A may be of the form $G(u) \cdot \nabla_x u$, for which $F = \widetilde{G}$ (see Lemma 1.2). Note that we do not make any assumption on the hyperbolicity of the system (2.1).

We shall construct solutions in the function space X_ρ defined by

$$X_\rho := \Big\{ u \ : \ Gen[u](\rho) < \infty \Big\}. \tag{2.3}$$

Theorem 2.1. *Let $\rho > 0$ and u_0 be in X_ρ. Then the Cauchy problem* (2.1) *with initial data u_0 has a unique solution $u(t)$ in $X_{\rho(t)}$ for positive times t as long as $\rho(t) = \rho - C_1 t$ remains positive, C_1 being some large positive constant depending on u_0.*

Proof. We shall construct solutions via the standard approximation. First, we let P_N be the projection of the first N Fourier modes; namely

$$P_N(f)(x) = \sum_{\alpha \in \mathbb{Z}^d, |\alpha| \leq N} e^{i\alpha x} f_\alpha. \tag{2.4}$$

Let $u_N(t)$ be a solution to the following regularized equations

$$\partial_t u_N = P_N A(P_N(u_N), \nabla_x P_N(u_N)) \tag{2.5}$$

with initial data $u_N(0,x) = P_N u_0(x)$ for all N. As the right hand side consists of only a finite number of Fourier modes, this equation is in fact an ordinary differential equation. Hence there exists a unique solution $u_N(t)$, defined for t small enough. It suffices to prove that $u_N(t)$ is a Cauchy sequence in $X_{\rho(t)}$, as long as $\rho(t)$ remains positive.

The Fourier coefficients of the solution to (2.1) satisfy

$$\partial_t u_{N,\alpha} = P_{N,\alpha} A(P_N u_N, \nabla_x P_N u_N)$$

where $P_{N,\alpha} f$ denotes the Fourier coefficient f_α if $|\alpha| \le N$, and zero, if otherwise. We therefore get

$$\sum_{|\alpha|\le N} \partial_t |u_{N,\alpha}| e^{z|\alpha|} \le \sum_{|\alpha|\le N} |\partial_t u_{N,\alpha}| e^{z|\alpha|} \le \sum_{|\alpha|\le N} |P_{N,\alpha} A(P_N u_N, \nabla_x P_N u_N)| e^{z|\alpha|}.$$

By definition, we note that

$$Gen[P_N(f)] \le Gen[f]. \tag{2.6}$$

Thus, using the assumption (2.2), we obtain the following Hopf-type differential inequality

$$\partial_t Gen[u_N] \le C_0 F(Gen[u_N]) \Big(1 + \partial_z Gen[u_N] \Big), \tag{2.7}$$

in which C_0 is independent of N. For convenience, we set

$$G_N(t,z) = Gen[u_N(t)](z).$$

The previous inequality yields

$$\partial_t G_N \le C_0 F(G_N)(1 + \partial_z G_N), \tag{2.8}$$

which is a differential inequality that we will now exploit in order to bound u_N and all its derivatives. Note that G_N is a finite sum and is therefore defined for all z. As N goes to $+\infty$, $G_N(0,z)$ converges to $Gen[u_0](z)$, which is defined for $|z| \le \rho$.

As usual with analytic solutions, the domain of analyticity shrinks with time, hence we introduce

$$F_N(t,z) = G_N(t, \theta(t) z)$$

for $t, z \ge 0$, where $\theta(\cdot)$ will be determined later. It follows that F_N satisfies

$$\partial_t F_N \le C_0 F(F_N) + C_0 (F(F_N) + \theta'(t) z) \partial_z F_N. \tag{2.9}$$

Note that F_N is defined for any z. In the limit $N \to +\infty$ we focus on $0 \le z \le \rho$.

We will choose $\theta(t)$ in such a way that the characteristics of (2.9) are outgoing on $[0,\rho]$, namely such that at $z = 0$, $F(F_N) > 0$ (which is always satisfied) and such that at $z = \rho$, $F(F_N) + \theta'(t)\rho < 0$. At $t = 0$, we choose $\theta(0) = 1$ and thus

$$F_N(0,z) = G_N(0,z) = Gen[P_N u_0](z) \le Gen[u_0](z),$$

which is well-defined on $[0,\rho]$. Set

$$M_0 = \sup_{0\le z\le \rho} Gen[u_0](z).$$

We will focus on times $0 \le t \le T_N$ such that $F_N(t,z) \le 2M_0$ for $0 \le t \le T_N$ and $0 \le z \le \rho$. We choose

$$\theta(t) = 1 - F(3M_0)\rho^{-1} t.$$

Observe that on $0 \le t \le T_N$, as long as $\theta(t) > 0$, $F(F_N) + \theta'(t)\rho < 0$. On such a time interval, (2.9) is a nonlinear transport equation with a source term and with outgoing characteristics at 0 and ρ. As a consequence, we have

$$\partial_t \sup_{0\le z\le 1} F_N(t,z) \le C_0 \sup_{0\le z\le 1} F(F_N(t,z)) \le C_0 F(2M_0),$$

hence

$$F_N(t,z) \le M_0 + C_0 t F(2M_0).$$

Classical arguments then lead to the fact that T_N is bounded away from 0, namely there exists some $T > 0$ such that $T_N \geq T$ for any N, and such that $\rho(t) > 0$ for any $t \leq T$. This implies that

$$F_N(t, z) \leq C(F_N(0, z)) \leq C(M_0)$$

for any $0 \leq z \leq \rho$ and any $0 \leq t \leq T$. Thus $u_N(t)$ is uniformly bounded in $X_{\rho(t)}$ for all N. As a consequence, u_N and all its derivatives of all orders are uniformly bounded in L^∞. Up to the extraction of a subsequence, u_N and all its derivatives converge uniformly, towards some function u, solution of (2.1). Moreover, classical arguments show that $u(t) \in X_{\rho(t)}$ for any $0 \leq t \leq T$, which ends the proof of the theorem. □

3. Euler equations

In this section, we apply the previous framework to construct analytic solutions to incompressible Euler equations on $\mathbb{T}^d$ or $\mathbb{R}^d$, $d \geq 2$. Namely, we consider

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= 0 \\ \nabla \cdot u &= 0 \end{aligned} \tag{3.1}$$

on $\mathbb{T}^d$ (the case $\mathbb{R}^d$ is treated similarly). Again, the existence result is classical (see, for instance, [3, 4, 12, 13]). The function space X_ρ is defined in (2.3). We have

Theorem 3.1. *Let $\rho > 0$ and u_0 be a divergence-free vector field in X_ρ. Then the Cauchy problem* (3.1) *with initial data u_0 has a solution $u(t)$ in $X_{\rho(t)}$ for positive times t as long as $\rho(t) = \rho - C_1 t$ remains positive, C_1 being a constant depending on u_0.*

Proof. Introduce the Leray projector $\mathbb{P}$, projection onto the divergence-free L^2 vector space. In Fourier coefficients, $\mathbb{P}_\alpha$ is an $d \times d$ matrix with entries

$$(\mathbb{P}_\alpha)_{jk} = \delta_{jk} - \frac{\alpha_j \alpha_k}{|\alpha|^2}.$$

In particular, $|\mathbb{P}_\alpha|$ is bounded, uniformly in $\alpha \in \mathbb{Z}^d$. Taking the Leray projection of (3.1), we obtain

$$\partial_t u = -\mathbb{P}(u \cdot \nabla u), \tag{3.2}$$

which falls into the previous abstract framework. It remains to check the assumption (2.2). Indeed, using (1.2) and the uniform boundedness of $|\mathbb{P}_\alpha|$ in α, we compute

$$Gen[\mathbb{P}(u \cdot \nabla u)] \leq Gen[u \cdot \nabla u] \leq Gen[u]Gen[\nabla u] \leq Gen[u]\partial_z Gen[u].$$

Note that the divergence-free condition is invariant under (3.2). Thus, applying the abstract framework introduced in the previous section to the evolution equation (3.2), we obtain Theorem 3.1. □

Remark 3.2. Note that there is no hyperbolicity assumption made in the first order evolution equation (2.1), which may in particular be illposed in Sobolev spaces. The abstract framework can also be applied to a variety of other physical relevant models (e.g., [7, 11]) that arise in a singular limit of Euler equations, Navier-Stokes equations, and Vlasov-Poisson systems.

4. Comments on other applications

In this section, we briefly highlight two recent applications to the use of generator functions to capture the instability of boundary layers [8] and prove the nonlinear Landau damping [9, 10], both of which have a different flavor from the previous short time existence theory. These applications [8–10] are a version of the large time Cauchy-Kovalevskaya theorem. The use of generator functions allows us to control all the derivatives uniformly in the small viscosity limit or in the large time limit. In particular, the work [9] provides an elementary proof of the nonlinear Landau damping that was first obtained by Mouhot and Villani [15] for analytic data and by Bedrossian, Masmoudi, and Mouhot [5] for Gevrey data.

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