

REDUCTION OF QUAD-EQUATIONS CONSISTENT AROUND A CUBOCTAHEDRON I: ADDITIVE CASE

NALINI JOSHI AND NOBUTAKA NAKAZONO

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ABSTRACT. In this paper, we consider a reduction of a new system of partial difference equations, which was obtained in our previous paper [*Classification of quad-equations on a cuboctahedron*, arXiv:1906.06650, 2019] and shown to be consistent around a cuboctahedron. We show that this system reduces to $A_2^{(1)*}$ -type discrete Painlevé equations by considering a periodic reduction of a three-dimensional lattice constructed from overlapping cuboctahedra.

1. INTRODUCTION

In this paper, we consider a system of partial difference equations (PΔEs) governing a function $u = u(\mathbf{l})$ taking values on the vertices of a face-centered cubic lattice Ω , given by

$$(1.1) \quad \Omega = \left\{ \mathbf{l} = \sum_{i=1}^3 l_i \boldsymbol{\epsilon}_i \mid l_i \in \mathbb{Z}, l_1 + l_2 + l_3 \in 2\mathbb{Z} \right\},$$

where $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3\}$ is a standard basis of \mathbb{R}^3 . The system consists of 6 equations:

$$(1.2) \quad \frac{u_{\bar{i}\bar{k}}}{u_{\underline{i}\bar{k}}} = \frac{(\alpha_{ij} + \gamma_i)u_{\bar{j}\bar{k}} - (\alpha_{ij} + \gamma_j - \gamma_k)u_{\underline{j}\bar{k}}}{(\alpha_{ij} - \gamma_j + \gamma_k)u_{\bar{j}\bar{k}} - (\alpha_{ij} - \gamma_i)u_{\underline{j}\bar{k}}}, \quad \frac{u_{\bar{j}\bar{k}}}{u_{\underline{j}\bar{k}}} = \frac{(\alpha_{ij} + \gamma_i)u_{\bar{i}\bar{k}} - (\alpha_{ij} - \gamma_j + \gamma_k)u_{\underline{i}\bar{k}}}{(\alpha_{ij} + \gamma_j - \gamma_k)u_{\bar{i}\bar{k}} - (\alpha_{ij} - \gamma_i)u_{\underline{i}\bar{k}}},$$

where $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$, and the bars \bar{i} and \underline{j} denote $\mathbf{l} \rightarrow \mathbf{l} + \boldsymbol{\epsilon}_i$ and $\mathbf{l} \rightarrow \mathbf{l} - \boldsymbol{\epsilon}_j$ respectively and the coefficients are given by

$$(1.3a) \quad \alpha_{ij} = \alpha_i(l_i) - \alpha_j(l_j), \quad \alpha_i(k) = \alpha_i(0) + k, \quad i, j \in \{1, 2, 3\}, k \in \mathbb{Z},$$

$$(1.3b) \quad \gamma_1 = -c + (-1)^{l_1+l_2}\delta_1, \quad \gamma_2 = -c + (-1)^{l_2+l_3}\delta_2, \quad \gamma_3 = -c + (-1)^{l_1+l_3}\delta_3,$$

with $\alpha_i(0)$, $i = 1, 2, 3$, c , and δ_j , $j = 1, 2, 3$, being complex parameters. Figure 1.1 shows a unit cell in Ω .

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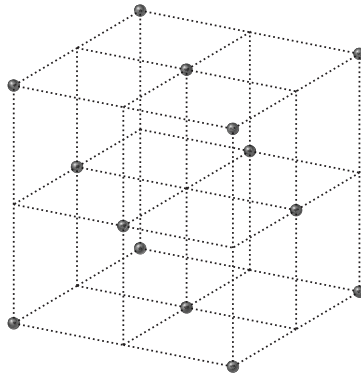


FIGURE 1.1. A unit cell of the Ω lattice

Our study is motivated by two considerations. Firstly, the system (1.2) satisfies the consistency around a cuboctahedron (CACO) property [10], which is a generalization of the famous consistency around a cube (CAC) property [17]. (See Appendix A for a summary of the details of the CACO property and §1.2 for those of the CAC property.) Secondly, we are motivated by finding relations between partial difference equations and ordinary difference equations known as the discrete Painlevé equations.

In this paper, we show that the system (1.2) reduces to discrete Painlevé equations with initial value space characterised as $A_2^{(1)*}$ in the sense of Sakai [22]. The latter equations have two forms in the literature given respectively by Tsuda [23] and Ramani *et al.* [21] and are explicitly given by:

$$(1.4a) \quad \begin{cases} (\underline{Y} + X)(X + Y) = \frac{((X + c_3)^2 - c_1)((X - c_3)^2 - c_2)}{(X + t)^2 - c_4}, \\ (\overline{X} + Y)(X + Y) = \frac{((Y - c_3)^2 - c_1)((Y + c_3)^2 - c_2)}{(Y + t + \frac{1}{2})^2 - c_5}, \end{cases}$$

$$(1.4b) \quad (\overline{X} + X)(\underline{X} + X) = \frac{(X^2 - c_1)(X^2 - c_2)}{(X + t)^2 - c_3}.$$

Here, $t \in \mathbb{C}$ is an independent variable, $c_i, i = 1 \dots, 5$, are complex parameters and X, Y are dependent variables:

$$(1.5) \quad X = X(t), \quad Y = Y(t), \quad \overline{X} = X(t + 1), \quad \underline{X} = X(t - 1), \quad \underline{Y} = Y(t - 1).$$

We note that discrete Painlevé equations admit special solutions when parameters take special values. For example, Equation (1.4a) has the special solution given by the generalized hypergeometric series ${}_3F_2$ when $4c_3 + 2\sqrt{c_4} + 2\sqrt{c_5} = 1$ [11].

Our main result is Theorem 1.1. To state the theorem, we first explain how to take the reduction on the lattice Ω . To be explicit, consider a vertex $\mathbf{l} \in \Omega$, given by $l_1\epsilon_1 + l_2\epsilon_2 + l_3\epsilon_3$. Define the plane $H_k \subset \Omega$ given by $l_3 = k$. We project the vertices of H_1 to the adjacent horizontal plane H_0 by taking $(l_1, l_2, 1) \mapsto (l_1 - 1, l_2 - 1, 0)$. The union of the projection with the lattice points on H_0 forms \mathbb{Z}^2 . We can define such a projection from every plane H_k to H_0 by the following:

$$(l_1, l_2, k) \mapsto (l_1 - k, l_2 - k, 0).$$

We call the result of this operation a $(1, 1, 1)$ -periodic reduction.

Theorem 1.1. *The $A_2^{(1)*}$ -type discrete Painlevé equations (1.4) can be obtained from the system of PΔEs (1.2) via the $(1, 1, 1)$ -periodic reduction.*

1.1. Notation and definitions. Throughout the paper, we use terminology to describe polynomials and quad-equations that is common in the literature. Readers who are unfamiliar with this notation may wish to consult [1, 8, 10]. We use $Q = Q(x, y, z, w)$ to denote a multivariable polynomial over \mathbb{C} . Under certain conditions, i.e., Q be affine linear and irreducible, we will refer to the equation $Q = 0$ as a *quad-equation* or sometimes, for succinctness, refer to the polynomial Q as a quad-equation. We remind the reader that the condition of irreducibility implies that $Q(x, y, z, w) = 0$ can be solved for each argument, and that the solution is a rational function of the other three arguments.

1.2. Background. Integrable systems are widely applicable models of science, occurring in fluid dynamics, particle physics and optics. The prototypical example is the famous Korteweg-de Vries (KdV) equation whose solitary wave-like solutions interact with elastically like particles, leading to the invention of the term *soliton*. It is then natural to ask what discrete versions of such equations are also integrable. This question turns out to be related to consistency conditions for polynomials associated to faces of cubes as we explain below.

Integrable discrete systems were discovered [15, 16, 18, 20] from mappings that turn out to be consistent on multi-dimensional cubes. (We note that there are additional systems that do not fall into this class; see e.g., [8, Chapter 3].) These are quad-equations in the sense in §1.1. In [1–4], Adler-Bobenko-Suris *et al.* classified quad-equations satisfying the consistency around a cube (CAC) property, which lead to integrable PΔEs. We refer to such PΔEs as ABS equations. It turns out that ABS equations contain many well known integrable PΔEs [9, 14–16].

Reductions of integrable PDEs lead to Painlevé equations, which first arose in the search for new transcendental functions in the early 1900’s [5, 6, 19]. Again a natural question is to ask whether discrete versions exist with analogous properties. This question led to the discovery of second-order difference equations called the discrete Painlevé equations [7, 13, 20].

It is now well-known that discrete Painlevé equations have initial value spaces with geometric structures that can be identified with root systems and affine Weyl groups [22]. Sakai showed that there are 22 types of initial value spaces as shown in Table 1.1.

TABLE 1.1. Types of spaces of initial values

Discrete type	Type of space of initial values
Elliptic	$A_0^{(1)}$
Multiplicative	$A_0^{(1)*}, A_1^{(1)}, A_2^{(1)}, A_3^{(1)}, \dots, A_8^{(1)}, A_7^{(1)'}$
Additive	$A_0^{(1)**}, A_1^{(1)*}, A_2^{(1)*}, D_4^{(1)}, \dots, D_8^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$

1.3. Outline of the paper. This paper is organized as follows. In §2, we show the extended affine Weyl group of type $E_6^{(1)}$ and its subgroup which forms that of

type $A_2^{(1)}$. Moreover, from those birational actions we obtain the discrete Painlevé equations (1.4) and the PΔEs (2.16), which are periodically reduced equations of the system (1.2). In §3, using the results in §2 we give the proof of Theorem 1.1. Finally, we give some concluding remarks in §4.

2. DERIVATION OF THE DISCRETE INTEGRABLE SYSTEMS FROM AN EXTENDED AFFINE WEYL GROUP OF TYPE $E_6^{(1)}$

In this section, we derive the partial/ordinary discrete integrable systems from the birational actions of an extended affine Weyl group of type $E_6^{(1)}$, denoted by $\widetilde{W}(E_6^{(1)})$. Note that details of $\widetilde{W}(E_6^{(1)})$ are given in Appendix B.

2.1. **Extended affine Weyl group of type $A_2^{(1)}$.** Let $a_i, i = 0, \dots, 6$, be parameters satisfying the condition

$$(2.1) \quad a_1 + 2a_2 + 3a_3 + 2a_4 + a_5 + 2a_6 + a_0 = 1,$$

and $\tau_j^{(i)}, i = 1, 2, 3, j = 0, 1, 2, 3$, be variables. Moreover, we define the transformations $s_i, i = 0, \dots, 6, \iota_j, j = 1, 2, 3$, by isomorphisms from the field of rational functions $K(\{\tau_j^{(i)}\})$, where $K = \mathbb{C}(\{a_i\})$, to itself. These transformations collectively form the extended affine Weyl group of type $E_6^{(1)}$, denoted by $\widetilde{W}(E_6^{(1)})$:

$$(2.2) \quad \widetilde{W}(E_6^{(1)}) = \langle s_0, \dots, s_6 \rangle \rtimes \langle \iota_1, \iota_2, \iota_3 \rangle.$$

See Appendix B for more details.

Let us define the transformations $w_i, i = 0, 1, 2$, and π by

$$(2.3) \quad w_0 = s_2 s_1 s_3 s_2, \quad w_1 = s_4 s_5 s_3 s_4, \quad w_2 = s_6 s_0 s_3 s_6, \quad \pi = \iota_3 \iota_1.$$

They collectively form the extended affine Weyl group of type $A_2^{(1)}$:

$$(2.4) \quad \widetilde{W}(A_2^{(1)}) = \langle w_0, w_1, w_2 \rangle \rtimes \langle \pi \rangle.$$

Indeed, the following fundamental relations hold:

$$(2.5) \quad (w_i w_j)^{a_{ij}} = 1, \quad i, j \in \{0, 1, 2\}, \quad \pi^3 = 1, \quad \pi w_{\{0,1,2\}} = w_{\{1,2,0\}} \pi,$$

where

$$(2.6) \quad (a_{ij})_{i,j=0}^2 = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}.$$

Introduce the parameters and variables that go well with $\widetilde{W}(A_2^{(1)})$ as follows. Let

$$(2.7a) \quad b_0 = a_1 + 2a_2 + a_3, \quad b_1 = a_3 + 2a_4 + a_5, \quad b_2 = a_3 + 2a_6 + a_0,$$

$$(2.7b) \quad c = \frac{a_0 + a_1 + a_3 + a_5}{2}, \quad d_{12} = \frac{a_0 + a_1 - a_3 - a_5}{2}, \quad d_{23} = \frac{a_0 - a_1 + a_3 - a_5}{2},$$

$$(2.7c) \quad d_{13} = \frac{a_0 - a_1 - a_3 + a_5}{2},$$

where $b_0 + b_1 + b_2 = 1$, and

$$(2.8) \quad y_1 = \frac{\tau_1^{(1)}}{\tau_0^{(1)}}, \quad y_2 = \frac{\tau_3^{(3)}}{\tau_2^{(3)}}, \quad y_3 = \frac{\tau_1^{(2)}}{\tau_0^{(2)}}, \quad y_4 = \frac{\tau_3^{(2)}}{\tau_2^{(2)}}, \quad y_5 = \frac{\tau_3^{(1)}}{\tau_2^{(1)}}, \quad y_6 = \frac{\tau_1^{(3)}}{\tau_0^{(3)}}.$$

Then, the actions of $\widetilde{W}(A_2^{(1)})$ on the parameters $b_0, b_1, b_2, c, d_{12}, d_{23}, d_{13}$ are given by

$$(2.9a) \quad w_i(b_j) = \begin{cases} -b_i & \text{if } i = j, \\ b_j + b_i & \text{if } i \neq j, \end{cases} \quad w_0 : (d_{12}, d_{23}) \mapsto (d_{23}, d_{12}),$$

$$(2.9b) \quad w_1 : (d_{23}, d_{13}) \mapsto (d_{13}, d_{23}), \quad w_2 : (d_{12}, d_{13}) \mapsto (-d_{13}, -d_{12}),$$

$$(2.9c) \quad \pi : (b_0, b_1, b_2, d_{12}, d_{23}, d_{13}) \mapsto (b_1, b_2, b_0, -d_{23}, -d_{13}, d_{12}),$$

where $i, j \in \mathbb{Z}/(3\mathbb{Z})$, while those on the y -variables $y_i, i = 1, \dots, 6$, are given by

$$(2.10a) \quad w_0 : \begin{pmatrix} y_1, & y_3 \\ y_5, & y_6 \end{pmatrix} \mapsto \begin{pmatrix} y_5, & \frac{(b_0 - c + d_{13})y_3 - (b_0 - d_{12} + d_{23})y_1}{(b_0 + d_{12} - d_{23})y_3 - (b_0 + c - d_{13})y_1} y_5 \\ y_1, & \frac{(b_0 - c + d_{13})y_6 - (b_0 - d_{12} + d_{23})y_1}{(b_0 + d_{12} - d_{23})y_6 - (b_0 + c - d_{13})y_1} y_5 \end{pmatrix},$$

$$(2.10b) \quad w_1 : \begin{pmatrix} y_1, & y_3 \\ y_4, & y_6 \end{pmatrix} \mapsto \begin{pmatrix} \frac{(b_1 - c + d_{12})y_1 - (b_1 - d_{13} + d_{23})y_3}{(b_1 + d_{13} - d_{23})y_1 - (b_1 + c - d_{12})y_3} y_4, & y_4 \\ y_3, & \frac{(b_1 - c + d_{12})y_6 - (b_1 - d_{13} + d_{23})y_3}{(b_1 + d_{13} - d_{23})y_6 - (b_1 + c - d_{12})y_3} y_4 \end{pmatrix},$$

$$(2.10c) \quad w_2 : \begin{pmatrix} y_1, & y_2 \\ y_3, & y_6 \end{pmatrix} \mapsto \begin{pmatrix} \frac{(b_2 - c - d_{23})y_1 - (b_2 - d_{12} - d_{13})y_6}{(b_2 + d_{12} + d_{13})y_1 - (b_2 + c + d_{23})y_6} y_2, & y_6 \\ \frac{(b_2 - d_{12} - d_{13})y_6 - (b_2 - c - d_{23})y_3}{(b_2 + c + d_{23})y_6 - (b_2 + d_{12} + d_{13})y_3} y_2, & y_2 \end{pmatrix},$$

$$(2.10d) \quad \pi : (y_1, y_2, y_3, y_4, y_5, y_6) \mapsto (y_3, y_5, y_6, y_2, y_4, y_1).$$

Remark 2.1. We follow the convention that the parameters and y -variables not explicitly included in the actions listed in Equations (2.9) and (2.10) are the ones that remain unchanged under the action of the corresponding transformation. That is, the transformation acts as an identity on those parameters or variables.

For later convenience, we here define the translations in $\widetilde{W}(A_2^{(1)})$ by

$$(2.11) \quad T_1 = w_1 w_2 \pi^2, \quad T_2 = w_2 w_0 \pi^2, \quad T_3 = w_0 w_1 \pi^2,$$

whose actions on the parameters $b_0, b_1, b_2, c, d_{12}, d_{23}, d_{13}$ are given by

$$(2.12a) \quad T_1 : (b_0, b_1, d_{12}, d_{13}) \mapsto (b_0 - 1, b_1 + 1, -d_{12}, -d_{13}),$$

$$(2.12b) \quad T_2 : (b_1, b_2, d_{12}, d_{23}) \mapsto (b_1 - 1, b_2 + 1, -d_{12}, -d_{23}),$$

$$(2.12c) \quad T_3 : (b_2, b_0, d_{23}, d_{13}) \mapsto (b_2 - 1, b_0 + 1, -d_{23}, -d_{13}).$$

Note that $T_1 T_2 T_3 = 1$ and $T_i T_j = T_j T_i$, where $i, j = 1, 2, 3$.

2.2. **Derivation of the partial difference equations from $\widetilde{W}(A_2^{(1)})$.** In this subsection, we derive the PΔEs (2.16) from the birational action of $\widetilde{W}(A_2^{(1)})$.

Let

$$(2.13) \quad u_{l_1, l_2, l_3} = T_1^{l_1} T_2^{l_2} T_3^{l_3}(y_2).$$

Note that

$$(2.14) \quad u_{0,1,1} = y_1, \quad u_{0,0,0} = y_2, \quad u_{1,0,0} = y_3, \quad u_{0,1,0} = y_4, \quad u_{1,1,0} = y_5, \quad u_{1,2,0} = y_6.$$

We assign the variable u_{l_1, l_2, l_3} on the vertices (l_1, l_2, l_3) of the triangle lattice

$$(2.15) \quad \mathbb{Z}^3 / (1, 1, 1) := \{ (l_1, l_2, l_3) \in \mathbb{Z}^3 \mid l_1 + l_2 + l_3 = 0 \}.$$

Then, we obtain Lemma 2.2.

Lemma 2.2. *On the triangle lattice there are three fundamental relations (essentially two):*

$$(2.16) \quad \frac{u_{\bar{i}}}{u_{\underline{i}}} = \frac{\left(b_{l_i, l_j}^{(i)} - c + (-1)^{l_i + l_j} d_{ij} \right) u_{\bar{j}} - \left(b_{l_i, l_j}^{(i)} - (-1)^{l_j + l_k} d_{jk} + (-1)^{l_i + l_k} d_{ik} \right) u_{\underline{j}}}{\left(b_{l_i, l_j}^{(i)} + (-1)^{l_j + l_k} d_{jk} - (-1)^{l_i + l_k} d_{ik} \right) u_{\bar{j}} - \left(b_{l_i, l_j}^{(i)} + c - (-1)^{l_i + l_j} d_{ij} \right) u_{\underline{j}}},$$

where $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ and

$$(2.17) \quad b_{l_1, l_2}^{(1)} = b_1 + l_1 - l_2, \quad b_{l_2, l_3}^{(2)} = b_2 + l_2 - l_3 - 1, \quad b_{l_1, l_3}^{(0)} = b_0 + l_3 - l_1.$$

Here, $u = u_{l_1, l_2, l_3}$ and the subscript \bar{i} (or, \underline{i}) for a function $u = u_{l_1, l_2, l_3}$ means $+1$ shift (or, -1 shift) in the l_i -direction.

Proof. Equation (2.16) with $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ are respectively obtained from the following actions:

$$(2.18a) \quad \frac{T_1(y_5)}{y_4} = \frac{(b_1 - c + d_{12})y_6 - (b_1 + d_{23} - d_{13})y_3}{(b_1 - d_{23} + d_{13})y_6 - (b_1 + c - d_{12})y_3},$$

$$(2.18b) \quad \frac{T_2(y_4)}{y_2} = \frac{(b_2 - c - d_{23})y_1 - (b_2 - d_{13} - d_{12})y_6}{(b_2 + d_{13} + d_{12})y_1 - (b_2 + c + d_{23})y_6},$$

$$(2.18c) \quad \frac{T_3(y_2)}{y_5} = \frac{(b_0 - c + d_{13})y_3 - (b_0 - d_{12} + d_{23})y_1}{(b_0 + d_{12} - d_{23})y_3 - (b_0 + c - d_{13})y_1}.$$

Moreover, we can easily verify that using Equations (2.16) we can express any u_{l_1, l_2, l_3} on the lattice by the six initial variables $y_i, i = 1, \dots, 6$, and one of the equations (2.16) can be obtained from the other two equations. Therefore, we have completed the proof. □

Remark 2.3. Because of the following relations:

(2.19a)

$$w_0(u_{l_1, l_2, l_3}) = u_{l_3, l_2, l_1}, \quad w_1(u_{l_1, l_2, l_3}) = u_{l_2, l_1, l_3}, \quad w_2(u_{l_1, l_2, l_3}) = u_{l_1+1, l_3+2, l_2},$$

(2.19b)

$$\pi(u_{l_1, l_2, l_3}) = u_{l_3+1, l_1+1, l_2},$$

which follow from

$$w_0 T_{\{1,2,3\}} = T_{\{3,2,1\}} w_0, \quad w_1 T_{\{1,2,3\}} = T_{\{2,1,3\}} w_1, \quad w_2 T_{\{1,2,3\}} = T_{\{1,3,2\}} w_2,$$

$$\pi T_{\{1,2,3\}} = T_{\{2,3,1\}} \pi, \quad w_0(u_{0,0,0}) = u_{0,0,0}, \quad w_1(u_{0,0,0}) = u_{0,0,0},$$

$$w_2(u_{0,0,0}) = u_{1,2,0}, \quad \pi(u_{0,0,0}) = u_{1,1,0},$$

the transformation group $\widetilde{W}(A_2^{(1)})$ can be also regarded as a symmetry of the triangle lattice (see Figure 2.1).

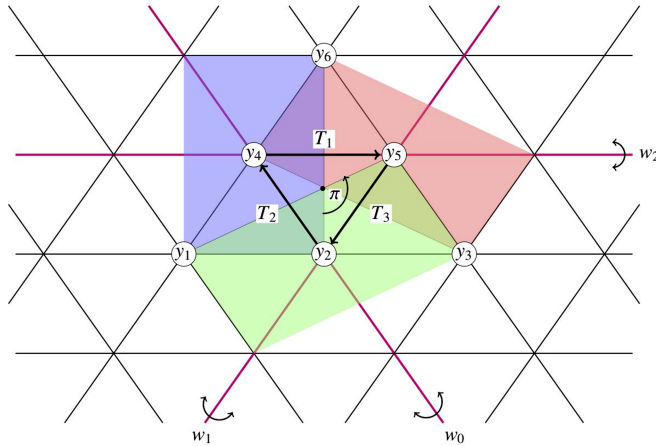


FIGURE 2.1. Triangle lattice. On the vertices the variables u_{l_1, l_2, l_3} are assigned, and on the quadrilaterals there exist quad-equations (2.16), e.g. Equations (2.18a), (2.18b) and (2.18c) are colored in red, blue and green, respectively.

2.3. Derivation of the $A_2^{(1)*}$ -type discrete Painlevé equations from $\widetilde{W}(A_2^{(1)})$.

In this subsection, we derive the $A_2^{(1)*}$ -type discrete Painlevé equations (1.4) from the birational action of $\widetilde{W}(A_2^{(1)})$.

Let

$$(2.20a) \quad f = \frac{(c - d_{12} + d_{23} - d_{13})y_1}{2(y_6 - y_1)} + \frac{b_2 + c - d_{12} + d_{23} - d_{13}}{4},$$

$$(2.20b) \quad g = \frac{(c - d_{12} + d_{23} - d_{13})y_3}{2(y_3 - y_6)} - \frac{b_2 + c - d_{12} + d_{23} - d_{13}}{4}.$$

Then, the action of $\widetilde{W}(A_2^{(1)})$ on the variables f and g is given by

$$\begin{aligned} w_0(f) &= f - \frac{b_0}{4}, \quad w_1(g) = g + \frac{b_1}{4}, \quad \pi(g) = f - \frac{b_2 + b_0}{4}, \\ &\frac{4(c - d_{12} + d_{23} - d_{13})}{4f - 2b_0 - b_2 - c - d_{12} + d_{23} + d_{13}} \left(w_0(g) + \frac{b_2 + b_0 + c + d_{12} - d_{23} - d_{13}}{4} \right) \\ &= \frac{(b_0 - d_{12} + d_{23})(4g + b_2 - c + d_{12} - d_{23} + d_{13})}{4f - b_2 + c - d_{12} + d_{23} - d_{13}} \\ &\quad - \frac{(b_0 - c + d_{13})(4g + b_2 + c - d_{12} + d_{23} - d_{13})}{4f - b_2 - c + d_{12} - d_{23} + d_{13}}, \\ &\frac{4(c - d_{12} + d_{23} - d_{13})}{4g + 2b_1 + b_2 + c - d_{12} - d_{23} + d_{13}} \left(w_1(f) - \frac{b_1 + b_2 + c - d_{12} - d_{23} + d_{13}}{4} \right) \\ &= \frac{(b_1 + d_{23} - d_{13})(4f - b_2 + c - d_{12} + d_{23} - d_{13})}{4g + b_2 - c + d_{12} - d_{23} + d_{13}} \\ &\quad - \frac{(b_1 - c + d_{12})(4f - b_2 - c + d_{12} - d_{23} + d_{13})}{4g + b_2 + c - d_{12} + d_{23} - d_{13}}, \\ \pi(f) &= -\frac{(4f - b_2 + c - d_{12} + d_{23} - d_{13})(4g + b_2 + c - d_{12} + d_{23} - d_{13})}{16(f + g)} \\ &\quad + \frac{b_0 + c - d_{12} + d_{23} - d_{13}}{4}. \end{aligned}$$

Using the transformation T_1^2 whose action on the parameter space $\{b_0, b_1, b_2, c, d_{12}, d_{23}, d_{13}\}$ is translational as $T_1^2 : (b_0, b_1) \mapsto (b_0 - 2, b_1 + 2)$ shows, we obtain the discrete Painlevé equation (1.4a) with the following correspondence:

$$(2.21a) \quad X = f, \quad Y = g, \quad \overline{X} = T_1^2(f), \quad \underline{Y} = T_1^{-2}(g), \quad t = \frac{2b_1 + b_2 - 2}{4},$$

$$(2.21b) \quad c_1 = \frac{(b_2 + c + d_{23})^2}{16}, \quad c_2 = \frac{(b_2 - c - d_{23})^2}{16}, \quad c_3 = \frac{d_{12} + d_{13}}{4},$$

$$(2.21c) \quad c_4 = \frac{(c + d_{12} - d_{23} - d_{13})^2}{16}, \quad c_5 = \frac{(c - d_{12} - d_{23} + d_{13})^2}{16}.$$

We can also obtain the discrete Painlevé equations from another operation on the parameter space as follows [12]. The action of T_1 on the parameter space:

$$T_1 : (b_0, b_1, d_{12}, d_{13}) \mapsto (b_0 - 1, b_1 + 1, -d_{12}, -d_{13}),$$

is not translational, but when the parameters take the special values $d_{12} = d_{13} = 0$, it becomes translational motion on the parameter sub-space $\{b_0, b_1, b_2, c, d_{23}\}$: $T_1 : (b_0, b_1) \mapsto (b_0 - 1, b_1 + 1)$. Under the specialization of the parameters, the action of T_1 gives the discrete Painlevé equation (1.4b) with the following correspondence:

$$(2.22a) \quad X = 2f, \quad \overline{X} = T_1(2f), \quad \underline{X} = T_1^{-1}(2f), \quad t = \frac{2b_1 + b_2 - 2}{2},$$

$$(2.22b) \quad c_1 = \frac{(b_2 + c + d_{23})^2}{4}, \quad c_2 = \frac{(b_2 - c - d_{23})^2}{4}, \quad c_3 = \frac{(c - d_{23})^2}{4}.$$

3. PROOF OF THEOREM 1.1

In this section, we give the proof of Theorem 1.1 via the reduction from the system of PΔEs (1.2) to the system of PΔEs (2.16).

Lemma 3.1 holds.

Lemma 3.1. *By imposing the (1, 1, 1)-periodic condition: $u(\mathbf{l} + \epsilon_1 + \epsilon_2 + \epsilon_3) = u(\mathbf{l})$ for $\mathbf{l} \in \Omega$, the system (1.2) can be reduced to the following system of PΔEs:*

$$(3.1) \quad \frac{u_{\bar{i}}}{u_{\underline{i}}} = \frac{(\alpha_{ij} - c + (-1)^{l_i+l_j} \delta_i)u_{\bar{j}} - (\alpha_{ij} - (-1)^{l_j+l_k} \delta_j + (-1)^{l_i+l_k} \delta_k)u_{\underline{j}}}{(\alpha_{ij} + (-1)^{l_j+l_k} \delta_j - (-1)^{l_i+l_k} \delta_k)u_{\bar{j}} - (\alpha_{ij} + c - (-1)^{l_i+l_j} \delta_i)u_{\underline{j}}},$$

where $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$, $u = u(\mathbf{l})$ and $\mathbf{l} = \sum_{i=1}^3 l_i \epsilon_i \in \mathbb{Z}^3 / (\epsilon_1 + \epsilon_2 + \epsilon_3)$.

Proof. Applying the (1, 1, 1)-periodic condition to the system (1.2), we obtain Equation (3.1) with $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$ from Equation (1.2) with $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$, respectively. Therefore, we have completed the proof. □

Remark 3.2.

- (i) The number of essential equations in the system (3.1) is two.
- (ii) By the (1, 1, 1)-reduction, each cuboctahedron is reduced to a hexagram (see Figure 3.1), which causes the reduction from the face-centred cubic lattice Ω to the triangle lattice $\mathbb{Z}^3 / (\epsilon_1 + \epsilon_2 + \epsilon_3)$.

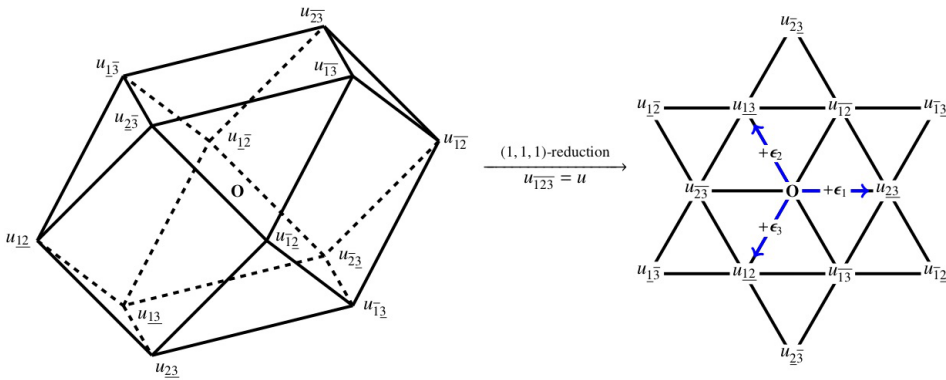


FIGURE 3.1. The (1, 1, 1)-reduction of the cuboctahedron

Lemma 3.3. *The reduced system (3.1) is equivalent to equations in the system (2.16).*

Proof. The statement follows from the following correspondences:

$$b_{l_1, l_2}^{(1)} = \alpha_{12}, \quad b_{l_2, l_3}^{(2)} = \alpha_{23}, \quad b_{l_1, l_3}^{(0)} = \alpha_{31}, \quad d_{12} = \delta_1, \quad d_{23} = \delta_2, \quad d_{13} = \delta_3,$$

$$u_{l_1, l_2, l_3} = u(l_1 \epsilon_1 + l_2 \epsilon_2 + l_3 \epsilon_3).$$

□

Remark 3.4. Lemma 3.3 means that the reduced system (3.1) can be obtained from the theory of the τ -function associated with $A_2^{(1)*}$ -type discrete Painlevé equations.

We are now ready to prove Theorem 1.1. The $(1, 1, 1)$ -periodic reduction from the system (1.2) to the system (3.1) given in Lemma 3.1, the relation between the system (3.1) and the system (2.16) given in Lemma 3.3, and that between the system (2.16) and the $A_2^{(1)*}$ -type discrete Painlevé equations (1.4) given in §2.2 and §2.3 collectively give the proof of Theorem 1.1.

4. CONCLUDING REMARKS

In this paper, we considered a reduction of a system of PΔEs, which is unusual in the sense that it has the CACO property but not the widely studied CAC property. We showed how the system (1.2) can be reduced to the $A_2^{(1)*}$ -type discrete Painlevé equations (1.4) using the affine Weyl group associated with the discrete Painlevé equations.

In a forthcoming paper (N. Joshi and N. Nakazono), we will show how another system of PΔEs, which also has the CACO property, can be reduced to the $A_2^{(1)}$ -type discrete Painlevé equations (see Table 1.1 for the distinction between $A_2^{(1)}$ and $A_2^{(1)*}$).

APPENDIX A. CONSISTENCY AROUND A CUBOCTAHEDRON PROPERTY

In this appendix, we recall the definition of *consistency around a cuboctahedron*. To define it, we also introduce an additional important property called *consistency around an octahedron*. We refer the reader to [10] for detailed information about these properties.

A.1. Consistency around an octahedron property. In this subsection, we give a definition of a consistency around an octahedron.

Let $u_i, i = 1, \dots, 6$, be variables and consider the octahedron shown in Figure A.1. The planes that pass through the vertices $\{u_4, u_2, u_1, u_5\}$, $\{u_2, u_6, u_5, u_3\}$ and $\{u_6, u_4, u_3, u_1\}$ give 3 quadrilaterals that lie in the interior of the octahedron and we assign the quad-equations $Q_i, i = 1, 2, 3$, to the quadrilaterals as the following:

$$(A.1) \quad Q_1(u_4, u_2, u_1, u_5) = 0, \quad Q_2(u_2, u_6, u_5, u_3) = 0, \quad Q_3(u_6, u_4, u_3, u_1) = 0.$$

The consistency around an octahedron property is defined by the following.

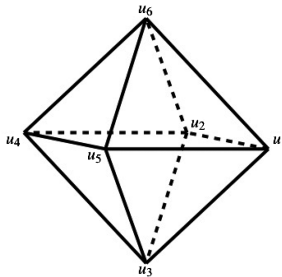


FIGURE A.1. An octahedron labelled with vertices $u_i, i = 1, \dots, 6$

Definition A.1 (CAO property [10]). The octahedron with quad-equations $\{Q_1, Q_2, Q_3\}$ is said to have a *consistency around an octahedron (CAO) property* if each quad-equation can be obtained from the other two equations. An octahedron is said to be a *CAO octahedron* if it has the CAO property.

A.2. Consistency around a cuboctahedron property. In this subsection, we give a definition of consistency around a cuboctahedron.

We consider the cuboctahedron centered around the origin whose twelve vertices are given by $V = \{\pm\epsilon_i \pm \epsilon_j \mid i, j \in \mathbb{Z}, 1 \leq i < j \leq 3\}$, where $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ form the standard basis of \mathbb{R}^3 . We assign the variables $u(\mathbf{l})$ to the vertices $\mathbf{l} \in V$ and impose the following relations:

$$(A.2a) \quad Q_1(u_5, u_1, v_5, v_4) = 0, \quad Q_2(v_2, v_1, u_2, u_4) = 0, \quad Q_3(u_3, u_5, v_3, v_2) = 0,$$

$$(A.2b) \quad Q_4(v_6, v_5, u_6, u_2) = 0, \quad Q_5(u_1, u_3, v_1, v_6) = 0, \quad Q_6(v_4, v_3, u_4, u_6) = 0,$$

$$(A.2c) \quad Q_7(u_4, u_2, u_1, u_5) = 0, \quad Q_8(u_2, u_6, u_5, u_3) = 0, \quad Q_9(u_6, u_4, u_3, u_1) = 0,$$

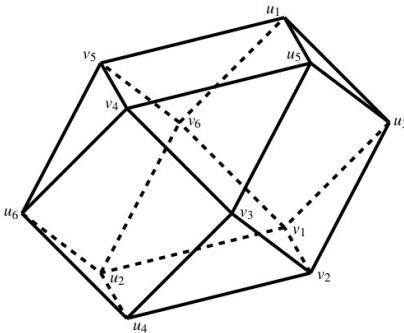
where $Q_i, i = 1, \dots, 9$, are quad-equations and

$$(A.3a) \quad u_1 = u(\epsilon_2 + \epsilon_3), \quad u_2 = u(-\epsilon_1 - \epsilon_3), \quad u_3 = u(\epsilon_1 + \epsilon_2), \quad u_4 = u(-\epsilon_2 - \epsilon_3),$$

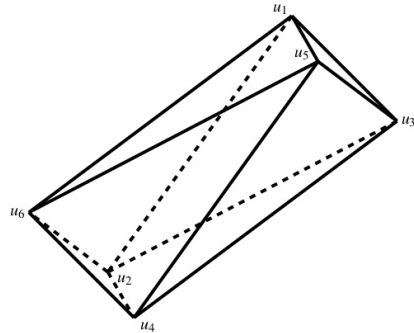
$$(A.3b) \quad u_5 = u(\epsilon_1 + \epsilon_3), \quad u_6 = u(-\epsilon_1 - \epsilon_2), \quad v_1 = u(\epsilon_2 - \epsilon_3), \quad v_2 = u(\epsilon_1 - \epsilon_3),$$

$$(A.3c) \quad v_3 = u(\epsilon_1 - \epsilon_2), \quad v_4 = u(-\epsilon_2 + \epsilon_3), \quad v_5 = u(-\epsilon_1 + \epsilon_3), \quad v_6 = u(-\epsilon_1 + \epsilon_2).$$

Note that quad-equations $Q_i, i = 1, \dots, 6$, are assigned to the faces of the cuboctahedron (see Figure A.2a). Moreover, $u_i, i = 1, \dots, 6$, collectively form the vertices of an octahedron and quad-equations $Q_i, i = 7, 8, 9$, are assigned to the quadrilaterals that appear as sections passing through four vertices of the octahedron (see Figure A.2b).



(a) A cuboctahedron labelled with vertices u_i and $v_j, i, j = 1, \dots, 6$



(a) An octahedron labelled with vertices $u_i, i = 1, \dots, 6$

FIGURE A.2. A cuboctahedron and an interior octahedron

We are now in a position to give the following definitions.

Definition A.2 (CACO property [10]). The cuboctahedron with quad-equations $\{Q_1, \dots, Q_9\}$ is said to have a *consistency around a cuboctahedron (CACO) property* if the following properties hold.

- (i) The octahedron with quad-equations $\{Q_7, Q_8, Q_9\}$ has the CAO property.
- (ii) Assume that u_1, \dots, u_6 are given so as to satisfy $Q_i = 0$, $i = 7, 8, 9$, and, in addition, v_k is given, for some $k \in \{1, \dots, 6\}$. Then, quad-equations Q_i , $i = 1 \dots, 6$, determine the variables v_j , $j \in \{1, \dots, 6\} \setminus \{k\}$, uniquely.

A cuboctahedron is said to be a *CACO cuboctahedron* if it has the CACO property.

Definition A.3 (Square property [10]). The CACO cuboctahedron with quad-equations $\{Q_1, \dots, Q_9\}$ is said to have a *square property* if there exist polynomials $K_i = K_i(x, y, z, w)$, $i = 1, 2, 3$, where $\deg_x K_i = \deg_w K_i = 1$ and $1 \leq \deg_y K_i, \deg_z K_i$, satisfying

$$(A.4) \quad K_1(v_1, u_1, u_4, v_4) = 0, \quad K_2(v_2, u_2, u_5, v_5) = 0, \quad K_3(v_3, u_3, u_6, v_6) = 0.$$

Then, each equation $K_i = 0$ is called a *square equation*.

A.3. CACO property of PΔEs. We now explain how to associate quad-equations with PΔEs in three-dimensional space by using the system of PΔEs (1.2) as an example. This requires us to consider overlapping cuboctahedra that lead to two-dimensional tessellations consisting of quadrilaterals. For each given cuboctahedron, there are twelve overlapping cuboctahedra.

The twelve overlapping cuboctahedra around a given one provide six directions of tiling by quadrilaterals. For later convenience, we label directions by $\epsilon_i \pm \epsilon_j$, $1 \leq i < j \leq 3$. Vertices labelled in this way form the set Ω given by (1.1). Such vertices are interpreted as being iterated on each successive cuboctahedron. We here consider the system of PΔEs (1.2). For simplicity, we abbreviate each respective equation in Equations (1.2) as

$$P_1(u_{\overline{13}}, u_{\overline{23}}, u_{\underline{13}}, u_{\underline{23}}) = 0, \quad P_2(u_{\overline{12}}, u_{\overline{13}}, u_{\overline{12}}, u_{\overline{13}}) = 0, \quad P_3(u_{\overline{23}}, u_{\overline{12}}, u_{\overline{23}}, u_{\overline{12}}) = 0,$$

$$P_4(u_{\underline{23}}, u_{\underline{13}}, u_{\underline{23}}, u_{\underline{13}}) = 0, \quad P_5(u_{\underline{13}}, u_{\underline{12}}, u_{\underline{13}}, u_{\underline{12}}) = 0, \quad P_6(u_{\underline{12}}, u_{\underline{23}}, u_{\underline{12}}, u_{\underline{23}}) = 0.$$

Conversely, given $\mathbf{l} \in \Omega$, we obtain the cuboctahedron centered around \mathbf{l} . We refer to its quad-equations as before by $\{Q_1(\mathbf{l}), \dots, Q_9(\mathbf{l})\}$. Moreover, the overlapped region gives an octahedron centred around $\mathbf{l} + \epsilon_3$, and we label its quad-equations by $\{\hat{Q}_1(\mathbf{l}), \hat{Q}_2(\mathbf{l}), \hat{Q}_3(\mathbf{l})\}$.

Each such quad-equation is identified with the 6 partial difference equations given in Equations (1.2) in the following way. For Q_1, \dots, Q_9 , we use

$$\begin{aligned} Q_1(\mathbf{l}) &= P_1(u_{\overline{13}}, u_{\overline{23}}, u_{\overline{1\overline{3}}}, u_{\overline{2\overline{3}}}) = 0, & Q_2(\mathbf{l}) &= P_1(u_{\overline{1\overline{3}}}, u_{\overline{2\overline{3}}}, u_{\overline{13}}, u_{\overline{23}}) = 0, \\ Q_3(\mathbf{l}) &= P_2(u_{\overline{1\overline{2}}, u_{\overline{1\overline{3}}}, u_{\overline{1\overline{2}}}, u_{\overline{1\overline{3}}}) = 0, & Q_4(\mathbf{l}) &= P_2(u_{\overline{1\overline{2}}}, u_{\overline{1\overline{3}}}, u_{\overline{12}}, u_{\overline{13}}) = 0, \\ Q_5(\mathbf{l}) &= P_3(u_{\overline{2\overline{3}}, u_{\overline{1\overline{2}}}, u_{\overline{2\overline{3}}}, u_{\overline{1\overline{2}}}) = 0, & Q_6(\mathbf{l}) &= P_3(u_{\overline{2\overline{3}}}, u_{\overline{1\overline{2}}}, u_{\overline{23}}, u_{\overline{12}}) = 0, \\ Q_7(\mathbf{l}) &= P_4(u_{\overline{23}}, u_{\overline{13}}, u_{\overline{2\overline{3}}}, u_{\overline{1\overline{3}}}) = 0, & Q_8(\mathbf{l}) &= P_5(u_{\overline{13}}, u_{\overline{12}}, u_{\overline{1\overline{3}}}, u_{\overline{1\overline{2}}}) = 0, \\ Q_9(\mathbf{l}) &= P_6(u_{\overline{12}}, u_{\overline{23}}, u_{\overline{1\overline{2}}}, u_{\overline{2\overline{3}}}) = 0, \end{aligned}$$

and for $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3$, we use

$$\begin{aligned} \hat{Q}_1(\mathbf{l}) &= P_1(u_{\overline{13}}, u_{\overline{23}}, u_{\overline{1\overline{3}}}, u_{\overline{2\overline{3}}}) = 0, & \hat{Q}_2(\mathbf{l}) &= P_2(u_{\overline{2\overline{3}}}, u_{\overline{3\overline{3}}}, u_{\overline{2\overline{3}}}, u) = 0, \\ \hat{Q}_3(\mathbf{l}) &= P_3(u_{\overline{3\overline{3}}}, u_{\overline{1\overline{3}}}, u, u_{\overline{1\overline{3}}}) = 0. \end{aligned}$$

Then, Proposition A.4 holds.

Proposition A.4 ([10]). *The system of PΔEs (1.2) has the CACO and square properties, that is, the following statements hold.*

- (i) *The cuboctahedra with quad-equations $\{Q_i(\mathbf{l})\}$ have the CACO and square properties.*
- (ii) *The square equations are consistent with the PΔEs (1.2).*
- (iii) *The octahedra with quad-equations $\{\hat{Q}_i(\mathbf{l})\}$ have the CAO property.*

APPENDIX B. EXTENDED AFFINE WEYL GROUP OF TYPE $E_6^{(1)}$ AND τ -VARIABLES

In this appendix, we review the action of the extended affine Weyl group of type $E_6^{(1)}$ given in [23], which is the symmetry group of $A_2^{(1)*}$ -type discrete Painlevé equations.

Let $a_i, i = 0, \dots, 6$, be parameters satisfying the condition (2.1) and $\tau_j^{(i)}, i = 1, 2, 3, j = 0, 1, 2, 3$, be variables. The actions of transformations $s_i, i = 0, \dots, 6$, and $\iota_j, j = 1, 2, 3$, on the parameters are given by

$$\begin{aligned} s_0 &: (a_0, a_6) \mapsto (-a_0, a_6 + a_0), & s_1 &: (a_1, a_2) \mapsto (-a_1, a_2 + a_1), \\ s_2 &: (a_1, a_2, a_3) \mapsto (a_1 + a_2, -a_2, a_3 + a_2), \\ s_3 &: (a_2, a_3, a_4, a_6) \mapsto (a_2 + a_3, -a_3, a_4 + a_3, a_6 + a_3), \\ s_4 &: (a_3, a_4, a_5) \mapsto (a_3 + a_4, -a_4, a_5 + a_4), & s_5 &: (a_4, a_5) \mapsto (a_4 + a_5, -a_5), \\ s_6 &: (a_0, a_3, a_6) \mapsto (a_0 + a_6, a_3 + a_6, -a_6), \\ \iota_1 a_{\{0,5,4,6\}} &\mapsto a_{\{5,0,4,6\}}, & \iota_2 a_{\{0,1,2,6\}} &\mapsto a_{\{1,0,6,2\}}, & \iota_3 a_{\{1,5,2,4\}} &\mapsto a_{\{5,1,4,2\}}, \end{aligned}$$

while those on the τ -variables $\tau_j^{(i)}$, $i = 1, 2, 3$, $j = 0, 1, 2, 3$, are given by

$$\begin{aligned}
 s_0 &: (\tau_2^{(3)}, \tau_3^{(3)}) \mapsto (\tau_3^{(3)}, \tau_2^{(3)}), & s_1 &: (\tau_2^{(1)}, \tau_3^{(1)}) \mapsto (\tau_3^{(1)}, \tau_2^{(1)}), \\
 s_2 &: (\tau_1^{(1)}, \tau_2^{(1)}) \mapsto (\tau_2^{(1)}, \tau_1^{(1)}), \\
 &: (\tau_0^{(2)}, \tau_0^{(3)}) \mapsto \left(\frac{(a_2 + a_3)\tau_1^{(1)}\tau_0^{(2)} - a_2\tau_1^{(2)}\tau_0^{(1)}}{a_3\tau_2^{(1)}}, \frac{(a_2 + a_3)\tau_1^{(1)}\tau_0^{(3)} - a_2\tau_1^{(3)}\tau_0^{(1)}}{a_3\tau_2^{(1)}} \right), \\
 s_3 &: (\tau_1^{(1)}, \tau_1^{(2)}, \tau_1^{(3)}, \tau_0^{(1)}, \tau_0^{(2)}, \tau_0^{(3)}) \mapsto (\tau_0^{(1)}, \tau_0^{(2)}, \tau_0^{(3)}, \tau_1^{(1)}, \tau_1^{(2)}, \tau_1^{(3)}), \\
 s_4 &: (\tau_1^{(2)}, \tau_2^{(2)}) \mapsto (\tau_2^{(2)}, \tau_1^{(2)}), \\
 &: (\tau_0^{(1)}, \tau_0^{(3)}) \mapsto \left(\frac{(a_3 + a_4)\tau_1^{(2)}\tau_0^{(1)} - a_4\tau_1^{(1)}\tau_0^{(2)}}{a_3\tau_2^{(2)}}, \frac{(a_3 + a_4)\tau_1^{(2)}\tau_0^{(3)} - a_4\tau_1^{(3)}\tau_0^{(2)}}{a_3\tau_2^{(2)}} \right), \\
 s_5 &: (\tau_2^{(2)}, \tau_3^{(2)}) \mapsto (\tau_3^{(2)}, \tau_2^{(2)}), \\
 s_6 &: (\tau_1^{(3)}, \tau_2^{(3)}) \mapsto (\tau_2^{(3)}, \tau_1^{(3)}), \\
 &: (\tau_0^{(1)}, \tau_0^{(2)}) \mapsto \left(\frac{(a_3 + a_6)\tau_1^{(3)}\tau_0^{(1)} - a_6\tau_1^{(1)}\tau_0^{(3)}}{a_3\tau_2^{(3)}}, \frac{(a_3 + a_6)\tau_1^{(3)}\tau_0^{(2)} - a_6\tau_1^{(2)}\tau_0^{(3)}}{a_3\tau_2^{(3)}} \right), \\
 \iota_1 &: (\tau_j^{(2)}, \tau_j^{(3)}) \mapsto (\tau_j^{(3)}, \tau_j^{(2)}), & \iota_2 &: (\tau_j^{(1)}, \tau_j^{(3)}) \mapsto (\tau_j^{(3)}, \tau_j^{(1)}), \\
 \iota_3 &: (\tau_j^{(1)}, \tau_j^{(2)}) \mapsto (\tau_j^{(2)}, \tau_j^{(1)}), & j &= 0, 1, 2, 3.
 \end{aligned}$$

Remark B.1.

- (i) Each transformation here defined is an isomorphism from the field of rational functions $K(\{\tau_j^{(i)}\})$, where $K = \mathbb{C}(\{a_i\})$, to itself.
- (ii) We follow the convention of Remark 2.1 for the above equations. That is, each transformation acts as an identity on parameters or variables not appearing in its definition.

The transformations collectively form the extended affine Weyl group of type $E_6^{(1)}$, denoted by (2.2). Indeed, the following fundamental relations hold:

(B.1a)

$$(s_i s_j)^{A_{ij}} = 1, \quad \iota_1^2 = \iota_2^2 = \iota_3^2 = 1, \quad \iota_1 \iota_2 = \iota_2 \iota_3 = \iota_3 \iota_1, \quad \iota_2 \iota_1 = \iota_3 \iota_2 = \iota_1 \iota_3,$$

(B.1b)

$$\iota_1 s_{\{0,5,4,6\}} = s_{\{5,0,4,6\}} \iota_1, \quad \iota_2 s_{\{0,1,2,6\}} = s_{\{1,0,6,2\}} \iota_2, \quad \iota_3 s_{\{1,5,2,4\}} = s_{\{5,1,4,2\}} \iota_3,$$

where $i, j \in \{0, 1, \dots, 6\}$ and

$$(B.2) \quad (A_{ij})_{i,j=0}^6 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 3 & 0 & 3 \\ 0 & 0 & 0 & 3 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 2 \end{pmatrix}.$$

Remark B.2. The correspondence between the notations in this paper and those in [23] is given by $\tau_j^{(i)} \rightarrow \tau_j^i$ and $\tau_0^{(i)} \rightarrow s_3(\tau_1^i)$, where $i, j = 1, 2, 3$.

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SCHOOL OF MATHEMATICS AND STATISTICS F07, THE UNIVERSITY OF SYDNEY, NEW SOUTH WALES 2006, AUSTRALIA

Email address: nalini.joshi@sydney.edu.au

INSTITUTE OF ENGINEERING, TOKYO UNIVERSITY OF AGRICULTURE AND TECHNOLOGY, 2-24-16 NAKACHO KOGANEI, TOKYO 184-8588, JAPAN

Email address: nakazono@go.tuat.ac.jp