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INTEGRABLE NONLOCAL NONLINEAR SCHRÖDINGER EQUATIONS ASSOCIATED WITH so$(3, \mathbb{R})$

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ABSTRACT. We construct integrable PT-symmetric nonlocal reductions for an integrable hierarchy associated with the special orthogonal Lie algebra so$(3, \mathbb{R})$. The resulting typical nonlocal integrable equations are integrable PT-symmetric nonlocal reverse-space, reverse-time and reverse-spacetime nonlinear Schrödinger equations associated with so$(3, \mathbb{R})$.

1. INTRODUCTION

The nonlinear Schrödinger (NLS) equation is an integrable model, frequently used in nonlinear optics and soft-condensed matter physics, and it appears as one of universal equations that describe the evolution of slowly varying packets of quasi-monochromatic waves in weakly nonlinear dispersive media [7]. On one hand, vector generalizations are introduced and studied in soliton theory, and generalizations with an external potential are used in modeling Bose-Einstein condensates as well as many other physical fields [1, 25]. On the other hand, nonlocal integrable NLS equations have been recently explored, which possess infinitely many symmetries and conservation laws, indeed [3, 4]. The resulting model equations can relate function values at point x in space to its function values at a mirror-reflection space point -x [4]. This has opened new avenues for studying NLS type integrable equations [20]. One popular example of nonlocal dynamics is pantograph modeling, which has long history in pantograph mechanics and pantograph transport [5].

It is known that integrable equations are a class of nonlinear partial differential equations, which are associated with matrix Lie algebras [1]. Lax pairs [10] pay a crucial role in the formulation of integrable equations and their solutions [1]. The trace identity [29] and the variational identity [17] can be used to establish Hamiltonian structures which exhibit the Liouville integrability of the underlying equations. Among the well-known integrable equations associated with simple Lie algebras are the KdV equation, the AKNS system of NLS equations, and the derivative NLS equation [1, 9]. More generally, there are integrable couplings associated with non-semisimple Lie algebras [22, 23], which bring hereditary recursion operators in block matrix form [13, 14].

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In our construction, we will use the special orthogonal Lie algebra $\mathfrak{g} = \mathrm{so}(3,\mathbb{R})$, presented by all 3×3 trace-free, skew-symmetric real matrices, with a basis:

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.1}$$

whose structure equations read

$$[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2. \tag{1.2}$$

Obviously, the derived algebra $[\mathfrak{g},\mathfrak{g}] = [\mathrm{so}(3,\mathbb{R}), \mathrm{so}(3,\mathbb{R})]$ is $\mathfrak{g} = \mathrm{so}(3,\mathbb{R})$ itself. The algebra is one of the only two three-dimensional real Lie algebras with this property. The other one is the special linear algebra $\mathrm{sl}(2,\mathbb{R})$, which lays a foundation for studying many integrable equations [2]. The corresponding matrix loop algebra that we will use is

$$\tilde{\mathfrak{g}} = \widetilde{\mathrm{so}}(3,\mathbb{R}) = \{M \in \mathrm{so}(3,\mathbb{R}) \,|\, \text{entries of } M \text{ - Laurent series in } \lambda\}, \tag{1.3}$$

where λ is a spectral parameter. The loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ contains matrices of the form $\lambda^{k_1} e_1 + \lambda^{k_2} e_2 + \lambda^{k_3} e_3$ with arbitrary integers $k_i,\ 1 \le i \le 3$. This matrix loop algebra has also been recently used to construct integrable equations [15, 16, 28].

In this paper, we would first like to revisit a hierarchy of integrable equations associated with $\mathrm{so}(3,\mathbb{R})$ [15]. We will then consider three classes of nonlocal PT-symmetric integrable reductions for the adopted spectral matrix, to generate three reduced hierarchies of scalar integrable equations. Three typical such reduced nonlocal PT-symmetric integrable equations are the reverse-space NLS equation

$$ip_t = p^*_{xx}(-x,t) - \frac{1}{2}p^2 p^*(-x,t) - \frac{1}{2}(p^*(-x,t))^3,$$

where p^* denotes the complex conjugate of p, the reverse-time NLS equation

$$ip_t = p_{xx}(x,-t) - \frac{1}{2}p^2 p(x,-t) - \frac{1}{2}(p(x,-t))^3,$$

and the reverse-spacetime NLS equation

$$ip_t = -p_{xx}(-x,-t) + \frac{1}{2}p^2 p(-x,-t) + \frac{1}{2}(p(-x,-t))^3,$$

which are all associated with so(3,$\mathbb{R}$).

2. The model integrable hierarchy revisited

2.1. **Integrable hierarchy.** We would like to revisit an integrable hierarchy associated with the matrix loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ [15]. We begin with a spatial matrix spectral problem

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \tag{2.1}$$

with

$$U = U(u,\lambda) = \lambda e_1 + p e_2 + q e_3 = \begin{bmatrix} 0 & -q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix}, \tag{2.2}$$

where i is the unit imaginary number, λ is a spectral parameter, $u = (p,q)^T$ is a potential and $\phi = (\phi_1, \phi_2, \phi_3)^T$ is a column eigenfunction. The spectral matrix for an integrable hierarchy in [15] is only U, but here we have adopted a spectral matrix iU, involving a constant factor i. This will bring us convenience in determining integrable nonlocal reductions.

As usual [11], we solve the stationary zero curvature equation

$$W_x = i[U, W] \tag{2.3}$$

for $W = W(u, \lambda) \in \widetilde{\mathrm{so}}(3, \mathbb{R})$. This is equivalent to

$$a_x = i(pc - qb),\ b_x = i(-\lambda c + qa),\ c_x = i(\lambda b - pa), \tag{2.4}$$

as long as W is given by

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{bmatrix} = \sum_{m\geq 0} W_{0,m}\lambda^{-m}, \tag{2.5}$$

with

$$W_{0,m} = a_m e_1 + b_m e_2 + c_m e_3 = \begin{bmatrix} 0 & -c_m & -a_m \\ c_m & 0 & -b_m \\ a_m & b_m & 0 \end{bmatrix},\ m \geq 0. \tag{2.6}$$

Upon taking the initial values

$$a_0 = -1,\ b_0 = c_0 = 0, \tag{2.7}$$

the system (2.4), being equivalent to

$$b_{m+1} = -ic_{m,x} + pa_m,\ c_{m+1} = ib_{m,x} + qa_m,\ a_{m+1,x} = i(pc_{m+1} - qb_{m+1}),\ m \geq 0, \tag{2.8}$$

defines the sequence of $\{a_m, b_m, c_m |\, m \geq 1\}$ uniquely, under the integration conditions

$$a_m|_{u=0} = b_m|_{u=0} = c_m|_{u=0} = 0,\ m \geq 1. \tag{2.9}$$

The first few sets are as follows:

$$\begin{aligned}
&b_1 = -p,\ c_1 = -q,\ a_1 = 0;\\
&b_2 = iq_x,\ c_2 = -ip_x,\ a_2 = \tfrac{1}{2}(p^2 + q^2);\\
&b_3 = -p_{xx} + \tfrac{1}{2}p^3 + \tfrac{1}{2}pq^2,\\
&c_3 = -q_{xx} + \tfrac{1}{2}p^2 q + \tfrac{1}{2}q^3,\\
&a_3 = i(p_x q - pq_x);\\
&b_4 = i(q_{xxx} - \tfrac{3}{2}p^2 q_x - \tfrac{3}{2}q^2 q_x),\\
&c_4 = i(-p_{xxx} + \tfrac{3}{2}p^2 p_x + \tfrac{3}{2}p_x q^2),\\
&a_4 = pp_{xx} + qq_{xx} - \tfrac{1}{2}p_x^2 - \tfrac{1}{2}q_x^2 - \tfrac{3}{8}(p^2 + q^2)^2.
\end{aligned}$$

Let us then set

$$V^{[m]} = (\lambda^m W)_+ = \sum_{l=0}^{m} W_{0,l}\lambda^{m-l},\ m \geq 0, \tag{2.10}$$

to introduce the temporal matrix spectral problems:

$$-i\phi_t = V^{[m]}\phi = V^{[m]}(u, \lambda)\phi,\ m \geq 0. \tag{2.11}$$

Finally, the compatibility conditions of (2.1) and (2.11), i.e., the zero curvature equations

$$U_{t_m} - V_x^{[m]} + i[U, V^{[m]}] = 0,\ m \ge 0, \tag{2.12}$$

give rise to a hierarchy of integrable equations:

$$u_{t_m} = K_m = i \begin{bmatrix} -c_{m+1} \\ b_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} iq \\ -ip \end{bmatrix},\ m \ge 0, \tag{2.13}$$

where the operator Φ is determined by (2.8)

$$\Phi = i \begin{bmatrix} q\partial^{-1}p & -\partial + q\partial^{-1}q \\ \partial - p\partial^{-1}p & -p\partial^{-1}q \end{bmatrix},\ \partial = \frac{\partial}{\partial x}. \tag{2.14}$$

2.2. **The Liouville intergrability.** Based on the trace identity [29] for our spectral matrix iU:

$$\frac{\delta}{\delta u} \int \mathrm{tr}(W \frac{\partial U}{\partial \lambda})\, dx = \lambda^{-\gamma} \frac{\lambda}{\partial \lambda} \lambda^{\gamma}\ \mathrm{tr}(W \frac{\partial U}{\partial u}), \tag{2.15}$$

where the constant γ is determined by

$$\gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\langle W, W \rangle|, \tag{2.16}$$

we can construct Hamiltonian structures which exhibit the Liouville integrability of the hierarchy (2.13). Obviously, the corresponding trace identity (2.15) reads

$$\frac{\delta}{\delta u} \int a\, dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{bmatrix} b \\ c \end{bmatrix}.$$

An application of it leads to the following Hamiltonian structures for the hierarchy (2.13):

$$u_{t_m} = K_m = i \begin{bmatrix} -c_{m+1} \\ b_{m+1} \end{bmatrix} = J \frac{\delta \mathcal{H}_m}{\delta u},\ m \ge 0, \tag{2.17}$$

with the Hamiltonian operator and the Hamiltonian functionals

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \mathcal{H}_m = \int (-\frac{ia_{m+2}}{m+1})\, dx,\ m \ge 0. \tag{2.18}$$

These yield infinitely many conservation laws of each system in the hierarchy (2.13), which can often be generated through symbolic computation by computer algebra systems (see, e.g., [8]).

To exhibit the Liouville integrability, we need to show that the operator Φ defined by (2.14) is a common hereditary recursion operator for the hierarchy (2.13).

First, a direct but lengthy computation can show that the operator Φ is hereditary (see [6] for definition), i.e., it satisfies

$$\Phi'(u)[\Phi K]S - \Phi\Phi'(u)[K]S = \Phi'(u)[\Phi S]K - \Phi\Phi'(u)[S]K, \tag{2.19}$$

and Φ is a recursion operator for $u_{t_0} = K_0$:

$$L_{K_0}\Phi = 0,\ K_0 = i \begin{bmatrix} q \\ -p \end{bmatrix},\ (L_K \Phi)S = \Phi[K, S] - [K, \Phi S], \tag{2.20}$$

where K and S are arbitrary vector fields and $[\cdot,\cdot]$ is the Lie bracket of vector fields. Another direct result is that J and

$$M = \Phi J = i \begin{bmatrix} -\partial + q\partial^{-1}q & -q\partial^{-1}p \\ -p\partial^{-1}q & -\partial + p\partial^{-1}p \end{bmatrix} \tag{2.21}$$

constitute a Hamiltonian pair (see [24] for details). The hereditary property (2.19) is equivalent to

$$L_{\Phi K}\Phi = \Phi L_K \Phi, \tag{2.22}$$

where K is an arbitrary vector field, and thus

$$L_{K_m}\Phi = L_{\Phi K_{m-1}}\Phi = \Phi L_{K_{m-1}}\Phi = 0,\ m \geq 1, \tag{2.23}$$

where the K_m's are given by (2.13). This implies that the operator Φ defined by (2.14) is a common hereditary recursion operator for the hierarchy (2.13).

Now, the hierarchy (2.13) is bi-Hamiltonian (see, e.g., [24, 26] for details):

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u},\ m \geq 1, \tag{2.24}$$

where J, M and $\mathcal{H}_m$ are defined by (2.18) and (2.21), and so, every member in the hierarchy is Liouville integrable, i.e., it possesses infinitely many commuting symmetries and conservation laws. In particular, we have the Abelian symmetry algebra:

$$[K_k, K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0,\ k, l \geq 0, \tag{2.25}$$

and the Abelian algebras of conserved functionals:

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J = \int (\frac{\delta \mathcal{H}_k}{\delta u})^T J \frac{\delta \mathcal{H}_l}{\delta u}\, dx = 0,\ k, l \geq 0, \tag{2.26}$$

and

$$\{\mathcal{H}_k, \mathcal{H}_l\}_M = \int (\frac{\delta \mathcal{H}_k}{\delta u})^T M \frac{\delta \mathcal{H}_l}{\delta u}\, dx = 0,\ k, l \geq 0. \tag{2.27}$$

The first nonlinear integrable system of equations in the hierarchy (2.13) is a system of NLS equations associated with so(3,ℝ):

$$p_{t_2} = i(q_{xx} - \frac{1}{2}p^2 q - \frac{1}{2}q^3),\ q_{t_2} = i(-p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2). \tag{2.28}$$

It possesses the following bi-Hamiltonian structure

$$u_{t_2} = K_2 = J\frac{\delta \mathcal{H}_2}{\delta u} = M\frac{\delta \mathcal{H}_1}{\delta u}, \tag{2.29}$$

where the Hamiltonian pair $\{J, M\}$ is defined by (2.18) and (2.21), and the Hamiltonian functionals, $\mathcal{H}_1$ and $\mathcal{H}_2$, are given by

$$\mathcal{H}_1 = -\frac{1}{2}\int (pq_x - p_x q)\, dx, \tag{2.30}$$

$$\mathcal{H}_2 = -\frac{i}{3}\int [pp_{xx} + qq_{xx} - \frac{1}{2}p_x^2 - \frac{1}{2}q_x^2 - \frac{3}{8}(p^2 + q^2)^2]\, dx. \tag{2.31}$$

3. Integrable nonlocal reductions

3.1. **Reverse-space reductions.** Let us first consider two specific reverse-space reductions for the spectral matrix:

$$U^\dagger(-x,t,-\lambda^*) = -CU(x,t,\lambda)C^{-1},\ C = \begin{bmatrix} 0 & 0 & \delta \\ 0 & 1 & 0 \\ \delta & 0 & 0 \end{bmatrix},\ \delta = \pm 1, \tag{3.1}$$

where $*$ and $\dagger$ stand for the complex conjugate and the Hermitian transpose, respectively. They lead to the potential reductions

$$p^*(-x,t) = -\delta q(x,t). \tag{3.2}$$

Under these potential reductions, one has the reduction property for W:

$$W^\dagger(-x,t,-\lambda^*) = CW(x,t,\lambda)C^{-1}, \tag{3.3}$$

since two matrices on both sides of the equation solve the stationary zero curvature equation (2.3) with the same initial values. This implies that

$$a^*(-x,t,-\lambda^*) = a(x,t,\lambda),\ b^*(-x,t,-\lambda^*) = \delta c(x,t,\lambda), \tag{3.4}$$

namely,

$$a_m^*(-x,t) = (-1)^m a_m(x,t),\ b_m^*(-x,t) = (-1)^m \delta c_m(x,t),\ m \geq 1. \tag{3.5}$$

Therefore, one obtains

$$(V^{[m]})^\dagger(-x,t,-\lambda^*) = (-1)^m CV^{[m]}(x,t,\lambda)C^{-1},\ m \geq 1, \tag{3.6}$$

and further

$$((U_t - V_x^{[2l]} + i[U,V^{[2l]}])(-x,t,-\lambda^*))^\dagger = -C(U_t - V_x^{[2l]} + i[U,V^{[2l]}])(x,t,\lambda)C^{-1},\ l \geq 1. \tag{3.7}$$

This tells that the potential reductions in (3.2) are compatible with the $2l$-th zero curvature equation of the integrable hierarchy (2.13). In this way, we obtain two reduced scalar integrable hierarchies associated with so(3,$\mathbb{R}$):

$$p_t = K_{2l,1}|_{q(x,t)=-\delta p^*(-x,t)},\ l \geq 1, \tag{3.8}$$

where $K_m = (K_{m,1}, K_{m,2})^T,\ m \geq 1$, are defined by (2.13). The infinitely many symmetries and conservation laws for the integrable hierarchy (2.13) are reduced to infinitely many ones for the above integrable hierarchies in (3.8).

With $\delta = 1$, the first nonlinear reduced scalar integrable equation is a nonlocal reverse-space PT-symmetric NLS equation associated with so(3,$\mathbb{R}$):

$$ip_t = p_{xx}^*(-x,t) - \frac{1}{2}p^2 p^*(-x,t) - \frac{1}{2}(p^*(-x,t))^3, \tag{3.9}$$

where p^* denotes the complex conjugate of p. Note that the two components of $K_m,\ m \geq 1$, have even and odd properties with respect to p and q. Actually, $K_{2l,1},\ l \geq 1$, are odd with respect to q and even with respect to p, and $K_{2l+1,1},\ l \geq 1$, are even with respect to q and odd with respect to p. Similarly, $K_{2l,2},\ l \geq 1$, are odd with respect to p and even with respect to q, and $K_{2l+1,2},\ l \geq 1$, are even with respect to p and odd with respect to q. Therefore, the first reduced scalar integrable equation with $\delta = -1$ in (3.8) has just a different sign from the nonlocal reverse-space NLS equation (3.9).

3.2. **Reverse-time reductions.** Secondly, let us consider two specific reverse-time reductions for the spectral matrix:

$$U^T(x,-t,-\lambda) = -CU(x,t,\lambda)C^{-1},\ C = \begin{bmatrix} 0 & 0 & \delta \\ 0 & 1 & 0 \\ \delta & 0 & 0 \end{bmatrix},\ \delta = \pm 1, \tag{3.10}$$

where T means taking the transpose of a matrix. They generate the potential reductions

$$p(x,-t) = -\delta q(x,t). \tag{3.11}$$

Similarly, under these potential reductions, we have the reduction property for W:

$$W^T(x,-t,-\lambda) = CW(x,t,\lambda)C^{-1}, \tag{3.12}$$

upon noting that two matrices on both sides of the equation solve the stationary zero curvature equation (2.3) with the same initial values. Thus, we have

$$a(x,-t,-\lambda) = a(x,t,\lambda),\ b(x,-t,-\lambda) = \delta c(x,t,\lambda), \tag{3.13}$$

namely,

$$a_m(x,-t) = (-1)^m a_m(x,t),\ b_m(x,-t) = (-1)^m \delta c_m(x,t),\ m \geq 1. \tag{3.14}$$

Then, we arrive at

$$(V^{[m]})^T(x,-t,-\lambda) = (-1)^m CV^{[m]}(x,t,\lambda)C^{-1},\ m \geq 1, \tag{3.15}$$

and further

$$((U_t - V_x^{[2l]} + i[U,V^{[2l]}])(-x,t,-\lambda))^T = C(U_t - V_x^{[2l]} + i[U,V^{[2l]}])(x,t,\lambda)C^{-1},\ l \geq 1. \tag{3.16}$$

This implies that the potential reductions in (3.11) are compatible with the $2l$-th zero curvature equation of the integrable hierarchy (2.13). Consequently, we obtain two reduced scalar integrable hierarchies associated with so(3,ℝ):

$$p_t = K_{2l,1}|_{q(x,t)=-\delta p(x,-t)},\ l \geq 1, \tag{3.17}$$

where $K_m = (K_{m,1}, K_{m,2})^T,\ m \geq 1$, are defined by (2.13). Infinitely many symmetries and conservation laws for the above hierarchies in (3.17) are obtained from the ones for the integrable hierarchy (2.13).

With $\delta = 1$, the first nonlinear reduced scalar integrable equation is a nonlocal reverse-time PT-symmetric NLS equation asscociated with so(3,ℝ):

$$ip_t = p_{xx}(x,-t) - \frac{1}{2}p^2 p(x,-t) - \frac{1}{2}(p(x,-t))^3. \tag{3.18}$$

Noting even and odd properties with respect to p and q in the two components of $K_m,\ m \geq 1$, we see that the reduced first scalar integrable equation with $\delta = -1$ in (3.17) has just a different sign from the nonlocal reverse-time NLS equation (3.18).

3.3. **Reverse-spacetime reductions.** Thirdly, let us consider two specific reverse-spacetime reductions for the spectral matrix:

$$U^T(-x,-t,\lambda)=CU(x,t,\lambda)C^{-1},\ C=\begin{bmatrix}0&0&\delta\\0&1&0\\\delta&0&0\end{bmatrix},\ \delta=\pm1, \tag{3.19}$$

where T means taking the transpose of a matrix again. They yield the potential reductions

$$p(-x,-t)=\delta q(x,t). \tag{3.20}$$

As before, under these potential reductions, W satisfies the following reduction property:

$$W^T(-x,-t,\lambda)=CW(x,t,\lambda)C^{-1}, \tag{3.21}$$

because two matrices solve the stationary zero curvature equation (2.3) with the same initial values. Thus, we have

$$a(-x,-t,\lambda)=a(x,t,\lambda),\ b(-x,-t,\lambda)=\delta c(x,t,\lambda), \tag{3.22}$$

namely,

$$a_m(-x,-t)=a_m(x,t),\ b_m(-x,-t)=\delta c_m(x,t),\ m\geq1. \tag{3.23}$$

Then, we obtain

$$(V^{[m]})^T(-x,-t,\lambda)=CV^{[m]}(x,t,\lambda)C^{-1},\ m\geq1, \tag{3.24}$$

and further

$$((U_t-V_x^{[m]}+i[U,V^{[m]}])(-x,-t,\lambda))^T=-C(U_t-V_x^{[m]}+i[U,V^{[m]}])(x,t,\lambda)C^{-1},\ m\geq1. \tag{3.25}$$

This implies that the potential reductions in (3.20) are compatible with the zero curvature equations of the integrable hierarchy (2.13).

In this way, we obtain two reduced scalar integrable hierarchies associated with so(3,ℝ):

$$p_t=K_{m,1}|_{q(x,t)=\delta p(-x,-t)},\ m\geq1, \tag{3.26}$$

where $K_m=(K_{m,1},K_{m,2})^T,\ m\geq1$, are defined by (2.13). Moreover, the infinitely many symmetries and conservation laws for the integrable hierarchy (2.13) are reduced to infinitely many ones for the above integrable hierarchies in (3.26).

With $\delta=1$, the first nonlinear reduced scalar integrable equation is a nonlocal reverse-spacetime PT-symmetric NLS equation associated with so(3,ℝ):

$$ip_t=-p_{xx}(-x,-t)+\frac{1}{2}p^2p(-x,-t)+\frac{1}{2}(p(-x,-t))^3. \tag{3.27}$$

Even and odd properties with respect to p and q in the two components of $K_m,\ m\geq1$, show that the first nonlinear reduced scalar integrable equation with $\delta=-1$ in (3.26) has just a different sign from the nonlocal reverse-spacetime NLS equation (3.27).

4. Concluding remarks

We have revisited a hierarchy of integrable equations based on zero curvature equations associated with so$(3,\mathbb{R})$ and presented three classes of integrable nonlocal PT-symmetric reductions for the hierarchy. Three examples among the reduced scalar integrable equations are a nonlocal reverse-space nonlinear Schrödinger (NLS) equation, a nonlocal reverse-time NLS equation, and a nonlocal reverse-spacetime NLS equation associated with the special orthogonal Lie algebra so(3,ℝ). Each class of nonlocal reductions contains two reductions, but every pair of reductions leads to two nonlocal integrable equations with only a sign difference. This is a new phenomenon for integrable equations associated with so$(3,\mathbb{R})$, different from the one for integrable equations associated with sl$(2,\mathbb{R})$.

There are interesting questions for integrable equations, both local and nonlocal, associated with the special orthogonal Lie algebras. First, what kind of general integrable hierarchies could exist? Some novel structures of integrable equations associated to so$(4,\mathbb{R})$ have been discussed [30]. Second, how can we formulate Riemann-Hilbert problems based on associated matrix spectral problems? The above spectral matrix iU in our analysis with zero potential has three eigenvalues, which brings difficulty in establishing relevant theories. The existing examples of Riemann-Hilbert problems belong to the class with two eigenvalues.

Integrable couplings are generated from zero curvature equations associated with non-semisimple Lie algebras, and their Hamiltonian structures could be furnished by applying the variational identity [18, 27]. Bi-integrable couplings and tri-integrable couplings are such examples and exhibit insightful thoughts about general structures of multi-component integrable equations [21]. Multi-integrable couplings provide abundant examples of recursion operators in block matrix form, indeed. There are rich mathematical structures related to integrable couplings [14, 21]. Nevertheless, non-semisimple matrix Lie algebras may not possess any non-degenerate and ad-invariant bilinear forms required in the variational identities [12, 19], and this causes much difficulty in establishing Hamiltonian structures for integrable couplings. For example, we do not even know whether there exists any Hamiltonian structure for a perturbation type coupling:

$$u = K(u),\ v = K'(u)[v],\ w_t = K'(u)[w].$$

In the KdV case, the question is if there is any Hamiltonian structure for the integrable couping:

$$u_t = 6uu_x + u_{xxx},\ v_t = 6(uv)_x + v_{xxx},\ w_t = 6(uw)_x + w_{xxx}.$$

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