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ANALYTIC CONTINUATION OF GENERALIZED TRIGONOMETRIC FUNCTIONS

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ABSTRACT. Via a unified geometric approach, certain generalized trigonometric functions with two parameters are analytically extended to maximal domains on which they are univalent. Some consequences are deduced concerning radius of convergence for the Maclaurin series, commutation with rotation, continuation beyond the domain of univalence, and periodicity.

1. Introduction

Inverses of functions of the form $y \mapsto \int_0^y (1-x^q)^{-1/p} dx$ for $y \in [0,1]$ with $p, q > 1$ have been of interest to analysts. See, for example, [1], [2], [5], [6]; also see [8] for an account of early work in this area. Herein, we study the complex-analytic aspects of a subclass of them and identify their maximal domain of univalence.

Throughout this note, n and k are integers with $n > 2$ and $1 \leq k < n$.

For $y \in [0,1]$, let

$$F_{n,k}(y) = \int_0^y \frac{1}{(1-x^n)^{k/n}} dx.$$

Denoting the number $F_{n,k}(1)$ by $\varphi_{n,k}$, define $S_{n,k} : [0, \varphi_{n,k}] \to [0,1]$ to be the inverse of $F_{n,k}$. These functions $S_{n,k}$ are often referred to as generalized sine functions. Primarily studied as real-valued functions, they are sometimes considered as analytic functions on $F_{n,k}[\mathbb{D}]$ ($\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$) due to the fact that $F_{n,k}$ is univalent as a complex-valued function on $\mathbb{D}$ (by the Noshiro-Warschawski theorem in [4]).

In this article, we identify, for each n and k, $S_{n,k}$'s "natural" domain of analyticity, which turns out to be a maximal domain on which $S_{n,k}$ is univalent. Our unified treatment also encompasses the circular sine function (if we allow $n = 2$), the historically important lemniscate sine function $S_{4,2}$, and the Dixon's elliptic function $S_{3,2}$. It is pertinent to note that, while we only treat the cases where k is an integer, the analysis herein is equally applicable if k is any positive real number less than n.

In §2, we introduce notation and state the main results, which are then proved in §3. In §4, we note several consequences, some of which concern further analytic continuation on larger domains beyond the domain of univalence.

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2. The main results

We often denote points in $\mathbb{C}$ by capital letters in the English alphabet; when we do so, we write AB for the (closed) line segment between point A and point B, whereas we write $[A, B)$ for the half-open segment $AB \setminus \{B\}$.

Now, let $n > 2$ be fixed. Let $\omega_n = e^{i2\pi/n}$. Denote the point $\varphi_{n,k}$ by $A_{n,k}$ and the point $\varphi_{n,k}\omega_n$ by $B_{n,k}$. We construct a closed set $\Pi_{n,k}$ for each $k < n$. There are two cases, $k = 1$ and $k > 1$, that need separate (but related) treatments.

We first treat the case $k = 1$ and construct $\Pi_{n,1}$. The boundary of $\Pi_{n,1}$ is the union of the two rays $\left\{\varphi_{n,1} + te^{i\pi/n} \mid t \geq 0\right\}$ and $\left\{\varphi_{n,1}\omega_n + te^{i\pi/n} \mid t \geq 0\right\}$ along with the two segments $OA_{n,1}$ and $OB_{n,1}$, whereas the interior of $\Pi_{n,1}$ is the component of $\mathbb{C} \setminus \partial\Pi_{n,1}$ that contains $\{te^{i\pi/n} \mid t > 0\}$ (the bisector of $\angle A_{n,1}OB_{n,1}$).

For $k > 1$, let $P_{n,k}$ denote the point on the bisector of $\angle A_{n,k}OB_{n,k}$ such that $A_{n,k}P_{n,k}$ has an angle of inclination $k\pi/n$. (If k were 1, such a point would only "exist" at ∞; cf. the construction of $\Pi_{n,1}$.) We denote by $\Pi_{n,k}$ the (compact) set enclosed by the polygon $OA_{n,k}P_{n,k}B_{n,k}$. Note that $\measuredangle OP_{n,k}A_{n,k} = (k-1)\pi/n$. Thus, in $\Pi_{n,k}$, the interior angle at the vertex $P_{n,k}$ exceeds π iff $k > \frac{n}{2} + 1$, in which case $\Pi_{n,k}$ is nonconvex, whereas the "interior angle" at $P_{n,k}$ becomes a straight angle (and $P_{n,k}$ is a degenerate vertex) iff $k = \frac{n}{2} + 1$. With a little geometry, one can verify that $P_{n,k}$ is the point

$$\varphi_{n,k} e^{i\pi/n} \left(\cos\frac{\pi}{n} + \sin\frac{\pi}{n} \cot\frac{(k-1)\pi}{n} \right) \; ; \tag{2.1}$$

note that $\cot\left[(k-1)\pi/n\right] < 0$ when $k > \frac{n}{2} + 1$, i.e., when the interior angle at the vertex $P_{n,k}$ exceeds π. Four cases of $\Pi_{n,k}$ are illustrated in Figures 1–4, some of which will be referred to in the proofs for Theorems 4.1, 4.5, and 4.7.

Let $V_n = \{re^{i\theta} \mid r > 0;\, \theta \in (0, 2\pi/n)\}$. We state a key lemma.

Lemma 2.1. *The analytic continuation of $F_{n,k}$ is a conformal equivalence from V_n to $\mathring{\Pi}_{n,k}$ (the interior of $\Pi_{n,k}$); its continuous extension to $\overline{V}_n$ (the closure of V_n) restricts to a homeomorphism from ∂V_n onto $\partial\Pi_{n,k}$ when $k = 1$ and onto $\partial\Pi_{n,k} \setminus \{P_{n,k}\}$ when $k > 1$.*

Definition 2.2. Define the domain $\Omega_{n,k}$ to be the interior of

$$\bigcup_{j=0}^{n-1} \left(\omega_n^j \cdot \Pi_{n,k}\right) \; ;$$

letting $J = [1, \infty)$, define Σ_n to be the plane with the n slits $\omega_n^j \cdot J$ ($j \in \mathbb{Z}$):

$$\Sigma_n = \mathbb{C} \setminus \left(\bigcup_{j=0}^{n-1} \left(\omega_n^j \cdot J\right) \right).$$

Theorem 2.3. *Concerning $S_{n,k}$, we have the following statements.*

(1) *The analytic continuation of $S_{n,k}$ is a conformal equivalence from $\Omega_{n,k}$ onto Σ_n.*

(2) *$S_{n,k} : \Omega_{n,k} \to \Sigma_n$ has a continuous extension $\tilde{S}_{n,k}$ with $\tilde{S}_{n,k}(\omega_n^j \varphi_{n,k}) = \omega_n^j$ ($j \in \mathbb{Z}$); moreover,*
   (a) *$\tilde{S}_{n,1}$ maps $\partial\Omega_{n,1} \setminus \cup_j\{\omega_n^j\varphi_{n,1}\}$ two-to-one onto $\cup_j \omega_n^j \cdot \mathring{J}$;*
   (b) *for $k > 1$, $\tilde{S}_{n,k}$ maps $\partial\Omega_{n,k} \setminus \cup_j\{\omega_n^j\varphi_{n,k}\, ,\, \omega_n^j \cdot P_{n,k}\}$ two-to-one onto $\cup_j \omega_n^j \cdot \mathring{J}$.*

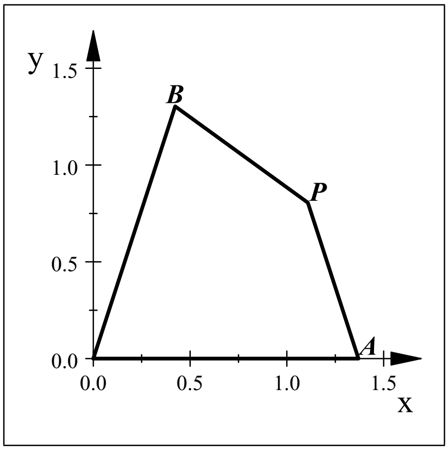

FIGURE 1. $\Pi_{5,3}$

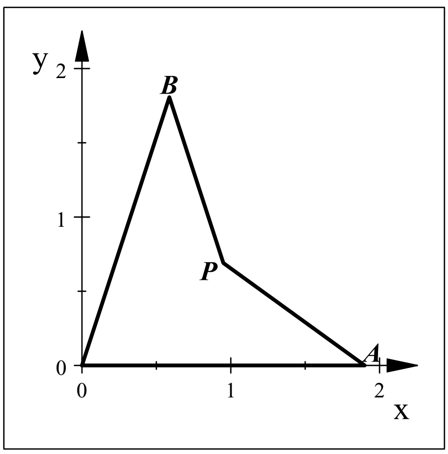

FIGURE 2. $\Pi_{5,4}$

(3) *$\Omega_{n,k}$ is the maximal domain containing* 0 *on which $S_{n,k}$ is univalent.*

We now set out to make the case.

3. ANALYTIC CONTINUATION OF F AND S; PROOF OF MAIN RESULTS

Let

$$K_{n,k}(z) = \frac{1}{(1-z^n)^{k/n}}$$

with the requirement that $K_{n,k}(0) = 1$ and that $K_{n,k}$ be continuous on $\overline{V}_n \setminus \{1, \omega_n\}$. Then, $K_{n,k}$ is analytic on V_n with a primitive

$$F_{n,k}(z) = \int_0^z K_{n,k}(\zeta) d\zeta,$$

where the integral is path-independent. We first examine the behavior of $F_{n,k}$ on ∂V_n.

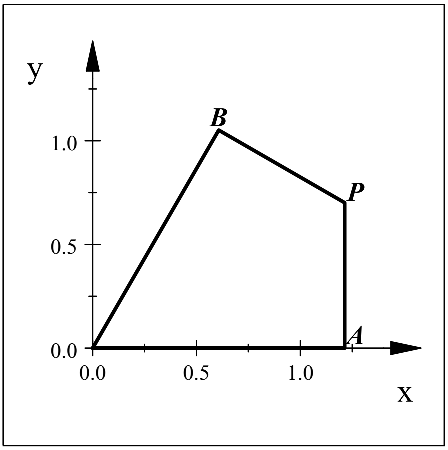

FIGURE 3. $\Pi_{6,3}$

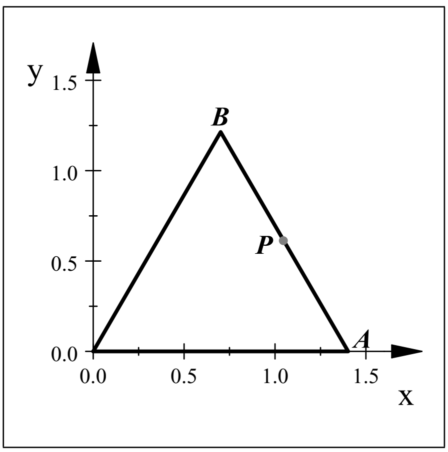

FIGURE 4. $\Pi_{6,4}$

In the following, $\sqrt[n]{a}$ stands for the principal nth root of a positive number a.

- For $x \in [0, 1)$, $F_{n,k}(x) = \int_0^x 1/\sqrt[n]{1-t^n}^k dt$. Thus,
$$F_{n,k}[[0, 1]] = OA_{n,k}.$$
- For $x > 1$, $K_{n,k}(x) = e^{ik\pi/n}/\sqrt[n]{x^n - 1}^k$ where the phase factor $e^{ik\pi/n}$ is dictated by the continuity of $K_{n,k}$. Then,
$$F_{n,k}(x) = \varphi_{n,k} + e^{ik\pi/n} \int_1^x \frac{1}{\sqrt[n]{t^n - 1}^k} dt.$$
In the case $k = 1$, since $\lim_{x\to\infty} \int_1^x \left(\sqrt[n]{t^n - 1}\right)^{-1} dt = \infty$,
$$F_{n,1}[J] = \left\{\varphi_{n,1} + te^{i\pi/n} \mid t \geq 0\right\} ,$$
which is a ray originating from $A_{n,1}$ with an angle of inclination π/n.

If $k > 1$, then

$$\lim_{x\to\infty} F_{n,k}(x) = \varphi_{n,k} + e^{ik\pi/n} I_{n,k} \quad \text{where } I_{n,k} = \int_1^\infty \frac{1}{\sqrt[n]{t^n - 1}^k} dt.$$

and

$$F_{n,k}[J] = \left\{\varphi_{n,k} + te^{ik\pi/n} \mid t \in [0, I_{n,k})\right\},$$

which is a segment $[A_{n,k}, P'_{n,k})$ of length $I_{n,k}$ with angle of inclination $k\pi/n$.

- For $z = t\omega_n$ with $t \in [0,1)$,

$$K_{n,k}(z) = (1 - (t\omega_n)^n)^{-k/n} = \left(\sqrt[n]{1 - t^n}\right)^{-k}$$

and

$$F_{n,k}(z) = \int_0^t K_{n,k}(\tau\omega_n)\omega_n d\tau = \omega_n \int_0^t \frac{1}{\sqrt[n]{1-\tau^n}^k} d\tau.$$

Thus,

$$F_{n,k}[\{t\omega_n \mid t \in [0,1]\}] = OB_{n,k}.$$

- For $z = t\omega_n$ with $t > 1$,

$$K_{n,k}(z) = e^{-i\pi k/n} \left(\sqrt[n]{t^n - 1}\right)^{-k},$$

where the phase factor is again dictated by the continuity of $K_{n,k}$, and

$$\begin{aligned} F_{n,k}(z) &= F_{n,k}(\omega_n) + \int_1^t K_{n,k}(\tau\omega_n)\omega_n d\tau \\ &= \varphi_{n,k}\omega_n + e^{-ik\pi/n}\omega_n \int_1^t \frac{1}{\sqrt[n]{\tau^n - 1}^k} d\tau. \end{aligned}$$

In the case $k = 1$,

$$F_{n,1}[\omega_n \cdot J] = \left\{\varphi_{n,1}\omega_n + te^{i\pi/n} \mid t \geq 0\right\}$$

which is a ray originating from $B_{n,1}$ with an angle of inclination π/n.

If $k > 1$, then

$$F_{n,k}[\omega_n \cdot J] = \left\{\varphi_{n,k}\omega_n + t\omega_n e^{-ik\pi/n} \mid t \in [0, I_{n,k})\right\}$$

is a segment $[B_{n,k}, P''_{n,k})$ of length $I_{n,k}$ whose angle with $\overrightarrow{OB}_{n,k}$, measured *clockwise from* $\overrightarrow{OB}_{n,k}$, has measure $k\pi/n$.

- We claim that, for $k > 1$,

$$\lim_{z\in\overline{V}_n,\ |z|\to\infty} F_{n,k}(z) = \lim_{r\to\infty} F_{n,k}(r), \tag{3.1}$$

which will imply that $P''_n = P'_n = P_n$. Let $\epsilon > 0$ be given. Note that

$$F_{n,k}(re^{i\theta}) - F_{n,k}(r) = \int_{\zeta = re^{it},\ t\in[0,\theta]} \frac{1}{(1-\zeta^n)^{k/n}} d\zeta,$$

whose modulus can be bounded by $\epsilon/2$ for sufficiently large r (since $k > 1$) and for all $\theta \in [0, 2\pi/n]$. Also $|F_{n,k}(r) - P'_{n,k}| < \epsilon/2$ for sufficiently large r as $P'_{n,k} = \lim_{r\to\infty} F_{n,k}(r)$. Thus, $|F_{n,k}(re^{i\theta}) - P'_{n,k}| < \epsilon$ for all sufficiently large r and for all $\theta \in [0, 2\pi/n]$, proving the claim.

Remark 3.1. Let $k > 1$. Consider $\triangle OA_{n,k}P_{n,k}$. On one hand, the point $P_{n,k}$ is located by (2.1); on the other hand, since $|A_{n,k}P_{n,k}| = I_{n,k}$, $P_{n,k}$ is also the point

$$\left(\varphi_{n,k}\cos\frac{\pi}{n} + I_{n,k}\cos\frac{(k-1)\pi}{n}\right)e^{i\pi/n}.$$

Comparing the two, we obtain the identity

$$\int_1^\infty \frac{1}{(t^n-1)^{k/n}}dt = \frac{\sin(\pi/n)}{\sin[(k-1)\pi/n]}\varphi_{n,k}.$$

Returning to the analysis of $F_{n,k}$ on V_n, we claim that $F_{n,k}$ maps V_n bijectively onto $\mathring{\Pi}_{n,k}$. Let $w \in \mathring{\Pi}_{n,k}$. Let γ_r be the positively oriented boundary of the circular sector with vertices 0, r, and $r\omega_n$. The preceding analysis shows that, for all sufficiently large r, $F_{n,k} \circ \gamma_r$ winds around w exactly once.

Thus, $F_{n,k} : V_n \to \mathring{\Pi}_{n,k}$ is a conformal equivalence. The boundary behavior of $F_{n,k}$ detailed above shows that its continuous extension is a homeomorphism from ∂V_n onto $\partial\Pi_{n,k}$ when $k = 1$ and onto $\partial\Pi_{n,k} \setminus \{P_{n,k}\}$ when $k > 1$.

This concludes the argument for Lemma 2.1.

At long last, we define $S_{n,k} : \mathring{\Pi}_{n,k} \to V_n$ to be the inverse of $F_{n,k}$. The boundary extension of $S_{n,k}$ mirrors that of $F_{n,k}$ and maps $OA_{n,k}$ and $OB_{n,k}$ to the two line segments $[0, 1]$ and $\{t\omega_n \mid t \in [0, 1]\}$ on ∂V_n. We may then apply the Schwarz reflection principle repeatedly to analytically continue $S_{n,k}$ over $\Omega_{n,k}$ with range Σ_n. This establishes Part (1) of Theorem 2.3.

Note that $S_{n,k}[\omega_n^j \cdot \mathring{\Pi}_{n,k}] = \omega_n^j \cdot V_n$. By the boundary behavior of $F_{n,k}$, the boundary extension of $S_{n,k}$ behaves as described by Part (2) of Theorem 2.3.

Finally, we argue Part (3) of Theorem 2.3, i.e., $\Omega_{n,k}$ is maximal among domains on which $S_{n,k}$ is univalent. Suppose that $S_{n,k}$ is analytically continued on an (open connected) domain properly containing $\Omega_{n,k}$. This domain necessarily contains a disc U around some line segment $L \subset \partial\Omega_{n,k}$ with L mapped by $\tilde{S}_{n,k}$ into a ray $\omega_n^j \cdot J$. By the Schwarz reflection principle, $S_{n,k}$ maps $U \setminus (\Omega_{n,k})$ into $\Sigma_n = S_{n,k}(\Omega_{n,k})$, ruining univalence.

4. Some consequences

From the main results, we deduce some notable consequences.

First, we consider the radius of convergence of the Maclaurin series for $S_{n,k}$.

Theorem 4.1. *Let $R_{n,k}$ be the radius of convergence for the Maclaurin series for $S_{n,k}$. Then,*

(1) $R_{n,1} = \varphi_{n,1}$;
(2) *for* $k > 1$, $R_{n,k} \le \left(\cos\frac{\pi}{n} + \sin\frac{\pi}{n}\cot\frac{(k-1)\pi}{n}\right)\varphi_{n,k}$;
(3) *for* $k \ge \frac{n}{2} + 1$, $R_{n,k} = \left(\cos\frac{\pi}{n} + \sin\frac{\pi}{n}\cot\frac{(k-1)\pi}{n}\right)\varphi_{n,k}$.

Proof. Recall that, by (2.1), $|OP_{n,k}| = \left(\cos\frac{\pi}{n} + \sin\frac{\pi}{n}\cot\frac{(k-1)\pi}{n}\right)\varphi_{n,k}$.

(1) Note that $\mathrm{dist}(0, \mathbb{C} \setminus \Omega_{n,1}) = \varphi_{n,1}$ and therefore $R_{n,1} \ge \varphi_{n,1}$. We claim that no extension of $S_{n,1}$ can be analytic at $\varphi_{n,1}$. To see this, first note that the two rays $\{\varphi_{n,1} + te^{i\pi/n} \mid t \ge 0\}$ and $\{\varphi_{n,1} + te^{-i\pi/n} \mid t \ge 0\}$ are mapped by $\tilde{S}_{n,1}$ onto J with $\tilde{S}_{n,1}(\varphi_{n,1}) = 1$. Within $\Omega_{n,1}$, these two rays make an interior angle of measure $2\pi(n-1)/n$. Thus, $\tilde{S}_{n,1}$ expands angle

at $\varphi_{n,1}$ by the non-integer factor $n/(n-1)$ and hence cannot be analytic there. (If we allow $n = 2$, $n/(n-1)$ is an integer and indeed the ordinary circular sine function *is* analytic at $\pi/2$!)

(2) Let $k > 1$. Since $|S_{n,k}(z)| \to \infty$ as $z \to P_{n,k}$ in $\Omega_{n,k}$, $R_{n,k} \leq |OP_{n,k}|$.

(3) When $k \geq \frac{n}{2} + 1$, the interior angle of the polygon $\overline{\Omega}_{n,k}$ at $P_{n,k}$ is at least π and hence $|OP_{n,k}| = \operatorname{dist}(0, \mathbb{C} \setminus \Omega_{n,k})$; see $\Pi_{5,4}$ and $\Pi_{6,4}$ in Figures 2 and 4. Hence, $R_{n,k} \geq |OP_{n,k}|$.

□

Next we examine the interaction between $S_{n,k}$ and certain rotations around O. We need a preliminary observation.

Let $L_n = \{te^{i\pi/n} \mid t \geq 0\}$.

Lemma 4.2. *When $k > 1$, $F_{n,k}[L_n] = [O, P_{n,k}) \subset L_n$, whereas $F_{n,1}[L_n] = L_n$.*

Proof. Integrating $K_{n,k}$ along L_n, we obtain

$$F_{n,k}(te^{i\pi/n}) = e^{i\pi/n} \int_0^t \frac{1}{\sqrt[n]{1+\tau^n}^k} d\tau.$$

The claim for $k > 1$ follows at once in light of (3.1), whereas the claim about $F_{n,1}$ is due to the divergence of $\int_0^\infty 1/\sqrt[n]{1+\tau^n}\, d\tau$. □

Remark 4.3. In light of Lemma 4.2, we make two observations. Let $k > 1$.

(1) The consideration given for Lemma 4.2 yields another expression for $|OP_{n,k}|$, i.e., $\int_0^\infty (1+t^n)^{-k/n}\, dt$, in addition to that given by (2.1). Comparing the two, we obtain the identity

$$\int_0^\infty \frac{1}{(1+t^n)^{k/n}} dt = \left(\cos\frac{\pi}{n} + \sin\frac{\pi}{n} \cot\frac{(k-1)\pi}{n} \right) \varphi_{n,k}.$$

(2) Define $\sqrt{V_n}$ to be $\{re^{i\theta} : r > 0; \theta \in (0, \pi/n)\}$. By Lemma 4.2, $F_{n,k}$ maps $\sqrt{V_n}$ conformally onto $\mathring{\triangle} OA_{n,k}P_{n,k}$, the interior of $\triangle OA_{n,k}P_{n,k}$. There is a *unique* conformal equivalence

$$\Psi_{n,k} : \{z \mid \operatorname{Im} z > 0\} \to \mathring{\triangle} OA_{n,k}P_{n,k}$$

whose continuous extension maps $\mathbb{R} \cup \{\pm\infty\}$ onto $\partial(\triangle OA_{n,k}P_{n,k})$ with $\Psi_{n,k}(0) = O$, $\Psi_{n,k}(1) = A_{n,k}$, and $\Psi_{n,k}(\pm\infty) = P_{n,k}$. By uniqueness, $F_{n,k}(z) = \Psi_{n,k}(z^n)$ for $z \in \sqrt{V_n}$. As $\Psi_{n,k}$ can be expressed by a Schwarz-Christoffel integral formula, this functional identity yields another integral expression for $F_{n,k}$ on $\sqrt{V_n}$.

We now show that each $S_{n,k}$ commutes with rotation around O by angle $2\pi/n$.

Theorem 4.4. *For $z \in \Omega_{n,k}$, $S_{n,k}(\omega_n z) = \omega_n S_{n,k}(z)$.*

Proof. For a line L in the plane, let R_L denote the reflection across L. Recall that the composition of reflections across two intersecting lines is a rotation around their point of intersection by an angle that is twice the angle between the two lines. It suffices to check this identity for z in the interior of $\triangle OA_{n,k}P_{n,k}$. By Lemma 4.2, $S_{n,k}$ maps $[O, P_{n,k}) \subset L_n$ into L_n. Applying the Schwarz reflection principle, we obtain

$$S_{n,k}\left(R_{L_n}(z)\right) = R_{L_n}\left(S_{n,k}(z)\right).$$

Applying this and the definition of $S_{n,k}$ on $\omega_n \cdot \mathring{\Pi}_{n,k}$, we have

$$\begin{aligned}\left(S_{n,k} \circ R_{(\omega_n \cdot \mathbb{R})}\right)\left(R_{L_n}(z)\right) &= \left(R_{(\omega_n \cdot \mathbb{R})} \circ S_{n,k}\right)\left(R_{L_n}(z)\right) \\ &= \left(R_{(\omega_n \cdot \mathbb{R})} \circ R_{L_n}\right)\left(S_{n,k}(z)\right);\end{aligned}$$

i.e.,

$$S_{n,k} \circ \left(R_{(\omega_n \cdot \mathbb{R})} \circ R_{L_n}\right) = \left(R_{(\omega_n \cdot \mathbb{R})} \circ R_{L_n}\right) \circ S_{n,k}.$$

Because $R_{\omega_n \cdot \mathbb{R}} \circ R_{L_n}(\zeta) = \omega_n \cdot \zeta$ for all $\zeta \in \mathbb{C}$, the result follows. □

Next we consider the possibility of further continuation of $S_{n,k}$.

Theorem 4.5. *Suppose that $n = 2k$. Then $S_{n,k}$ can be analytically continued to a function (also denoted by $S_{n,k}$) on the interior of*

$$\cup_{j=0}^{n-1} \left[\omega_n^j \cdot \left(\cup_{m \in \mathbb{Z}} \left(2m\varphi_{n,k} + \overline{\Omega}_{n,k}\right)\right)\right] \;;$$

for $z \in \omega_n^j \cdot \left(\cup_{m \in \mathbb{Z}} \left(2m\varphi_{n,k} + \overline{\Omega}_{n,k}\right)\right)$,

$$S_{n,k}(z + 4\omega_n^j \varphi_{n,k}) = S_{n,k}(z).$$

Furthermore, $S_{n,k}$ has multiplicity 2 at $\omega_n^j \varphi_{n,k}$ ($j \in \mathbb{Z}$).

Proof. When $k = n/2$, $\measuredangle O A_{n,k} P_{n,k} = \pi/2$; see $\Pi_{6,3}$ in Figure 3. Therefore, in the polygon $\overline{\Omega}_{n,k}$, the interior angle at $A_{n,k}$ becomes a straight angle. Let $P_{n,k}^*$ denote the complex conjugate of $P_{n,k}$. Note that $\tilde{S}_{n,k}$ folds up the open segment $(P_{n,k}, P_{n,k}^*)$ into J with $\tilde{S}_{n,k}(A_{n,k}) = 1$. We may apply the Schwarz reflection principle to continue $S_{n,k}$ across $(P_{n,k}, P_{n,k}^*)$, and then across $2\varphi_{n,k} + (P_{n,k}, P_{n,k}^*)$, and so on. Recall that the composition of reflections across two parallel lines is a translation by twice the distance between them. This implies that $4\varphi_{n,k}$ is a period for $S_{n,k}$ on $\cup_{m \in \mathbb{Z}} \left(2m\varphi_{n,k} + \overline{\Omega}_{n,k}\right)$. Applying the identity in Theorem 4.4, we can extend $S_{n,k}$ analytically on the domain claimed by the statement. Finally, note that $S_{n,k}$ is two-to-one on some disc centered at $A_{n,k}$ (again due to the Schwarz reflection principle), proving the final claim. □

Remark 4.6. Note that $S_{4,2}$ is the historically important lemniscate sine function sl (and $2\varphi_{4,2}$ is the lemniscate constant). Since $\overline{\Omega}_{4,2}$ is a square each of whose sides is mapped into $\mathbb{R}$ or $i\mathbb{R}$, repeated application of Schwarz reflection principle allows $S_{4,2}$ to be analytically continued to an elliptic function on $\mathbb{C}$, a well-known classical result now encompassed by Theorem 4.5.

By considering the behavior of $\tilde{S}_{n,k}$ near $P_{n,k}$, we deduce the following.

Theorem 4.7. *Suppose that n is even and $k = \frac{n}{2} + 1$. Then, $(S_{n,k})^{n/2}$ can be continued to a meromorphic function on a neighborhood of $P_{n,k}$, whereas $S_{n,k}$ cannot.*

Proof. When $k = \frac{n}{2} + 1$, $P_{n,k}$ becomes a degenerate vertex of $\overline{\Omega}_{n,k}$ as the interior angle at $P_{n,k}$ becomes a straight angle; see $\Pi_{6,4}$ in Figure 4. Recall that $\tilde{S}_{n,k}$ maps $[A_{n,k}, P_{n,k})$ onto J and that it maps $[B_{n,k}, P_{n,k})$ onto the ray $\omega_n \cdot J$. If $S_{n,k}$ were continued to a meromorphic function on a disc around $P_{n,k}$, then Schwarz reflection across $[A_{n,k}, P_{n,k})$ and across $[B_{n,k}, P_{n,k})$ would entail a contradiction. Note that $(S_{n,k})^{n/2}$ maps $[B_{n,k}, P_{n,k})$ onto $(-\infty, 1]$. Thus, $A_{n,k}B_{n,k} \setminus \{P_{n,k}\}$ is mapped into the x-axis by $(S_{n,k})^{n/2}$. The Schwarz reflection principle can then be applied to analytically extend $(S_{n,k})^{n/2}$ on some punctured disc centered at $P_{n,k}$. Thus, $P_{n,k}$ is an isolated singularity of $(S_{n,k})^{n/2}$ and in fact a pole. □

Remark 4.8. Our approach unifies various individual cases and provides insights into their properties. For example, as the complex plane can be tessellated by regular hexagons and $\overline{\Omega}_{3,2}$ is a regular hexagon, it is easy to show that $S_{3,2}$ is extendable to an elliptic function; this is known in [3] and $S_{3,2}$ is the Dixon's elliptic function sm. Similarly, it can be shown, based upon Theorem 4.7 and the fact that $\overline{\Omega}_{4,3}$ is a square, that $(S_{4,3})^2$ can be continued to an elliptic function; this is established in [7] but via an entirely different (and less elementary) approach. As a related matter, our main results readily imply that $S_{n,n-1}$ is extendable to a real-analytic function on $\mathbb{R}$ if n is even and on the interval

$$\left(-\frac{1}{2}\varphi_{n,n-1}\sec\frac{\pi}{n},\ \varphi_{n,n-1}+\frac{1}{2}\varphi_{n,n-1}\sec\frac{\pi}{n}\right)$$

if n is odd.

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