

CONFORMAL IMAGES OF CARLESON CURVES

CHRISTOPHER J. BISHOP

(Communicated by Filippo Bracci)

ABSTRACT. We show that if γ is a curve in the unit disk, then arclength on γ is a Carleson measure iff the image of γ has finite length under every conformal map of the disk onto a bounded domain with a rectifiable boundary.

In this note we characterize curves in \mathbb{D} for which arclength is a Carleson measure, in terms of conformal maps onto rectifiable domains, answering a question asked by Percy Deift (personal communication) arising from his work on Riemann-Hilbert problems. The question seems natural and the proof follows from standard techniques, but I have not been able to locate this result in the literature.

Recall that a positive measure μ on the open unit disk, \mathbb{D} , is called a Carleson measure if

$$\|\mu\|_C = \sup_{|z|=1, r>0} \frac{\mu(D(z, r))}{r} < \infty.$$

The left hand side is called the Carleson norm of the measure.

Theorem 1. *If γ is a curve in the unit disk, then arclength on γ is a Carleson measure iff the image of γ has finite length under every conformal map onto a bounded domain with rectifiable boundary.*

Proof. One direction is an easy consequence of known facts. If f is a conformal map onto a rectifiable domain, then the F. and M. Riesz theorem (e.g., [2, Theorem VI.1.2]) says that its derivative is in the Hardy space H^1 . For a Jordan domain, the H^1 norm of f' is the length of the image's boundary. If the boundary is not a Jordan curve then we may replace “length” by “1-dimensional Hausdorff measure” (also denoted by ℓ) and get $\ell(\partial\Omega) \leq \|f'\|_{H^1} \leq 2\ell(\partial\Omega)$. For any H^p function g on the unit disk

$$\int |g|^p d\mu \leq C_p \|\mu\|_C \|g\|_{H^p}$$

(e.g., [1, Theorem II.3.9]), where $\|\cdot\|_{H^p}$ is the Hardy space norm. Thus taking $g = f'$ we see that

$$\ell(f(\gamma)) = \int_{\gamma} |f'| ds \leq C_1 \|\mu\|_C \cdot \ell(\partial f(\mathbb{D})),$$

where μ denotes arclength measure on γ .

Received by the editors July 20, 2020, and, in revised form, September 27, 2020, and November 17, 2020.

2020 *Mathematics Subject Classification.* Primary 30H10.

Key words and phrases. Carleson measures, conformal maps, rectifiable, Hardy spaces.

The author was partially supported by NSF Grant DMS 1906259.

The converse requires more work. [1, Theorem II.3.9] implies that if μ is not Carleson, then there is a $g \in H^1$ so that $\int |g| d\mu = \infty$. By the usual factorization theorems for Hardy spaces (e.g., [1, Corollary II.5.7]), we can assume g never vanishes in \mathbb{D} , but this is not quite enough to deduce that $g = h'$ for some conformal map h . Instead, we will explicitly construct a conformal map h onto a rectifiable domain so that $\int |h'| d\mu = \infty$.

Our conformal map h will be built as a limit of compositions from a collection of conformal maps defined as follows. Suppose $0 < a < 1$ and let $\Omega_{a,\epsilon} = \mathbb{D} \cup D(1+a, (1+\epsilon)a)$ be the overlapping union of the unit disk \mathbb{D} and a smaller disk centered outside of \mathbb{D} . See Figure 1. The conformal map $\mathbb{D} \rightarrow \Omega_{a,\epsilon}$ is a composition of Möbius transformations and power functions, but we will not need the explicit formula. We will only use the following facts.

Lemma 2. *There is a constant $0 < c < 1$ so that given any $0 < a < 1$ and $0 < \delta < 1/2$, there exists a $0 < r_a < 1$ so that the following holds. For any $0 < r < r_a$ there is an $\epsilon > 0$ and a conformal map $f : \mathbb{D} \rightarrow \Omega_{a,\epsilon}$ such that:*

- (1) $f(0) = 0$ and f is symmetric with respect to \mathbb{R} ,
- (2) $f(1-r) = 1+a$,
- (3) $|f'| \geq ca/r$ on $D(1, r)$.
- (4) f has a conformal extension across \mathbb{T} ,
- (5) $|f(z) - z| < \delta$ and $|f'(z) - 1| < \delta$ on $\overline{\mathbb{D}} \setminus D(1, \delta)$.

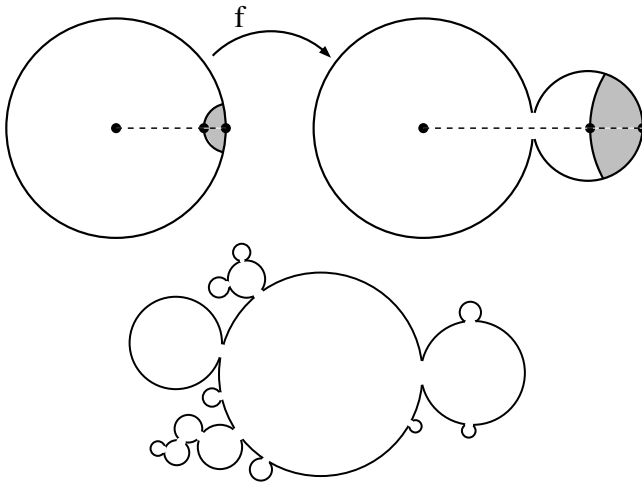


FIGURE 1. The top picture shows the domain $\Omega_{a,\epsilon}$ which is a small disk attached to the unit disk. A properly placed Carleson region is expanded by this map to a size comparable to the added “bubble” and $|f'|$ is comparable to the ratio of the diameters of the region and its image. By composing maps of this form, we build a sequence of domains that look like the lower picture, except that in the proof the sizes of the “bubbles” shrink much more dramatically.

The lemma can be proven by an explicit calculation of f , or by applying symmetry and distortion properties of conformal maps (e.g., Koebe’s $\frac{1}{4}$ -theorem). The

idea for (2) is that the hyperbolic distance between 0 and a is a continuous function of ϵ and it goes to ∞ as ϵ goes to zero. For a given a, ϵ we can choose r so the image is $> 1 + a$, but the image tends to 1 as $\epsilon \searrow 0$, so there is an intermediate choice of ϵ where r maps to $1 + a$. By replacing $f(z)$ by $f(sz)$ for s very close to 1, we can assume f has a conformal extension across \mathbb{T} and the previous conditions still hold. We leave the details to the reader.

By conjugating f with a rotation of \mathbb{D} (i.e., replace $f(z)$ by $f(\lambda z)/\lambda$, $|\lambda| = 1$), we can clearly make $|f'|$ large on any sufficiently small Carleson disk, not just those centered at 1.

Let μ denote arclength measure on a curve γ and suppose this is not a Carleson measure. Then there must be sequence of disks centered at points $\{x_n\}$ on the unit circle and radii $\rho_n \rightarrow 0$ so that

$$\mu(D(x_n, \rho_n)) \geq n\rho_n.$$

Fix one such disk $D = D(x, r)$ and let $W_t = D \cap \{|z| < t\}$. Since $D \cap \mathbb{D}$ is the union of the W_t 's as $t \nearrow 1$, we can choose a t so that $\mu(W_t) \geq \frac{1}{2}\mu(D)$. For each disk in our sequence, make such a choice and inductively define a subsequence of sets $\{W_n\}$ so that $\mu(W_n) \geq nd_n$ and $d_{n+1} \leq 2^{-n} \cdot \text{dist}(W_n, \mathbb{T})$, where $d_n = \text{diam}(W_n)$ (Euclidean diameter). We now proceed by induction to construct a sequence of conformal maps $\{h_j\}$ on \mathbb{D} that map our non-Carleson curve γ to curves with longer and longer length. The limiting map h will map γ to a curve of infinite length.

Start with $a = \delta = 1/2$ and let r_a be as in the lemma. Choose k_1 so large that the region $W_{k_1} \subset D(x_{k_1}, \rho_{k_1})$ has diameter less than r_a . By the lemma, we can choose a point $a_1 = a \cdot x_{k_1}$ outside \mathbb{D} , an $\epsilon_1 > 0$, and a conformal map $f_1 : \mathbb{D} \rightarrow \Omega_{a_1, \epsilon_1}$ so that $|f'_1| \geq ca_1/\rho_{k_1}$ on W_{k_1} , and f_1 extends to be analytic on $\{|z| < 1 + s_1\}$ for some positive s_1 . Let $h_1 = f_1$.

In general, assume we have used the lemma to choose conformal maps f_1, \dots, f_{n-1} and that all of them have a conformal extension to $\{|z| < 1 + s_{n-1}\}$ for some positive s_{n-1} . Let $h_{n-1} = f_1 \circ \dots \circ f_{n-1}$. Let $M_{n-1} = \max|h'_{n-1}|$ over the closed unit disk (since h_{n-1} has a holomorphic extension across the boundary, this maximum is certainly finite). Similarly, let $m_n = \min|h'_{n-1}| > 0$. Choose $0 < a_n < s_{n-1}$ and $\epsilon_n > 0$ so small that $a_n M_{n-1} \leq 2^{-n}$ and so that the conformal map f_n given by the lemma satisfies both

$$|f_n(z) - z| \leq s_{n-1}/2, \text{ and } |f'_n - 1| \leq 2^{-n},$$

on $\overline{\mathbb{D}} \setminus D(1, s_{n-1})$. Moreover, $|f'_n| \geq c/(a_n \rho_{k_n})$ on $D(1, r_n)$, where $r_n = r_{a_n}$ as given by the lemma.

Now choose k_n so large that the region W_{k_n} satisfies:

- (6) $\text{diam}(W_{k_n}) < r_{a_n}$ (r_a as given by the lemma),
- (7) The minimum and maximum of $|h'_{n-1}|$ over W_{k_n} differ by at most a factor of 2 (this is possible by the distortion theorem for conformal maps if $\text{diam}(W_{k_n})$ is small enough).
- (8) $k_n \geq c/(m_n a_n)$.

By the definition of W_n , Condition (8) implies

$$\mu(W_{k_n}) / \text{diam}(W_{k_n}) \geq k_n \geq c/(m_n a_n)$$

or

$$\mu(W_{k_n}) \geq \frac{c \cdot \text{diam}(W_{k_n})}{m_n a_n}.$$

By conjugating f_n by an appropriate rotation, we get a function (also called f_n) so that $|f'_n| \geq ca_n/\rho_{k_n}$ on W_{k_n} . This implies that the length of σ inside W_{k_n} is expanded to approximately unit length under f_n . We want to show this is also true for the composition $h_n = h_{n-1} \circ f_n = f_1 \circ \cdots \circ f_{n-1} \circ f_n$ and show these maps have a limit h with the same property.

By construction, the image of each map f_j lies inside a disk where the map f_{j-1} is defined and conformal so the composition is well defined and conformal on \mathbb{D} . Since the maps f_j converge uniformly to the identity on compact subsets of \mathbb{D} (as rapidly as we wish), the limiting map h exists and is conformal on \mathbb{D} . Next we check that $h(\gamma)$ has infinite length and that $h(\mathbb{T})$ is rectifiable.

On each W_{k_j} we have

$$|h'_n| \geq |h'_j| \left(\prod_{m=j+1}^n (1 - 2^{-m}) \right) \geq c|h'_j|.$$

Thus later generations of the construction do not greatly affect the expansion we have already created on earlier regions. Since $h_n \rightarrow h$ uniformly on compact sets, we also have $h'_n \rightarrow h'$ uniformly on compact sets and hence

$$\int_K |h'| d\mu = \lim_n \int_K |h'_n| d\mu,$$

for any compact $K \subset \mathbb{D}$. In particular we can let $K = W_{k_1} \cup \cdots \cup W_{k_n}$ be a finite union of the sets W_{k_j} and note that

$$\int_K |h'_n| d\mu \geq c \sum_{j=1}^n \int_{W_{k_j}} |h'_j| d\mu \gtrsim \sum_{j=1}^n m_{n-1} \cdot \frac{1}{|a_j| \rho_{k_j}} \cdot \rho_{k_j} k_j \gtrsim \sum_{j=1}^n 1 \rightarrow \infty$$

by our choice of k_j in Condition (8). Thus $h(\gamma)$ has infinite length.

Finally, we have to check that h maps \mathbb{D} to a domain with rectifiable boundary. However, the domain $h_n(\mathbb{D})$ is obtained by taking the union of \mathbb{D} with disk of diameter a_n and composing with the map h_{n-1} and then dilating the map very slightly to make sure it has a conformal extension across the unit circle. Adding the disk adds length $O(a_n)$ and composing with h_{n-1} gives a curve which is in the union of $\partial h_{n-1}(\mathbb{D})$ and the image of the small disk. This image has length $O(M_{n-1}a_n) = O(2^{-n})$. Dilating shortens the length of the boundary curve (since $|f'|$ is subharmonic the length of $f(|z| = r)$ is always less than the length of $f(|z| = 1)$ for any conformal map). Thus we can choose $|a_n| \searrow 0$ so rapidly that the length of $\partial h_n(\mathbb{D})$ is uniformly bounded above by some $L < \infty$.

Next, note that the length of $\partial h(\mathbb{D})$ is equal to

$$\sup_{0 < r < 1} \int |h'(re^{i\theta})| d\theta.$$

On the other hand, for any fixed r , h_n converges uniformly to h on the compact set $\{|z| = r\}$ and hence its derivative converges uniformly to h' on this set. Thus for a fixed $0 < r < 1$,

$$\int |h'(re^{i\theta})| d\theta \leq \sup_n \int |h'_n(re^{i\theta})| d\theta \leq L.$$

Taking the sup over r we see $h' \in H^1$ and so $h(\mathbb{T})$ is rectifiable. \square

Although Deift's question concerned curves, we never used this, and we have actually proven that a positive measure μ on the disk is Carleson iff $\int |f'|d\mu < \infty$ for any conformal map f onto a rectifiable domain.

ACKNOWLEDGMENTS

I thank the anonymous referee for several helpful comments and suggestions that clarified the argument and improved the exposition of this note. Also thanks to Percy Deift for raising the problem originally and encouraging me to record its solution.

REFERENCES

- [1] John B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics, vol. 96, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981. MR628971
- [2] John B. Garnett and Donald E. Marshall, *Harmonic measure*, New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2008. Reprint of the 2005 original. MR2450237

DEPARTMENT OF MATHEMATICS, STONY BROOK UNIVERSITY, STONY BROOK, NEW YORK 11794-3651

Email address: `bishop@math.sunysb.edu`