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BOSHERNITZAN'S CONDITION, FACTOR COMPLEXITY, AND AN APPLICATION

VAN CYR AND BRYNA KRA

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Abstract. Boshernitzan gave a decay condition on the measure of cylinder sets that implies unique ergodicity for minimal subshifts. Interest in the properties of subshifts satisfying this condition has grown recently, due to a connection with discrete Schrödinger operators, and of particular interest is how restrictive the Boshernitzan condition is. While it implies zero topological entropy, our main theorem shows how to construct minimal subshifts satisfying the condition, and whose factor complexity grows faster than any pre-assigned subexponential rate. As an application, via a theorem of Damanik and Lenz, we show that there is no subexponentially growing sequence for which the spectra of all discrete Schrödinger operators associated with subshifts whose complexity grows faster than the given sequence have only finitely many gaps.

1. Boshernitzan's complexity conditions

For a symbolic dynamical system (X, σ), many of the isomorphism invariants we have are statements about the growth rate of the *word complexity function* $P_X(n)$, which counts the number of distinct cylinder sets determined by words of length n having nonempty intersection with X. For example, the exponential growth of $P_X(n)$ is the topological entropy of (X, σ), while the linear growth rate of $P_X(n)$ gives an invariant to begin distinguishing between zero entropy systems. Of course there are different senses in which the growth of $P_X(n)$ could be said to be linear and different invariants arise from them. For example, one can consider systems with *linear limit inferior growth*, meaning $\liminf_{n\to\infty} P_X(n)/n < \infty$, or the stronger condition of *linear limit superior growth*, meaning $\limsup_{n\to\infty} P_X(n)/n < \infty$. (There exist systems satisfying the first condition but not satisfying the second.)

Under the assumption of linear limit inferior growth, and with a further hypothesis that the system (X, σ) is minimal, Boshernitzan [2] showed that the system only supports finitely many σ-invariant ergodic probability measures. Boshernitzan also considered another version of linear complexity on a minimal shift, studying *linear measure growth*, also referred to in the literature (see for example [7]) as *condition* (B): if μ is a σ-invariant Borel probability measure on X, assume that there exists a sequence of integers $n_k \to \infty$ such that

$$\inf_k \min_{|w|=n_k} n_k \mu([w]) > 0,$$

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where $\mu([w])$ denotes the measure of the cylinder set determined by the word w and $|w|$ denotes the length of the word w. Boshernitzan showed that linear measure growth implies that the minimal subshift (X, σ) is uniquely ergodic. Another consequence of linear measure growth, for word complexity, is that $\liminf_{n\to\infty} P_X(n)/n$ is finite.

Each of these three linear complexity assumptions, linear limit inferior growth, linear limit superior growth, and linear measure growth, immediately implies that the associated system has zero topological entropy. It is natural to ask which of these conditions imply any of the others. One of our main results is that while linear measure growth implies linear limit inferior growth, it does not imply linear limit superior growth. In fact, we show linear measure growth is flexible enough that examples satisfying it can be constructed with limit superior growth faster than any pre-assigned subexponential growth rate.

A second motivation for the construction we give comes from a question on the spectra of discrete Schrödinger operators that arise from a subshift. If (X, σ) is a shift, then each $x = (x_n)_{n\in\mathbb{Z}} \in X$ defines a discrete Schrödinger operator $H_x \colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by

$$(H_x u)(n) := u(n+1) + u(n-1) + x_n u(n)$$

(and x is called the *potential function* for this operator). Characterizing the spectra of discrete Schrödinger operators is an active field of study (e.g., [1, 4, 7]) and we refer the reader to [5, 6] for excellent surveys on the theory of discrete Schrödinger operators associated with symbolic systems. For operators built in this way, the dynamical properties of (X, σ) can influence the spectral properties of H_x for any $x \in X$. When (X, σ) is minimal, Damanik (personal communication) asked whether the condition that $h_{top}(X) > 0$ implies that the spectrum of H_x can have only finitely many gaps. Our example shows that the assumption of positive entropy in this question cannot be relaxed to just ask that $P_X(n)$ grow "nearly exponentially" infinitely often: for any subexponential rate $\{a_n\}_{n=1}^{\infty}$ our example, via a theorem of Damanik and Lenz [7], gives a Schrödinger operator whose spectrum has infinitely many gaps and whose complexity is larger than $\{a_n\}_{n=1}^{\infty}$ infinitely often.

We turn to stating our main theorem. For a word w in the language of a subshift (X, σ), we denote the cylinder set starting at zero it determines by $[w]_0^+$ and we denote the words of length n in the language of the subshift by $\mathcal{L}_n(X)$ (for further discussion of the definitions, see Section 2.1):

Theorem 1.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers satisfying*

$$\limsup_{n\to\infty} \frac{1}{n} \log a_n = 0. \tag{1}$$

There exists a minimal and uniquely ergodic subshift (Y, σ) such that

$$\limsup_{n\to\infty} \frac{P_Y(n)}{a_n} = \infty$$

and such that the unique invariant measure μ has the property that there is a sequence $\{n_k\}_{k=1}^{\infty}$ satisfying

$$\liminf_{k\to\infty} \min\left\{\mu([w]_0^+) \cdot n_k \colon w \in \mathcal{L}_{n_k}(Y)\right\} > 0. \tag{2}$$

The hypothesis in this theorem is a type of subexponential growth on the sequence $\{a_n\}_{n=1}^{\infty}$ and the constructed system is a zero entropy system satisfying the

Boshernitzan condition while the factor complexity grows faster than the given sequence. To prove the theorem, it suffices to show that the system (Y, σ) supports a measure μ satisfying the property (2), as it then follows from Boshernitzan [3, Theorem 1.2] that the system is uniquely ergodic.

An immediate corollary of Theorem 1.1, combined with a theorem of Damanik and Lenz [7, Theorem 2], is the following:

Corollary 1.2. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers which grows subexponentially in the sense of* (1). *There exists a Cantor set $\Sigma \subset \mathbb{R}$, of Lebesgue measure zero, and a minimal subshift (Y, σ) such that*

$$\limsup_{n\to\infty} \frac{P_Y(n)}{a_n} = \infty$$

and for every $y \in Y$ the discrete Schrödinger operator $H_y \colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ given by

$$(H_y u)(n) := u(n-1) + u(n+1) + y_n u(n)$$

has spectrum exactly Σ.

2. Background

2.1. **Symbolic systems.** We work over the alphabet $\mathcal{A} = \{0, 1\} \subset \mathbb{R}$ and consider $\mathcal{A}^{\mathbb{Z}}$. We denote $x \in \mathcal{A}^{\mathbb{Z}}$ as $x = (x_n)_{n\in\mathbb{Z}}$ and we endow $\mathcal{A}^{\mathbb{Z}}$ with the topology induced by the metric $d(x, y) = 2^{-\inf\{|i| \colon x_i \neq y_i\}}$. The *left shift* $\sigma \colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $(\sigma x)_n = x_{n+1}$ for all $n \in \mathbb{Z}$. If $X \subset \mathcal{A}^{\mathbb{Z}}$ is closed and σ-invariant, then (X, σ) is a *subshift*.

If $w = (a_{-n}, \ldots, a_0, \ldots, a_n) \in \mathcal{A}^{2n+1}$, then the *central cylinder set* $[w]_0$ *determined by* w is defined to be

$$\{x \in X \colon x_j = a_j \text{ for } j = -n, \ldots, n\}.$$

If $w = (a_0, a_1, \ldots, a_n) \in \mathcal{A}^{n+1}$, then the *one-sided cylinder set* $[w]_0^+$ *determined by* w is defined to be

$$\{x \in X \colon x_j = a_j \text{ for } j = 0, \ldots, n\}.$$

If (X, σ) is a subshift and $n \in \mathbb{N}$, the *words* $\mathcal{L}_n(X)$ *of length* n are defined to be the collection of all $w \in \mathcal{A}^n$ such that $[w]_0^+ \neq \emptyset$, and the *language* $\mathcal{L}(X)$ of the subshift is the union of all the words:

$$\mathcal{L}(X) = \bigcup_{n=1}^{\infty} \mathcal{L}_n(X).$$

If $w \in \mathcal{L}(X)$ is a word, we say that $u \in \mathcal{L}(X)$ *is a subword of* w if $w = w_1 u w_2$ for some (possibly empty) words $w_1, w_2 \in \mathcal{L}(X)$.

For a subshift (X, σ), the *word complexity* $P_X(n) \colon \mathbb{N} \to \mathbb{N}$ is defined to be the number of words of length n in the language:

$$P_X(n) = |\mathcal{L}_n(X)|.$$

2.2. **Well approximable irrationals.** A key ingredient in our construction is Theorem 2.1 of V. Sos [9] (formerly known as the Steinhaus Conjecture).

Theorem 2.1 (Three Gap Theorem)**.** *Assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $n \in \mathbb{N}$, the partition of the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ determined by the points $\{0, \alpha, 2\alpha, \ldots, (n-1)\alpha\}$, with all points taken* (mod 1)*. Then the subintervals determined by this partition have* *at most three distinct lengths, and when there are three distinct lengths, the largest length is the sum of the other two.*

Given an integer $n \geq 1$ and irrational α, we refer to the partition determined by the points $\{0, \alpha, 2\alpha, \dots, (n-1)\alpha\}$ of the unit circle as the *n-step partition*, and make use of it for well chosen α. An irrational real number α is *well approximable* if there exists a sequence $\{n_k\}_{k=1}^{\infty}$ of integers such that for each n_k, the associated n_k-step partition in the Three Gap Theorem has three distinct lengths and the ratio of the smallest to the largest length in such a partition tends to zero as $k \to \infty$, and without loss of generality we can assume that shortest length in the n_k-step partition is not present in the n_{k-1}-step partition. (This sequence is obtained as the denominators in the regular continued fraction expansion of α, and this can be rephrased as unbounded partial quotients.) Furthermore, we can choose the sequence $\{n_k\}_{k=1}^{\infty}$ such that the smallest length present in the n_k-step partition is not present for in the $(n_k - 1)$-step partition.

An irrational that is not well approximable is said to be *badly approximable*, and the set of badly approximable reals has Lebesgue measure zero. Notice that if α is well approximable and $\{n_k\}_{k=1}^{\infty}$ is the associated sequence, then the $(n_k - 1)$-step partition in the Three Gap Theorem has only two distinct lengths and the ratio of their lengths tends to 1 as $k \to \infty$.

2.3. **Sturmian systems.** To make use of the approximations determined by the Three Gap Theorem, we use Sturmian sequences. To define this notion, let α be an irrational real number and consider the partition $\mathcal{P} = \{[0, \alpha), [\alpha, 1)\}$ of $[0, 1)$ and let T_α denote the rotation $T(x) = x + \alpha \pmod 1$. For any $x \in [0, 1)$ and each $n \in \mathbb{Z}$, define

$$c_n(x) = \begin{cases} 0 & \text{if } x + n\alpha \pmod 1 \in [0, \alpha); \\ 1 & \text{otherwise.} \end{cases}$$

Let $X_\alpha \subseteq \{0, 1\}^{\mathbb{Z}}$ be closure of the set of all sequences of the form

$$(\dots, c_{-2}(x), c_{-1}(x), c_0(x), c_1(x), c_2(x), \dots).$$

Then X_α is called the *Sturmian shift* with *rotation angle* α. A classical fact is that the system (X_α, σ) is minimal, uniquely ergodic, and $P_{X_\alpha}(n) = n + 1$ for all $n \in \mathbb{N}$.

Moreover, words $w \in \mathcal{L}_{X_\alpha}(X)$ correspond to the cells of $\bigvee_{i=0}^{|w|-1} T_\alpha^{-i}\mathcal{P}$, and with respect to the unique invariant measure ν_α, the measure of the cylinder set $[w]_0^+$ is the Lebesgue measure λ of the cell of $\bigvee_{i=0}^{|w|-1} T_\alpha^{-i}\mathcal{P}$ corresponding to w. In other words, there is a bijection

$$\varphi_n \colon \mathcal{L}_n(X_\alpha) \to \mathcal{P}_n \tag{3}$$

such that for any word $w \in \mathcal{L}_n(X)$, we have

$$\nu_i([w]_0^+) = \lambda(\varphi_n(w)).$$

In view of the discussion in Section 2.2, if α is well approximable, there exists a sequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \frac{\min\{\nu_\alpha([w]_0^+) \colon w \in \mathcal{L}_{n_k-1}(X_\alpha)\}}{\max\{\nu_\alpha([w]_0^+) \colon w \in \mathcal{L}_{n_k-1}(X_\alpha)\}} = 1. \tag{4}$$

Recall that (X, σ) is *uniquely ergodic* if there exists a unique Borel probability σ-invariant measure on X. Recasting this definition in terms of the language, the subshift (X, σ) is uniquely ergodic if and only if for any $w \in \mathcal{L}(X)$, there exists

$\delta \geq 0$ such that for any $\varepsilon > 0$ there is an integer $N \geq 1$ with the property that for all $u \in \mathcal{L}(X)$ with $|u| \geq N$, we have

$$\left| \frac{\#\text{ of occurrences of } w \text{ as a subword of } u}{|u|} - \delta \right| < \varepsilon.$$

In this case, δ is the measure of the cylinder set $[w]_0^+$ with respect to the unique invariant measure on X.

3. The construction

We construct a minimal subshift $X \subseteq \{0,1\}^{\mathbb{Z}}$ such that

$$\limsup_{n\to\infty} \frac{P_X(n)}{n} = \infty \tag{5}$$

and for which there exists an invariant measure μ supported on X and a sequence $\{n_k\}_{k=1}^{\infty}$ satisfying

$$\liminf_{k\to\infty} \min\left\{\mu([w]_0^+)\cdot n_k \colon w \in \mathcal{L}_{n_k}(X)\right\} > 0. \tag{6}$$

3.1. **Setup.** We fix $\varepsilon = 1/8$ (any value $\leq 1/8$ suffices) and choose a well approximable real number α satisfying

$$\left|\frac{1}{2} - \alpha\right| < \varepsilon. \tag{7}$$

Let X_α denote the Sturmian shift with rotation angle α and let ν denote the (unique) invariant measure supported on X_α (see Section 2.3). For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ denote the partition of $[0,1)$ into subintervals whose endpoints are given by the set

$$\{0, \alpha, 2\alpha, \dots, (n-1)\alpha\},$$

where, as usual, all points are taken in $[0,1)$, meaning modulo 1.

Using (4) derived from the well approximability of α, there exists $n \in \mathbb{N}$ satisfying

$$\frac{\lambda(\text{shortest subinterval in } \mathcal{P}_n)}{\lambda(\text{longest subinterval in } \mathcal{P}_n)} > 1 - \varepsilon \tag{8}$$

(in fact there exist infinitely many such n). The partition $\mathcal{P}_n$ is obtained from the partition $\mathcal{P}_{n-1}$ by subdividing one of the subintervals in $\mathcal{P}_{n-1}$ into two pieces. Thus the length of the longest subinterval in $\mathcal{P}_{n-1}$ is at most twice the length of the longest subinterval in $\mathcal{P}_n$. Similarly the length of the shortest subinterval in $\mathcal{P}_{n-1}$ is at least as long as the length of the shortest subinterval in $\mathcal{P}_n$. Therefore we also have

$$\frac{\lambda(\text{shortest subinterval in } \mathcal{P}_{n-1})}{\lambda(\text{longest subinterval in } \mathcal{P}_{n-1})} \geq \frac{\lambda(\text{shortest subinterval in } \mathcal{P}_n)}{2\cdot\lambda(\text{longest subinterval in } \mathcal{P}_n)} > \frac{1}{2} - \frac{\varepsilon}{2}.$$

We are now ready to begin our construction.

$$\text{Fix some } n \text{ satisfying (8)} \tag{9}$$

and let $\varphi_{n-1} \colon \mathcal{L}_{n-1}(X_\alpha) \to \mathcal{P}_{n-1}$ and $\varphi_n \colon \mathcal{L}_n(X_\alpha) \to \mathcal{P}_n$ denote the bijections defined in (3). Then

$$\frac{\min\{\nu([w]_0^+) \colon w \in \mathcal{L}_n(X_\alpha)\}}{\max\{\nu([w]_0^+) \colon w \in \mathcal{L}_n(X_\alpha)\}} > 1 - \varepsilon \tag{10}$$

and

$$\frac{\min\{\nu([w]_0^+)\colon w\in\mathcal{L}_{n-1}(X_\alpha)\}}{\max\{\nu([w]_0^+)\colon w\in\mathcal{L}_{n-1}(X_\alpha)\}} > \frac{1}{2}-\frac{\varepsilon}{2}. \tag{11}$$

Since X_α is uniquely ergodic, we can choose $N\in\mathbb{N}$ such that for any $m\geq N$ and any word $w\in\mathcal{L}_{n-1}(X_\alpha)\cup\mathcal{L}_n(X_\alpha)$ and any word $u\in\mathcal{L}_m(X_\alpha)$, we have

$$\left|\frac{\#\text{ of occurrences of } w \text{ as a subword of } u}{|u|}-\nu([w]_0^+)\right|<\varepsilon\cdot\nu([w])/3 \tag{12}$$

(here we use the fact that X_α is minimal and so $\nu([w])>0$). Since X_α is Sturmian, we have $P_{X_\alpha}(\ell)=\ell+1$ for all $\ell\in\mathbb{N}$. Equivalently, this means that $P_{X_\alpha}(\ell+1)=P_{X_\alpha}(\ell)+1$ for all $\ell\in\mathbb{N}$. In particular, for all ℓ there is a unique word $w_\ell\in\mathcal{L}_\ell(X_\alpha)$ for which both $w_\ell 0$ and $w_\ell 1$ are elements of $\mathcal{L}_{\ell+1}(X_\alpha)$. Let $w\in\mathcal{L}_n(X_\alpha)$ be the unique word of length n with this property and let $\tilde{w}\in\mathcal{L}_N(X_\alpha)$ be the unique word of length N with this property. Note that for any $m\geq N$, the unique word $u\in\mathcal{L}_m(X_\alpha)$ with this property has w as its rightmost subword of length n and $\tilde{w}$ as its rightmost subword of length N. (Note that we refer to subwords as being *right* or *left* in another word, where this is meant in the natural sense of ordering: a subword a is the rightmost subword of the word b if $b=ca$ for some word c, with the analogous interpretation for other directional indicators.)

Since X_α is minimal and not periodic, all sufficiently long words in $\mathcal{L}(X_\alpha)$ contain every word of length $|\tilde{w}|+1$ (which is $N+1$) as a subword and there is a uniform gap g (which depends only on N) between consecutive occurrences of any word in $\mathcal{L}_{N+1}(X_\alpha)$, and by assuming that we consider sufficiently long words, we can also assume that none of the words has period at most $|\tilde{w}|$. Let $m\geq N(1+\varepsilon)+3g+3|\tilde{w}|$ be sufficiently large such that this holds (thus we have a uniform gap between occurrences of any word in $\mathcal{L}_{N+1}(X_\alpha)$ and no words of length m have period at most $|\tilde{w}|$) and such that the unique word $u\in\mathcal{L}_m(X_\alpha)$ for which both $u0$ and $u1$ are in $\mathcal{L}_{m+1}(X_\alpha)$ has this property. Then the rightmost subword of u of length $|\tilde{w}|$ is $\tilde{w}$ and there is an occurrence of $\tilde{w}0$ within distance g of the left edge of u. Define a to be the subword of u that begins with the leftmost occurrence of $\tilde{w}0$ and ends just before the rightmost occurrence of $\tilde{w}$ (meaning we remove the rightmost $N=|\tilde{w}|$ letters of u to obtain the end of the word a). Note that $|a|\geq\max\{2g+2|\tilde{w}|,N(1+\varepsilon)\}\geq\max\{2g+2|\tilde{w}|,N\}$ and so (12) holds for all words in $\mathcal{L}_{|\tilde{w}|}(X_\alpha)$ and $u=a$ (because its length is at least N). Since $|a|\geq 2g+2|\tilde{w}|\geq 2g$, every word in $\mathcal{L}_{|\tilde{w}|}(X_\alpha)$ occurs as a subword of a. Moreover every subword of aa of length $|\tilde{w}|$ is an element of $\mathcal{L}(X_\alpha)$, since $a\tilde{w}0\in\mathcal{L}_{|a\tilde{w}0|}(X_\alpha)$ and the leftmost subword of length $|\tilde{w}0|$ in a is $\tilde{w}0$. Since X_α is aperiodic, there exists an integer e such that

$$\underbrace{aa\cdots a}_{e\text{ times}}\notin\mathcal{L}(X_\alpha),$$

and assume that e is sufficiently large such that no word of length $|a|$ can be concatenated with itself e times and still be in $\mathcal{L}(X_\alpha)$. Set

$$A:=\underbrace{aa\cdots a}_{2e+1\text{ times}}.$$

Let $k\geq 3|A|+3g+3|\tilde{w}|$ and let $v\in\mathcal{L}_k(X_\alpha)$ be the unique word for which both $v0$ and $v1$ are elements of $\mathcal{L}_{k+1}(X_\alpha)$. Since $k\geq 2g+|\tilde{w}|$, every element of $\mathcal{L}_{|\tilde{w}|+1}(X_\alpha)$ occurs as a subword of v. Let b be the subword of v that begins at the leftmost occurrence of $\tilde{w}1$ and ends just before the rightmost occurrence of $\tilde{w}$. Then $|b| \geq \max\{2g+2|\tilde{w}|, 3|A|\} \geq \max\{2g+2|\tilde{w}|, N(1+\varepsilon)\} \geq \max\{2g+2|\tilde{w}|, N\}$ and so (12) holds for the word w when taking $u = b$. Since $|b| \geq |g|+|\tilde{w}|$, every word in $\mathcal{L}_N(X_\alpha)$ occurs as a subword of b, Moreover, every subword of length N that occurs in ab, ba, and bb is in $\mathcal{L}_N(X_\alpha)$, since $a\tilde{w}1, b\tilde{w}0, b\tilde{w}1 \in \mathcal{L}(X_\alpha)$, the leftmost subword of a is $\tilde{w}0$, and the leftmost subword of b is $\tilde{w}1$. Finally we define two words:

$$\begin{aligned} x &:= \underbrace{AA\cdots A}_{|b| \text{ times}}, \\ y &:= \underbrace{bb\cdots b}_{|A| \text{ times}}. \end{aligned}$$

By construction both of these words are periodic and we let p denote the minimal period of x and let q denote the minimal period of y. These words have the following properties:

(1) $|x| = |y|$;
(2) A does not occur as a subword of y. Namely since $b \in \mathcal{L}(X_\alpha)$, $|b| > |A|$, and so an occurrence of A would occur as a subword of bb. Since the word $A = aa\cdots a$ repeated $2e+1$ times and so an occurrence of A in bb would force the word $aa\cdots a$ (repeated e times) to occur as a subword of b, contradicting the fact that $aa\cdots a$ (repeated e times) is not in $\mathcal{L}(X_\alpha)$;
(3) x does not occur as a subword of yy (because $|y| \geq 3|A|$ and if x occurred in yy it would force an occurrence of A in y);
(4) y does not occur as a subword of xx (again because such an occurrence would force an occurrence of A in y);
(5) x occurs exactly once as a subword of xy. To check this, note that x cannot overlap y by $|A|$, or more, symbols without forcing an occurrence of A in y. So an occurrence of x (the word) must overlap x (the leftmost subword of xy) on more than $|A| \geq 2p$ symbols. This means that the occurrence of x (the word) has to be offset from the beginning of xy by a multiple of p. If this occurrence of x overlaps y by at least $|\tilde{w}0|$ many symbols, then since $|x|$ is a multiple of p (where $|x|$ is the length of the leftmost x in xy), and x (now the hypothetical nontrivial occurrence of x in xy) is offset from the beginning of xy by a multiple of p, this implies that the leftmost $|\tilde{w}0|$ letters of y must agree with the leftmost $|\tilde{w}0|$ letters of x. But x begins with the word $\tilde{w}0$ and y begins with the word $\tilde{w}1$, a contradiction. This means that x (the subword) overlaps y on at most $|\tilde{w}|$ symbols. and so this occurrence of x is offset from the beginning of xy by at most $|\tilde{w}|$ symbols, which implies that $p \leq |\tilde{w}|$. But x is the concatenation of copies of a and the length $|a| > 2|\tilde{w}|$, and so a has period bounded by the length $|\tilde{w}|$. Since $a \in \mathcal{L}_m(X_\alpha)$, we have a contradiction since no word this long in this language (by definition of m) has period less than or equal to the length $|\tilde{w}|$.

   Thus we cannot have both $\tilde{w}0$ and $\tilde{w}1$ occurring as subwords of x, contradicting the fact that A is a subword of x, a is a subword of A, and every word in $\mathcal{L}_{|\tilde{w}|+1}(X_\alpha)$ occurs as a subword of a.

Next we define two more words, s and t, as follows:

$$\begin{aligned} s &:= xxyxx; \\ t &:= xxyyx. \end{aligned}$$

Note that all of the words ss, st, ts, and tt contain $xxxy$ at least once as a subword. Consider where such a subword could occur:

$$\begin{aligned} ss &:= xxyxxxxyxx; \\ st &:= xxyxxxxyyx; \\ ts &:= xxyyxxxyxx; \\ tt &:= xxyyxxxyyx. \end{aligned}$$

We analyze where it can occur in ss, and the analysis for the other three cases is similar. Since y does not occur as a subword of xx, the prefix xxx (in $xxxy$) cannot completely overlap the leftmost y in ss. This means that the farthest to the left that this prefix can occur is if it begins one letter after the beginning of the leftmost y in ss. But since the word y in $xxxy$ cannot be completely contained in the central $xxxx$ of ss, the farthest to the left $xxxy$ can occur in ss is to have the y at least partially overlap the rightmost y in ss. Also, since the only place in xy that x can occur is at the leftmost edge, the y in $xxxy$ cannot occur anywhere farther to the right in ss than the rightmost y (otherwise it would force an occurrence of x in xy which would guarantee that one of the subwords xx in $xxxy$ exactly overlaps the rightmost xy in ss, which is impossible since $x \neq y$). Therefore any occurrence of $xxxy$ in ss must have the y in $xxxy$ partially overlap the rightmost y in ss, but not extend any farther to the right than this occurrence of y. If it did not exactly overlap the rightmost y in ss, then the x immediately preceding the rightmost y (in ss) occurs in a nontrivial place within the rightmost xy in $xxxy$, a contradiction.

Lemma 3.1. *Any element $z \in \{0,1\}^{\mathbb{Z}}$ that can be written as a bi-infinite concatenation of the words s and t can be written in a unique way as such a concatenation.*

A shift with this property is sometimes known as a *uniquely decipherable* coded shift.

Proof. Note that s and t have the same length and are not the same word. We have already noted that the word $xxxy$ occurs in each of ss, st, ts, and tt and, moreover, it occurs exactly once in each such word. If z can be written as a bi-infinite concatenation of the words s and t, then there must be an occurrence of $xxxy$ within distance $|ss|$ of the origin. Choose a way to write z as a concatenation of s and t and mark the locations in $\mathbb{Z}$ where this choice places the beginnings of these words. Find an occurrence of $xxxy$ within distance $|ss|$ of the origin. Since $|xxxy| < |s| = |t|$, this occurrence must be contained in one of the words ss, st, ts, or tt that begins from our marked set of integers. But $xxxy$ occurs exactly once in any such word and its location is always exactly $4|x|$ symbols from where the word started. This allows us to determine where the marked integers in this occurrence of ss, st, ts, or tt are located. In turn, this means that we can read off the sequence of words s and t that were concatenated to produce z by starting from one of the marked integers and looking at blocks of size $|s|$ moving to the right and left.

Thus, once we find an occurrence of $xxxy$ within distance $|ss|$ of the origin in z, the locations (in $\mathbb{Z}$) where the words s and t begin are determined, and once these locations are determined, the bi-infinite sequence of s and t is also determined. In other words, there is a unique way to write z as such a bi-infinite concatenation. $\square$

Lemma 3.2. *Let $Y \subseteq \{0,1\}^{\mathbb{Z}}$ be the subshift consisting of all elements of $\{0,1\}^{\mathbb{Z}}$ that can be written as bi-infinite concatenations of the words s and t. Let $Z \subseteq Y$ be any subshift of Y and let μ be any σ-invariant probability measure on Z. Recall that $n \in \mathbb{N}$ is defined in (9) and N is fixed to guarantee that (12) holds. Then for any $k \leq N$ we have $\mathcal{L}_k(Z) = \mathcal{L}_k(X_\alpha)$ and we have*

$$\frac{\min\{\mu([w]_0^+)\colon w \in \mathcal{L}_n(Z)\}}{\max\{\mu([w]_0^+)\colon w \in \mathcal{L}_n(Z)\}} > 1 - 2\varepsilon$$

and

$$\frac{\min\{\mu([w]_0^+)\colon w \in \mathcal{L}_{n-1}(Z)\}}{\max\{\mu([w]_0^+)\colon w \in \mathcal{L}_{n-1}(Z)\}} > \frac{1}{2} - \varepsilon.$$

Proof. By construction of the words a and b (from which s and t are built), the claim about equality of the languages holds because $|a|, |b| > n$ and every subword of length N that occurs in any of aa, ab, ba, or bb is in the language $\mathcal{L}_N(X_\alpha)$. To see the claim about measure, by the ergodic decomposition theorem it suffices to show the analogous claim holds when μ is replaced by any ergodic measure supported on Z. Let $c \in \mathcal{L}_n(Z) \cup \mathcal{L}_{n-1}(Z) = \mathcal{L}_n(X_\alpha) \cup \mathcal{L}_{n-1}(X_\alpha)$ be fixed. Fix an ergodic measure $\zeta \in \mathcal{M}(Z)$ and fix a generic element $z \in Z$. Let $\mathbf{1}_{[c]}(\cdot)$ denote the indicator function of the cylinder set determined by c. Then for all sufficiently large S, we have

$$\left|\zeta([c]) - \frac{1}{2S+1}\sum_{k=-S}^{S}\mathbf{1}_{[c]}(\sigma^k z)\right| < \frac{\varepsilon}{3}\cdot\nu([c]). \tag{13}$$

We know that z can be parsed into an infinite concatenation of the words a and b. Fixing S, there exists a subword $z_{-S}z_{-S+1}\cdots z_{S-1}z_S$ of z whose length is at least $2S+1-2|a|$ that is a concatenation of the words a and b. Let $i < j$ be the indices where this subword starts and ends. Then

$$\frac{1}{2S+1}\sum_{k=-S}^{S}\mathbf{1}_{[c]}(\sigma^k z) = \\ \frac{1}{2S+1}\sum_{k=i}^{j}\mathbf{1}_{[c]}(\sigma^k z) + \frac{1}{2S+1}\sum_{k=-S}^{i-1}\mathbf{1}_{[c]}(\sigma^k z) + \frac{1}{2S+1}\sum_{k=j}^{S}\mathbf{1}_{[c]}(\sigma^k z).$$

The second and third of these averages tend to zero as $S \to \infty$, and so for sufficiently large S we can assume they are both at most $\varepsilon \cdot \nu([c])/12$. The first average can be rewritten as:

$$\frac{1}{2S+1}\sum_{k=i}^{j}\mathbf{1}_{[c]}(\sigma^k z) = \frac{1}{2S+1}\sum_{\ell=0}^{(j-i)/|a|-1}\sum_{m=0}^{|a|-1}\mathbf{1}_{[c]}(\sigma^{i+\ell|a|+m}z).$$

Since $|a| \geq N$, $c \in \mathcal{L}_n(X_\alpha) \cup \mathcal{L}_{n-1}(X_\alpha) = \mathcal{L}_n(Z) \cup \mathcal{L}_{n-1}(Z)$, and since any subword of length N that occurs in any of aa, ab, ba, and bb is in the language $\mathcal{L}_N(X_\alpha)$, then for each fixed ℓ we have

$$\left|\frac{1}{|a|}\sum_{m=0}^{|a|-1}\mathbf{1}_{[c]}(\sigma^{\ell|a|+m}z) - \nu([c])\right| < \frac{\varepsilon}{3}\cdot\nu([c])$$

by (12). Therefore, recalling that S is sufficiently large, we have that

$$\left|\frac{1}{2S+1}\sum_{k=-S}^{S}\mathbf{1}_{[c]}(\sigma^k z)-\nu([c])\right|<\frac{\varepsilon}{3}\cdot\nu([c])+\frac{\varepsilon}{6}\cdot\nu([c]).$$

Combining this with (13), it follows that

$$|\zeta([c])-\nu([c])|<\frac{5\varepsilon}{6}\cdot\nu([c]).$$

By (10) and (11) we know that

$$\frac{\min\{\nu([w]_0^+)\colon w\in\mathcal{L}_n(Z)\}}{\max\{\nu([w]_0^+)\colon w\in\mathcal{L}_n(Z)\}}>1-\varepsilon$$

and

$$\frac{\min\{\nu([w]_0^+)\colon w\in\mathcal{L}_{n-1}(Z)\}}{\max\{\nu([w]_0^+)\colon w\in\mathcal{L}_{n-1}(Z)\}}>\frac{1}{2}-2\varepsilon.$$

Thus we have

$$\frac{\min\{\zeta([w]_0^+)\colon w\in\mathcal{L}_n(Z)\}}{\max\{\zeta([w]_0^+)\colon w\in\mathcal{L}_n(Z)\}}\geq\frac{\min\{\nu([w]_0^+)\colon w\in\mathcal{L}_n(Z)\}\cdot(1-5\varepsilon/6)}{\max\{\nu([w]_0^+)\colon w\in\mathcal{L}_n(Z)\}\cdot(1+5\varepsilon/6)}>1-2\varepsilon$$

since $\varepsilon<1/8$. Similarly,

$$\frac{\min\{\zeta([w]_0^+)\colon w\in\mathcal{L}_{n-1}(Z)\}}{\max\{\zeta([w]_0^+)\colon w\in\mathcal{L}_{n-1}(Z)\}}$$
$$\geq\frac{\min\{\nu([w]_0^+)\colon w\in\mathcal{L}_{n-1}(Z)\}\cdot(1-5\varepsilon/6)}{\max\{\nu([w]_0^+)\colon w\in\mathcal{L}_{n-1}(Z)\}\cdot(1+5\varepsilon/6)}>\frac{1}{2}-\frac{\varepsilon}{2}.\qquad\square$$

Lemma 3.3. *The shift (Y,σ) defined in Lemma 3.2 satisfies*

$$h_{top}(Y)=\log(2)/|s|>0.$$

Proof. The number of distinct words whose length n is any particular multiple of $2|s|$ is at least $|s|\cdot 2^{n/|s|}$ and at most $4|s|\cdot 2^{n/|s|}$: as a word of this length must contain an occurrence of $xxxy$, this identifies how this word is parsed as a subword in the concatenation of the words s and t (other than perhaps the leftmost and rightmost words in the concatenation). $\square$

We fix a subexponentially growing sequence $\{a_n\}_{n=1}^{\infty}$. It follows from Lemma 3.3 that

$$P_Y(m)>a_m \tag{14}$$

for all but finitely many $m\in\mathbb{N}$. Thus,

(15)
given the subexponentially growing sequence $\{a_m\}_{m=1}^{\infty}$ and a bound $B\geq|s|$,

we can choose some $m>B$ such that $P_Y(m)>a_m$, and then fix two (distinct) words $u,v\in\mathcal{L}(Y)$, of equal length, that each contain every element of $\mathcal{L}_m(Y)$ as a subword and which can be written as a concatenation of the words s and t. Finally define

$$0_* \;:=\; uuvuuvuuvvvvuuvuuvuuv \tag{16}$$

$$1_* \;:=\; uuvuuvuuvvvvvvvvuuvuuv. \tag{17}$$

Arguing as with words that can be written as bi-infinite concatenations of s and t, observe that any element of $\{0,1\}^{\mathbb{Z}}$ that can be written as a bi-infinite concatenation of 0_* and 1_* can be written in a unique way as such a concatenation. Let Z be the subshift consisting of all elements of $\{0,1\}^{\mathbb{Z}}$ that can be written as a bi-infinite concatenation of the word 0_* and 1_*. Then $Z \subseteq Y$, meaning Lemma 3.2 applies to any σ-invariant probability measure on Z. Furthermore, by (14), we have that

$$P_Z(|0_*|) > a_{|0_*|}. \tag{18}$$

3.2. **Inflated subshifts.** The subshift Z constructed in the last paragraph of Section 3.1 and satisfying (18) consists of all bi-infinite concatenations of the words 0_* and 1_*. In this way we can think of Z has being a "full shift" on the "symbols" 0_* and 1_*. Since 0_* and 1_* are actually words, rather than symbols, we prove two lemmas that show Z nevertheless does have some properties that a full shift would have, such as a unique representation of every element of Z as a concatenation of 0_* and 1_*. In what follows, we abstract the construction so that it can then be applied in an inductive construction. Note that if the words u and v in Lemma 3.4 are taken to be the words u and v from Section 3.1, then β_0 and β_1 would be exactly the words 0_* and 1_* defined in (16) and (17).

Lemma 3.4. *Assume that $u, v \in \{0,1\}^*$ are two distinct words of equal length. Let*

$$\begin{aligned} \beta_0 &:= uuvuuvuuvvvvuuvuuvuuv; \\ \beta_1 &:= uuvuuvuuvvvvvvvuuvuuv. \end{aligned}$$

Then for any $x \in \{0,1\}^{\mathbb{Z}}$ that can be written as a bi-infinite concatenation of the words β_0 and β_1, there is a unique way to write it as such a concatenation. Furthermore, for such an $x \in \{0,1\}^{\mathbb{Z}}$ and for $i \in \{0,1\}$, whenever β_i occurs as a subword of x, there exists a sequence $\ldots, j_{-2}, j_{-1}, j_0, j_1, j_2, \ldots$ such that x is (a finite shift of) the sequence

$$\cdots \beta_{j_{-2}}\beta_{j_{-1}}\beta_i\beta_{j_0}\beta_{j_1}\beta_{j_2}\cdots. \tag{19}$$

Proof. Let $x \in \{0,1\}^{\mathbb{Z}}$ be such that it can be written as a bi-infinite concatenation of the words β_0 and β_1; choose some way to do this. Since β_0 and β_1 have equal length, there exists $k \in \{0,1,\ldots,|\beta_0|-1\}$ such that, when x is written as a bi-infinite concatenation of β_0 and β_1, all of the concatenated words begin at indices of x that are congruent to k modulo $|\beta_0|$. Now fix $i \in \{0,1\}$ and suppose β_i occurs as a subword of x, beginning at index $m \in \mathbb{Z}$.

If $m \equiv k \pmod{|\beta_0|}$, then we have already found the decomposition appearing in (19). We aim to show it is not possible that $m \not\equiv k \pmod{|\beta_0|}$, which implies the uniqueness of the decomposition of x into a bi-infinite concatenation of β_0 and β_1 and also provides the decomposition appearing in (19). Note that our chosen way to write x as a bi-infinite concatenation of β_0 and β_1 implies that there exist $m_1, m_2 \in \{0,1\}$ such that β_i arises as a subword of $\beta_{m_1}\beta_{m_2}$ in a location that is neither the leftmost nor the rightmost.

Let $p \in \{0,1\}^{\mathbb{Z}}$ be the periodic word

$$p := \cdots uuvuuvuuvuuvuuv \cdots.$$

Then $|uuv|$ is a period of p. We claim it is the minimal period of p. To see this, note that the minimal period of p must divide $|uuv|$. If the minimal period of p is $\frac{1}{2}|uuv|$, then the leftmost subword of length $\frac{1}{2}|u|$ occurring in u is equal to the rightmost subword of u and the period of p implies that $u = v$, a contradiction. If the minimal period of p is $\frac{1}{3}|uuv|$, then it is immediate that $u = v$ and again we have a contradiction. Therefore the minimal period of p is at most $\frac{1}{4}|uuv|$. The minimal period of p is also a period of the word uu, but since $|u|$ is also a period of uu, the Fine-Wilf Theorem [8] implies that the minimal period of p divides $|u|$. Therefore $u = v$ and we have a contradiction. Thus, the minimal period of p is $|uuv|$.

Now, β_i occurs as a subword of $\beta_{m_1}\beta_{m_2}$ in a location that is neither the leftmost nor the rightmost. This occurrence of β_i overlaps at least one of β_{m_1} and β_{m_2} (in their obvious positions within $\beta_{m_1}\beta_{m_2}$) on at least $\frac{1}{2}|\beta_0|$ letters. We analyze the case when β_i overlaps β_{m_1} on at least $\frac{1}{2}|\beta_0|$ of its letters (the other case is similar). Note that β_0 and β_1 are both comprised of seven blocks of length $|uuv|$ and both start with $uuvuuvuuv$ and end with $uuvuuv$. In between these common prefixes and suffixes, β_0 has the block vvv followed by uuv and β_1 has the block $vvvvvv$. In both of these words, we refer to the word vvv and $vvvvvv$, respectively, as their "block of v's." Note that the word $\beta_{m_1}\beta_{m_2}$ contains the word $(uuv)^5$ (uuv self-concatenated five times) beginning to the right of the block of v's in β_{m_1}. The block of v's in β_i does not occur as a subword of $(uuv)^5$: namely, since we have shown that the minimal period of $(uuv)^5$ is $|uuv| = |vvv|$, and since the minimal period of vvv is $|v|$ which is a proper divisor of $|uuv|$, the word vvv cannot occur as a subword of $(uuv)^5$. In fact, in the case that $i = 1$, the rightmost vvv in $vvvvvv$ does not occur in $(uuv)^5$ and so the location where β_i occurs in $\beta_{m_1}\beta_{m_2}$ is far enough to the left that the block of v's in β_i extends by at most $|vvv| - 1$ to the right of the block of v's in β_{m_1}. This means the suffix $uuvuuv$ in β_i completely overlaps (meaning it contains the occurrence of) the suffix $uuvuuv$ in β_{m_1}. Since the minimal period of $uuvuuv$ is $|uuv|$, this guarantees that the word β_i appears in $\beta_{m_1}\beta_{m_2}$ a multiple of $|uuv|$ letters from the left. If it appears exactly $|uuv|$ letters from the left, then the leftmost vvv in the block of v's of β_{m_1} overlaps the word uuv in β_i, implying that $u = v$ and we have a contradiction. Otherwise the leftmost vvv in the block of v's of β_i appears as a subword of $(uuv)^5$ in $\beta_{m_1}\beta_{m_2}$, again implying that $u = v$ and a contradiction. Therefore β_i cannot have occurred in $\beta_{m_1}\beta_{m_2}$ in any location other than the leftmost or the rightmost. $\square$

Lemma 3.5 shows that if we pick a subshift $W \subseteq \{0,1\}^{\mathbb{Z}}$ which has desirable ergodic properties, and "inflate" it by making the replacements $0 \mapsto 0_*$, $1 \mapsto 1_*$ and taking the orbit closure of all elements of $\{0,1\}^{\mathbb{Z}}$ obtained in this way, then the "inflated" shift also has desirable ergodic properties.

Lemma 3.5. *Let $W \subseteq \{0,1\}^{\mathbb{Z}}$ be a subshift. Assume there exist an integer $m > 3$ and a positive constant C_1 such that for any ergodic measure η supported on W, we have*

$$C_1 < \frac{\min\{\eta([w]_0^+) \colon w \in \mathcal{L}_m(W)\}}{\max\{\eta([w]_0^+) \colon w \in \mathcal{L}_m(W)\}}. \tag{20}$$

Assume that $u, v \in \{0,1\}^*$ are two distinct words of equal length and set

$$\begin{aligned} \beta_0 &:= uuvuuvuuvvvvuuvuuvuuv; \\ \beta_1 &:= uuvuuvuuvvvvvvuuvuuv. \end{aligned}$$

Let $X \subseteq \{0,1\}^{\mathbb{Z}}$ be the subshift consisting of all elements of $\{0,1\}^{\mathbb{Z}}$ that can be written as a bi-infinite concatenation

$$\cdots \beta_{i_{-2}}\beta_{i_{-1}}\beta_{i_0}\beta_{i_1}\beta_{i_2}\cdots,$$

where $(\dots, i_{-2}, i_{-1}, i_0, i_1, i_2, \dots) \in W$. If μ is any ergodic measure supported on X, then

$$\frac{C_1^2}{4} < \frac{\min\{\mu([w]_0^+) \colon w \in \mathcal{L}_{|\beta_0|\cdot(m-1)}(X)\}}{\max\{\mu([w]_0^+) \colon w \in \mathcal{L}_{|\beta_0|\cdot(m-1)}(X)\}}.$$

Proof. Let $z \in W$ be any fixed element and let $w \in \mathcal{L}_m(W)$. Then

$$\begin{aligned}
\frac{C_1}{P_W(m)} &\leq \min\{\eta([w]_0^+) \colon \eta \text{ is an ergodic measure supported on } W\} \\
&= \min\{\eta([w]_0^+) \colon \eta \text{ is an invariant measure supported on } W\} \\
&\leq \liminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1_{[w]_0^+}(\sigma^k z). \qquad (21)
\end{aligned}$$

Let μ be an ergodic measure supported on X and let $x \in X$ be a generic point for the measure μ. Choose $u_1, u_2 \in \mathcal{L}_{(m-1)|\beta_0|}(X)$ such that

$$\mu([u_1]_0^+) = \max\{\mu([w]_0^+) \colon w \in \mathcal{L}_{(m-1)|\beta_0|}(X)\}$$

and

$$\mu([u_2]_0^+) = \min\{\mu([w]_0^+) \colon w \in \mathcal{L}_{(m-1)|\beta_0|}(X)\}.$$

Since μ is ergodic,

$$\mu([u_1]_0^+) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{[u_1]_0^+}(\sigma^k x)$$

and

$$\mu([u_2]_0^+) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{[u_2]_0^+}(\sigma^k x).$$

We analyze occurrences of u_1 in x, noting that the same analysis applies for occurrences of u_2. Any occurrence of u_1 in x occurs in a concatenation of the words β_0 and β_1, some number of times. Since $m \geq 4$ and $|u_1| = (m-1)|\beta_0|$, whenever u_1 occurs it must completely contain at least one of the words $\beta_0\beta_0$, $\beta_0\beta_1$, $\beta_1\beta_0$, or $\beta_1\beta_1$ as a subword.

Any occurrence of u_1 in x occurs in a concatenation of m words which are all β_0 and β_1. Since $m \geq 4$, at least two of these concatenated words occur as a subword of u_1 and so the location of u_1 in the concatenation the words β_0 and β_1 is uniquely determined by Lemma 3.4. Thus each occurrence of u_1 occurs either as an exact concatenation of $m-1$ many of the words β_0 and β_1, or as a subword of a concatenation of exactly m of the words β_0 and β_1 (which is neither the leftmost nor rightmost subword of length $|u_1|$) and all but (perhaps) the first and last of the words β_0 and β_1 can be determined from the word u_1. In the first case, when u_1 is itself an exact concatenation of $m-1$ of the words β_0 and β_1, we adopt the convention of viewing it as a subword of the concatenation of m many of β_0 and β_1 by appending the extra β_0 or β_1 onto its right when it occurs. Therefore there are at most four ways to concatenate the words β_0 and β_1 such that u_1 occurs as a subword, corresponding to the ambiguity of the edge (first and last) concatenated words and that there are at most two choices for each of these edge words, or the ambiguity of the rightmost word when u_1 is an exact concatenation of $m-1$ of the words β_0 and β_1. This means that the asymptotic frequency with which u_1 occurs as a subword of x is $1/|\beta_0|$ times the sum of the measures of the cylinder sets of length m (in the shift W) that determine occurrences of u_1 in X. But by (21), the frequency with which this cylinder set of length m occurs (in W) is at most $1/C_1P_W(m)$. Therefore

$$\mu([u_1]_0^+) \le 4/(C_1P_W(m)\cdot|\beta_0|).$$

Similarly we have

$$\mu([u_2]_0^+) \ge C_1/(P_W(m)\cdot|\beta_0|),$$

where this time there is at least one word which is a concatenation of exactly m many of the words β_0 and β_1 that imply an occurrence of u_2. Therefore, we have

$$\frac{C_1^2}{4} < \frac{\min\{\mu([w]_0^+)\colon w\in\mathcal{L}_{|\beta_0|\cdot(m-1)}(X)\}}{\max\{\mu([w]_0^+)\colon w\in\mathcal{L}_{|\beta_0|\cdot(m-1)}(X)\}}. \qquad \square$$

3.3. **Induction.** Let $\{a_m\}_{m=1}^{\infty}$ be a subexponentially growing sequence of positive integers, meaning

$$\limsup_{m\to\infty}\frac{1}{m}\log a_m = 0.$$

We inductively construct a sequence of shifts U_1, U_2, U_3, ... and ultimately define our subshift Y from Theorem 1.1. Let $n_1\in\mathbb{N}$ be the smallest integer that satisfies (8) and let, by taking $n=n_1$ in (9), $U_1\subseteq\{0,1\}^{\mathbb{Z}}$ be the subshift constructed at the end of Section 3.1 (where it was called Z). Let N be the parameter arising in Section 3.1 and let 0_1 and 1_1 be the words defined in Equations (16) and (17) constructed from the sequence $\{m\cdot a_m\}_{m=1}^{\infty}$ and $B\ge N$ in (15) and let $N_1\ge B$ be the parameter m in the sentence following (15). Then we have $P_{U_1}(N_1) > N_1\cdot a_{N_1}$.

Now suppose we have constructed a nested sequence of subshifts

$$U_1\supseteq U_2\supseteq U_3\supseteq\cdots\supseteq U_k,$$

a sequence of positive integers $n_1<N_1<n_2<N_2<\cdots<n_k<N_k$, and a sequence of words 0_1, 1_1, 0_2, 1_2, ..., 0_k, 1_k where 0_i, $1_i\in\mathcal{L}(U_i)$ are two distinct words of equal length (and this common length is at least N_k). We suppose that for each $i\le k$, U_i is the subshift obtained by taking all possible bi-infinite concatenations of the words 0_i and 1_i. Suppose further that for any $i\le k$ and any $j\le i$ we have $P_{U_i}(N_j) > N_j\cdot a_{N_j}$ and

$$\frac{1}{16} < \frac{\min\{\mu([w]_0^+)\colon w\in\mathcal{L}_{n_j}(U_i)\}}{\max\{\mu([w]_0^+)\colon w\in\mathcal{L}_{n_j}(U_i)\}} < 1$$

for any ergodic measure μ supported on U_i. Take $n_{k+1} > |0_k|$ to be an integer satisfying (8) and let $N_{k+1} = |0_*|$ when n is chosen to be n_{k+1} in (9). We produce the subshift Z, constructed in Section 3.1 and the words 0_* and 1_* with this choice of n. Next, we apply Lemma 3.5 with $W=Z$, $m=n_{k+1}$, $C_1=1/4$, $u=0_k$, and $v=1_k$ to produce a new subshift X. Note that since Z consisted of all possible concatenations of 0_* and 1_*, it follows that X consists of all possible concatenations of the "inflated" versions of the words 0_* and 1_*, where $0\mapsto 0_k$ and $1\mapsto 1_k$. We define 0_{k+1} and 1_{k+1} to be the words 0_* and 1_* after this replacement is applied. Note that every element of U_{k+1} can be written as a bi-infinite concatenation of 0_k and 1_k, and so $U_{k+1}\subseteq U_k$.

Note that since $N_{k+1} = |0_*|$, by Equation (18) and taking $b_n = na_n$, we have guaranteed that $P_{X_{k+1}}(N_j) \geq N_j a_{N_j}$ for all $j \leq k+1$. Finally, by Lemmas 3.2 and 3.5, we have

$$\frac{1}{64} < \frac{\min\{\mu([w]_0^+) \colon w \in \mathcal{L}_{n_j}(U_{k+1})\}}{\max\{\mu([w]_0^+) \colon w \in \mathcal{L}_{n_j}(U_{k+1})\}}$$

for all $j \leq k+1$.

Finally, define

$$Y := \bigcap_{k=1}^{\infty} U_k.$$

Then since the subshifts are nested, Y is nonempty. Since each pattern occurs syndetically, the system Y is minimal. Finally, by construction, we obtain a subshift satisfying $P_Y(N_j) > N_j a_{N_j}$ for all $j \in \mathbb{N}$, in particular

$$\limsup_{n\to\infty} \frac{P_Y(n)}{a_n} = \infty,$$

and such that

$$\frac{1}{64} < \frac{\min\{\mu([w]_0^+) \colon w \in \mathcal{L}_{n_j}(Y)\}}{\max\{\mu([w]_0^+) \colon w \in \mathcal{L}_{n_j}(Y)\}}$$

for all $j \in \mathbb{N}$. This concludes the proof of Theorem 1.1.

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Department of Mathematics, Bucknell University, 380 Olin Science Building, Lewisburg, Pennsylvania, 17837
Email address: van.cyr@bucknell.edu

Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL, 60208
Email address: kra@math.northwestern.edu