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A RADON-NIKODYM THEOREM FOR NONLINEAR FUNCTIONALS ON BANACH LATTICES

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Abstract. A Radon-Nikodym theorem is established for a class of nonlinear orthogonally additive monotone functionals on Dedekind complete Banach lattices. A functional S is absolutely continuous with respect to T if $T(f) = 0$ implies $S(f) = 0$ for f in the domain. It is shown that S is absolutely continuous with respect to T implies S is equal to the composition of an extension of T with an appropriate generalized orthomorphism. In the special case that S and T are linear, the generalized orthomorphism reduces to a multiplication operator consistent with the classical formulation of this theorem.

1. Introduction

In this note, we will consider nonlinear and more specifically, orthogonally additive monotone functionals on Banach lattices with quasi-interior points.

We will review a few of the salient features of Banach lattices we will need for this study. For further details, one may consult reference such as [8] or [11]. Since we are in the setting of vector lattices (Riesz spaces), we use the usual notations of $<, \leq, \wedge$ for infimum, $\vee$ for supremum, and interval notation such as $[f, g] = \{h : f \leq h \leq g\}$. We recall that an element e in a vector lattice E is an *order unit* if the order ideal $I(e)$ generated by e, i.e., $\cup_{n=1}^{\infty}[-ne, ne]$, is equal to E. In the vector lattice $C(X)$, all continuous real-valued functions on a compact space X, the function of constant value 1 is an order unit. For a Banach lattice E, an element e is a *quasi-interior point* if the order ideal generated by e is dense in E. Many of the classical L^p spaces are then Banach lattices with quasi-interior points. Throughout this note, E will denote a Dedekind complete Banach lattice with quasi-interior point. Using the representation theory for Banach lattices (e.g., see [11]), there exists an extremally disconnected compact topological space X so that E is lattice isomorphic to an order ideal in $C^{\infty}(X)$, the collection of all the extended continuous real-valued functions each finite on a dense subset of X. This order ideal contains $C(X)$. An order ideal I is a vector subspace with the property that if $|f| \leq |g|$ and $g \in I$ then $f \in I$. Further, the order ideal $I(e)$ generated by the quasi-interior point e will correspond to $C(X)$ and the image of e will be the constant function 1. In what follows, we identify E with its representation as functions on X.

The role of nonlinear operators in analysis has a rich history, notably the Urysohn operators in integral equations (presentations in [1] provides an overview). The

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Urysohn operator defined by $Tf(x) = \int K(x, y, f(y))dy$ with appropriate conditions on the kernel K is orthogonally additive (defined below). Subsequent extensive studies of nonlinear operators include ongoing analysis of orthogonally additive and generalized Urysohn operators (e.g., [5], [7] and [9]). In a variety of applications, nonlinear operators related to linear operator play a significant role. Given a linear operator L from E to another vector lattice, as a straight forward example, we can consider $T(f) = S(f^2)$ which is then nonlinear and orthogonally additive.

Radon-Nikodym type theorems that extend the measure theoretic results have been studied both for linear operators and nonlinear operators on vector lattices. This author in [4] analyzed absolutely continuous operators between order unit spaces (i.e, spaces of all continuous functions on a compact space X). The proofs there depended on the classical results of the Radon-Nikodym theorem and Riesz representation theorem. The analysis in [2] and [10] provide insights to an operator T being absolutely continuous with respect to S related to relationship of Sf to Tf for each f in the domain. The first investigations of extensions of Radon-Nikodym theorems to linear maps on vector lattices (Riesz spaces) not directly defined by measure theory was provided by Luxemburg and Schep in [6] (their proofs used spectral theory). For functionals, they characterized order continuous linear operators absolutely continuous with one another. The present analysis extends these results for nonlinear functionals and our Corollary 1 is very much akin to the characterization in [6].

In more generality, we will establish that a nonlinear functional S absolutely continuous with a nonlinear functional T can be characterized in a manner quite similar to the result in classical measure theory where S is realized as T composed with a multiplication operator.

Letting E^+ denote the positive cone of E, we will consider functionals T from E^+ to $\mathbb{R}^+$ that are

(i) monotone, i.e., $T(f) \leq T(g)$ whenever $f \leq g$ and
(ii) orthogonally additive, i.e., $T(f+g) = T(f)+T(g)$ whenever f is orthogonal to g (i.e., $f \wedge g = 0$).

In the remainder of this note, S and T will denote monotone, orthogonally additive functionals on E^+

We begin with a definition in this setting analogous to that in measure theory. Here, S and T are monotone orthogonally additive functionals on E^+.

Definition 1. Given S and T functionals on E^+, the functional S is *absolutely continuous* with respect T ($S << T$) if $Tf = 0$ for $f \in E^+$ implies $Sf = 0$.

We will show in Theorem 2 that if $S << T$, there exists a generalized orthomorphism φ (defined below) with domain E^+ so that $S(f) = \hat{T}(\varphi(f))$ where $\hat{T}$ is an extension of T to extended continuous real-valued functions on X. Theorem 2 expresses this in a bit more generality. Given the absence of linearity, our proofs will not use measure theory. We also discuss consequences for more restrictive situation of S being dominated by T (defined in Definition 7). We conclude with a corollary for the situation where the functionals are linear. In this linear case, in analogy to the classical result, we have $\varphi(f) = gf$ for a fixed element $g \in C^\infty(X)^+$ so that $Sf = T(gf)$.

2. Radon-Nikodym theorem

Definition 2. The functional T on E^+ is *unconstrained if* $T(f) > 0$ *implies that* $\vee\{T(nf) : n \in N\} = \infty$ *and* $0 < \alpha < \beta$ *implies* $0 < T(\alpha f) < T(\beta f)$.

Let $\mathcal{K}$ denote all clopen (open and closed) subsets of X. We set

$$\mathcal{G} = \{g \in E^+ : \forall K \in \mathcal{K},\ T(g\chi_K) \leq S(\chi_K)\}.$$

Lemma 1. *Given* g_1 *and* g_2 *in* $\mathcal{G}$*, then* $(g_1 \vee g_2)$ *is in* $\mathcal{G}$

Proof. Given K and g_1, g_2 in $\mathcal{G}$, let $K_1 = \overline{\{x : g_1(x) > g_2(x)\}} \cap K$ and $K_2 = K - K_1$. We have $(g_1 \vee g_2)(\chi_K) = g_1(\chi_{K_1}) + g_2(\chi_{K_2})$. Then $T(g_1\chi_{K_1}) \leq S(\chi_{K_1})$ and $T(g_2\chi_{K_2}) \leq S(\chi_{K_2})$. Now orthogonal additivity implies that
$T(g_1 \vee g_2)(\chi_K) = T(g_1\chi_{K_1}) + T(g_2\chi_{K_2}) \leq S(\chi_{K_1}) + S(\chi_{K_2}) = S(\chi_K)$. □

Clearly $\mathcal{G}$ is not empty since it contains the zero functional. We will consider

$$\hat{g} = \vee\{g \in \mathcal{G}\},$$

where the supremum is in the space $C(X, \mathbb{R}^*)$ of all continuous functions from X (the representation space for E) to $\mathbb{R}^* = [0, \infty]$. We verify that $C(X, \mathbb{R}^*)$ is Dedekind complete. Consider the order isomorphism from $\mathbb{R}^*$ to $[0, 1/2]$ defined by $\rho(x) = \frac{1}{1+e^{-x}} - \frac{1}{2}$ for $x \neq \infty$ and $\rho(\infty) = \frac{1}{2}$ and define the order isomorphis ω from $C(X, \mathbb{R}^*)$ to $C([0, \frac{1}{2}])$ by $\omega(f) = f \circ \rho^{-1}$. Since $C(X)$ (X extremally disconnected) is Dedekind complete, $C(X, \mathbb{R}^*)$ is as well.

Definition 3. For the functional T, we define an extension of T to a map from $C(X, \mathbb{R}^*)$ to $\mathbb{R}^*$ by

$$\hat{T}(h) = \vee_n T(h \wedge ne)$$

for each h in $C(X, \mathbb{R}^*)$.

In this context, we will use the following version of order continuity.

Definition 4. T is *order continuous* if given $\{f_\alpha\}$ increasing to f, then $T(f_\alpha)$ converge to $T(f)$

Lemma 2. *Let* T *be an unconstrained functional and order continuous on* $C(X, \mathbb{R}^+)$*. Then* $\hat{T}$ *is monotone, orthogonally additive and*

$$\hat{T}(\hat{g}) = \vee\{Tg : g \in \mathcal{G}\}.$$

There exists a unique $\hat{g}^* \in (C^\infty(X))^+$ *with the properties that if* $\hat{T}(\hat{g}^*\chi_K) = 0$ *for a clopen set* $K \subset X$*, then* $\hat{g}^*\chi_K = 0$ *and for every clopen* $K \subset X$*,*

$$\hat{T}(\hat{g}\chi_K) = \hat{T}(\hat{g}^*\chi_K).$$

Proof. It is clear that $\hat{T}$ is monotone. If h_1 is orthogonal to h_2 in $C(X, \mathbb{R}^*)$, then $T((h_1 + h_2) \wedge ne) = T(h_1 \wedge ne) + T(h_2 \wedge ne)$ since T is orthogonally additive. It follows directly that $\hat{T}$ is orthogonally additive. $\mathcal{G}$ can be viewed as an increasing net (in light of Lemma 1).

$$T((\vee_{g \in \mathcal{G}} g) \wedge ne) = T(\vee_{g \in \mathcal{G}}(g \wedge ne)) = \vee_{g \in G} T(g \wedge ne)$$

since T is order continuous. In turn, $\vee_{g \in G} T(g \wedge ne) \leq \vee_{g \in \mathcal{G}} T(g)$. This tells us that $T((\vee_{g \in \mathcal{G}} g) \wedge ne) \leq \vee_{g \in \mathcal{G}} T(g)$ and taking the supremum in n, we have

$\hat{T}(\hat{g}) \leq \vee_{g\in\mathcal{G}} T(g)$. On the other hand, $T(g) = \vee_n T(g \wedge ne)$ by order continuity and $\vee_n T(g \wedge ne) \leq \vee_n T(\hat{g} \wedge ne) = \hat{T}(\hat{g})$. Thus $T(g) \leq \hat{T}(\hat{g})$ and, in turn, $\vee_{g\in\mathcal{G}} T(g) \leq \hat{T}(\hat{g})$ establishing the equality.

Let $H = \overline{\cup\{K : \hat{T}(\hat{g}\chi_K) = 0\}}$ where $K \subset X$ is clopen. $\hat{T}(\hat{g}\chi_H) = \vee_n T(\hat{g}\chi_H \wedge ne)$. Let (K_α) be an increasing net of clopen sets whose union is dense in H. By order continuity $T(\hat{g}\chi_H\chi_{K_\alpha} \wedge ne) = 0$ and is convergent to $T(\hat{g}\chi_H \wedge ne)$ and in turn, $\hat{T}(\hat{g}\chi_H) = 0$. Now, for any clopen $K \subset X$, we have shown that $\hat{T}$ is orthogonally additive and thus

$$\hat{T}(\hat{g}\chi_K) = \hat{T}(\hat{g}\chi_{K\cap H}) + \hat{T}(\hat{g}\chi_{K\cap(H)^c}) = \hat{T}(\hat{g}\chi_{K\cap(H)^c}) = \hat{T}(g\chi_K\chi_{H^c})$$

since $\hat{T}(\hat{g}\chi_H) = 0$ (therefore, $\hat{T}(\hat{g}\chi_{K\cap H} = 0)$. Setting $\hat{g}^* = \hat{g}\chi_{(H)^c}$ for any K, we have $\hat{T}(\hat{g}\chi_K) = \hat{T}(\hat{g}^*\chi_K)$.

To verify that $\hat{g}^*$ is in $(C^\infty(X))^+$, let $M = \overline{\{x : \hat{g}^*(x) = \infty\}^o}$. If the interior of M is not empty, then for any non-zero $g \in \mathcal{G}$, any clopen $K \subset M^o$, and any $n \in N$, we have $ng\chi_K \leq \hat{g}^*$. Since an increasing net $(ng\chi_{K_\alpha})$ can be chosen with supremum $ng\chi_M$, we have $ng\chi_M \leq \hat{g}^*$. Assume $T(g\chi_M) > 0$. Then $T(ng\chi_M) \leq \hat{T}(\hat{g}\chi_M) = \vee_{g\in\mathcal{G}} T(g\chi_M) \leq S(\chi_M)$, but $S(\chi_M)$ is finite while $T(ng\chi_M)$ is unbounded as n increases (since T is unconstrained), a contradiction. Thus $\hat{T}(\hat{g}\chi_M) = 0$ so that $M \subset H$ and thus $\hat{g}^*$ is an element of $C^\infty(X)$.

We observe that given $\hat{T}(\hat{g}^*\chi_K) = 0$, we have $K \subset H$ and hence $\hat{g}^*\chi_K = 0$.

To verify the uniqueness, assume $\hat{g}_1^*$ and $\hat{g}_2^*$ both satisfy the conditions of the Lemma. Assume that $\hat{g}_1^* > \hat{g}_2^*$ (and not equal). Let K be such that $\hat{g}_1^*\chi_K > a\hat{g}_2^*\chi_K$ for a number $a > 1$. We observe that $K \cap H = \emptyset$ (H as above). If not, then $T(\hat{g}_2^*\chi_{K\cap H}) = 0$ and $T(\hat{g}_1^*\chi_{K\cap H}) = 0$ which implies that $\hat{g}_1^*\chi_{K\cap H} = \hat{g}_2^*\chi_{K\cap H} = 0$ which contradicts our assumption. By considering a clopen subset of K if necessary, we can assume that $((\hat{g}_1^* \wedge me)\chi_K) > a((\hat{g}_2^* \wedge me)\chi_K) > 0$ and $\hat{g}_2^*\chi_K = (\hat{g}_2^* \wedge me)\chi_K$. We note that even with a subset of K, $T(\hat{g}_2^*\chi_K) > 0$ (if zero, then $K \subset H$). The unconstrained condition tells us that $\hat{T}(\hat{g}_1^*\chi_K) \geq T((\hat{g}_1^* \wedge me)\chi_K \geq T(a((\hat{g}_2^* \wedge me)) > T((\hat{g}_2^* \wedge me)\chi_K) = \hat{T}(\hat{g}_2^*\chi_K)$, a contradiction. □

Proposition 1. *Given T order continuous on E^+, the functional $\hat{T}$ is order continuous.*

Proof. Let $\hat{g}_\alpha$ be directed up and order converge to $\hat{g}$. If $\hat{T}(\hat{g})$ is finite, then since $\hat{T}(\hat{g}) = \vee T(\hat{g} \wedge ne)$, given $\epsilon > 0$, there exists N so that $|\hat{T}(\hat{g}) - T(\hat{g} \wedge Ne)| < \epsilon$. Further, there exists α_0 so that $|T(\hat{g} \wedge Ne) - T(\hat{g}_{\alpha_0} \wedge Ne)| < \epsilon$ since T is order continuous. Now, for $\alpha > \alpha_0$, we have $\hat{T}(\hat{g}) \geq \hat{T}(\hat{g}_\alpha) \geq T(\hat{g}_{\alpha_0} \wedge Ne)$ and thus $|\hat{T}(\hat{g}) - \hat{T}(\hat{g}_\alpha)| \leq |\hat{T}\hat{g} - T(\hat{g}_{\alpha_0} \wedge Ne)| \leq 2\epsilon$, establishing the convergence. If $\hat{T}(\hat{g})$ is infinite, then for any number M, there exists a N so that $T(\hat{g} \wedge Ne) > M$. Then by order convergence, there exists a α_0 with $T(\hat{g}_{\alpha_0} \wedge Ne) > M$. It follows, for $\alpha > \alpha_0$, that $\hat{T}(\hat{g}_\alpha) \geq \hat{T}(\hat{g}_{\alpha_0}) > T(\hat{g}_{\alpha_0} \wedge Ne)) > M$ establishing the convergence. □

In the absence of linearity, we adopt the following:

Definition 5. The map T is *uniformly continuous* if for every $\epsilon > 0$, there exist $\delta > 0$ so that if $\|f - g\| < \delta$ for any f and g in the domain, then $|Tf - Tg| < \epsilon$.

Examples: We note that given a linear functional L that is uniformly continuous, there are a variety of associated nonlinear functionals that will also be uniformly continuous.

For example, we can consider $T_1(f) = L(f \wedge n)$ for a fixed $n \in \mathbb{N}$ (here n represents ne, the constant function n). T_1 is not linear but is uniformly continuous since if $\|f - g\| < \delta$ in the formulation of uniform continuity, then $|(f \wedge n) - (g \wedge n)| \leq |f - g|$ and since the norm is monotone $\|(f \wedge n) - (g \wedge n)\| < \delta$ so that $\|T_1(f) - T_1(g)\| = \|L(f \wedge n) - L(g \wedge n)\| < \epsilon$.

For another example among analogous types of compositions, we let $T_2(f) = L((f \wedge 1)^2)$. Given $\|f - g\| < \delta/2$ in the inequality for the uniform continuity of L, we have $|(f \wedge 1)^2 - (g \wedge 1)^2| = |((f \wedge 1) + (g \wedge 1))((f \wedge 1) - (g \wedge 1))| \leq 2|f - g|$. Since the norm is monotone, if $\|f - g\| < \delta/2$, then $\|(f \wedge 1)^2 - (g \wedge 1)^2\| \leq \delta$ so that $|T_2(f) - T_2(g)| \leq \epsilon$.

Theorem 1. *Let S and T be order continuous functionals on E^+ with T unconstrained and uniformly continuous. If S is absolutely continuous with respect to T, then*

$$S(\chi_K) = \hat{T}(\hat{g}^* \chi_K)$$

for every clopen set $K \subset X$ where $\hat{g}^*$ is as described in Lemma 2.

Proof. We have by definition, $T(g\chi_K) \leq S(\chi_K)$ for each K and by order continuity $\hat{T}(\hat{g}\chi_K) = \hat{T}(\hat{g}^*\chi_K) \leq S(\chi_K)$. We assume that the equality is not satisfied. This means there exists a clopen set K^* such that $\hat{T}(\hat{g}\chi_{K^*})$ is strictly less than $S(\chi_{K^*})$ and we let α be such that $\hat{T}(\hat{g}\chi_{K^*}) < \alpha < S(\chi_{K^*})$. For any $g \in \mathcal{G}$, $\|(g + (1/j)e - g)\| \leq \|(1/j)e\|$. Thus for a sufficiently large j, we have $|T(g + (1/j)e) - T(g)|$ is as small as desired for all $g \in E$ by the uniform continuity assumption. Therefore, for sufficiently large j, $T((g + (1/j)e)\chi_{K^*}) \leq \alpha < S(\chi_{K^*})$ and in turn, $\hat{T}((\hat{g} + (1/j)e)\chi_{K^*}) \leq \alpha < S(\chi_{K^*})$. We note that $\vee_{g\in\mathcal{G}}(g + (1/j)e) = (\vee_{g\in\mathcal{G}} g) + (1/j)e$. For a fixed j satisfying the above and any clopen K, we define

$$M(\chi_K) = S(\chi_K) - \hat{T}((\hat{g} + (1/j)e)\chi_K).$$

M is orthogonally additive and $M(\chi_{K^*}) > 0$. We will demonstrate that there is a $\hat{K} \subset K^*$ such that $M(\chi_K) \geq 0$ for every clopen K contained in $\hat{K}$.

We consider $W = \overline{\cup\{K : T(\chi_K) = 0\}}$. We can assume that $S \neq 0$ since if equal to zero, then the theorem is trivial. Thus $W \neq X$ (since $T(\chi_K) = 0$ implies $S(\chi_K) = 0$ and both are order continuous). It follows that $\hat{T}(\hat{g}^*\chi_K) = \hat{T}(\hat{g}^*\chi_K\chi_W) + \hat{T}(\hat{g}^*\chi_K\chi_{W^c})$. The last term is 0 since it is less than or equal to $S(\chi_{W^c})$. In our following argument, we can assume that $K^* \subset W^c$.

We first note that

$$* \quad (\hat{g} + (1/j)e)\chi_K) \neq \hat{g}\chi_K$$

for every clopen set $K \subset K^*$ (here, $T(\chi_K) \neq 0$). If $(\hat{g} + (1/j)e)\chi_K) = \hat{g}\chi_K$, then $\hat{g}\chi_K = \infty\chi_K$. Then for any m, we have $T((\hat{g} \wedge me)\chi_K) = T(m\chi_K) \leq S(\chi_K)$ which is not possible as $\{T(m\chi_K)\}$ is unbounded by the unconstrained assumption. Let $t_1 = \wedge\{M(\chi_K) : K \subset K^*\}$. We note that since $\hat{T}((\hat{g} + (1/j)e)\chi_K) \leq \hat{T}((\hat{g} + (1/j)e)\chi_{K^*})$ which we observed above is less than or equal to α, it follows that $t_1 > -\alpha$. If $t_1 \geq 0$, then we would have $M(\chi_K) \geq 0$ for every $K \subset K^*$ as desired. Thus we can assume $t_1 < 0$. Choose $K_1 \subset K^*$ with $M(\chi_{K_1}) < t_1/2$. Clearly K_1 is a proper subset of K^*. Continuing inductively, let $t_{n+1} = \wedge\{M(\chi_K) : K \subset K^* - \cup_{i=1}^n K_i\}$ and choose $K_{n+1} \subset K^* - \cup_{i=1}^n K$ with the property that $M(\chi_{K_{n+1}}) < t_{n+1}/2$. Again, we can assume each t_n is less than zero (otherwise we would have the desired result). We note that $K^* - \cup_{i=1}^n K_i$ is not empty. Indeed, if this set is empty, then by orthogonal additive, we would have $\sum_{i=1}^n M(\chi_{K_i}) =$

$M(\chi_{K^*})$. However, the left side of the above equality is negative while the right side is positive. Further, $\cup_{i=1}^{\infty} K_i$ is not dense in K^*. If it were, then since the partial sums $\sum_{i=1}^{m} \chi_{K_i}$ increase to $\chi_{\overline{\cup_{i=1}^{\infty} K_i}}$ and $\{K_i\}$ are pairwise disjoint, order continuity of S and $\hat{T}$ (together with the fact that the range of S is $[0,\infty)$) would imply that $\sum_{i=1}^{m} M(\chi_{K_i}) = M(\cup_{i=1}^{m} K_i)$ converges to $M(\chi_{K^*})$. It then would follow that $M(\chi_{K^*}) \leq 0$ which is not that case.

We set $\hat{K} = K^* - \overline{\cup_{i=1}^{\infty} K_i}$. Now for any $K \subset \hat{K}$, we have $K \subset K^* - \cup_{i=1}^{n} K_i$ so that $M(\chi_K) > t_n$ for each n. We next verify that t_n converges to zero. Assume to the contrary that there is an $\beta < 0$ so that for every N, there is an $n > N$ with $t_n < \beta < 0$. Letting $H_i = \cup_{l=1}^{i} K_l$, we then have, using the order continuity,

$$\begin{aligned} M(\chi_{\overline{\cup_{l=1}^{\infty} K_l}}) &= S(\chi_{\overline{\cup_{i=1}^{\infty} H_i}}) - \hat{T}((\hat{g} + (1/j)e)(\chi_{\overline{\cup_{i=1}^{\infty} H_i}})) \\ &= \lim_{i\to\infty} S(\chi_{H_i}) - \lim_{i\to\infty} \hat{T}((\hat{g} + (1/j)e)\chi_{H_i}). \end{aligned}$$

Since $\sum_{i=1}^{\infty} M(\chi_{K_i}) = -\infty$, we have $\lim_{i\to\infty} M(\chi_{H_i}) = -\infty$. It follows that $M(\chi_{\overline{\cup_{i=1}^{\infty} K_i}}) = -\infty$. Now, $M(\chi_{K^*}) = M(\chi_{K^* - \overline{\cup_{i=1}^{\infty} K_i}}) + M(\chi_{\overline{\cup_{i=1}^{\infty} K_i}})$. We note that from the definition of M, since S is finite, $M(\chi_{K^* - \overline{\cup_{i=1}^{\infty} K_i}}) < +\infty$. Thus the right hand side of the above equation is negative while the left side, $M(\chi_{K^*})$, is positive. Therefore, we conclude that $t_n \to 0$ and $M(\chi_K) \geq 0$ for all $K \subset \hat{K}$ or $\hat{T}((\hat{g} + (1/j)e)\chi_K) \leq S(\chi_K)$. We will now see that this contradicts the maximality of $\mathcal{G}$. Assume that all the vectors $g' = (g + (1/j)e)\chi_{\hat{K}} + g\chi_{(X-\hat{K})}$ for $g \in \mathcal{G}$ are in $\mathcal{G}$. Then $\vee_{g\in\mathcal{G}}(g + (1/j)e)\chi_{\hat{K}} = (\hat{g} + (1/j)e)\chi_{\hat{K}} \leq \hat{g}\chi_{\hat{K}}$, a contradiction to the inequality (*) above. Thus $\hat{T}(\hat{g}\chi_K) = S(\chi_K)$. In view of Lemma 2, we can replace $\hat{g}$ with $\hat{g}^*$. □

In order to extend our result to all vectors in E^+, we first establish the following.

Here we define a generalized orthomorphism similar to the formulation in [5]. An extended analysis of nonlinear orthomorphisms can be found in [3]. A operator φ is an orthomorphism on a vector lattice E if φ is an order bounded operator from E to E with the property that if $|f| \wedge |g| = 0$ for f, g in E, then $|\varphi(f)| \wedge |g| = 0$.

Definition 6. A monotone map φ from E^+ to $C(X, \mathbb{R}^*)$ is a *generalized orthomorphism* if $f \wedge h = 0$ for f, h in E^+ implies that $\varphi f \wedge h = 0$.

We are now able to establish that if $S << T$, then $S(f) = \hat{T}(\varphi(f))$ expressed in a bit more generality.

Theorem 2. *Let S and T be order continuous functionals on E^+ with T unconstrained and uniformly continuous. If S absolutely continuous with respect to T, then there exists a generalized orthomorphism φ from E^+ to $C(X, \mathbb{R}^*)^+$ so that for each $f \in E^+$ and clopen $K \subset X$,*

$$S(f\chi_K) = \hat{T}(\varphi(f)\chi_K).$$

Proof. We first establish that $S(f) = \hat{T}(\varphi(f))$ for appropriately defined φ. Given $f \in E^+$, it was established in [5], that there is an increasing sequence of vector f_n with supremum f with the property that each f_n can be expressed as $f_n = \sum_{i=1}^{m_n} \alpha_i \chi_{K_i}$ where $\{K_i\}$ is a finite collection of disjoint clopen sets and each α_i is a real number. For a fixed α_i and each $h \in E^+$, we define $S_i(h) = S(\alpha_i h)$ and $T_i(h) = T(\alpha_i h)$ and note that $S_i << T_i$. Theorem 1 tells us that there exists

$\hat{g}_i^*$ with $S_i(\chi_K) = \hat{T}_i(\hat{g}_i^* \chi_K)$ for each clopen K. Now for a fixed n, we define $\hat{g}_n^* = \sum_{i=1}^{m_n} \alpha_i \hat{g}_n^* \chi_{K_i}$. Then, since S and T are orthogonally additive, we have

$$* \quad \hat{T}(\hat{g}_n^*) = \sum_{i=1}^{m} \hat{T}(\hat{g}_i^* \chi_{K_i}) = \sum_{i=1}^{m} S(\alpha_i \chi_{K_i}) = S(\sum_{i=1}^{m} \alpha_i \chi_{K_i}) = S(f_n).$$

We will verify that $\hat{g}_n^* \leq \hat{g}_{n+1}^*$ for all n. Assume to the contrary that there exists a point z with $\hat{g}_n^*(z) > \hat{g}_{n+1}^*(z)$. Let H be a clopen set with $\hat{g}_n^* \chi_H > c\hat{g}_{n+1}^* \chi_H$ for a constant $c > 1$. We can also choose H as a subset of some K_i. Restricting H further if necessary, we can assume that $\hat{g}_n^*(x) < \infty$ for all $x \in H$. We will use the notation that $f_n \chi_H(x) = \alpha$ and $f_{n+1} \chi_H(x) = \beta$ for all $x \in X$. If $\hat{T}(\hat{g}_{n+1}^* \chi_H) > 0$, then since T is unconstrained, we have $S(\alpha \chi_H) = \hat{T}(\hat{g}_n^* \chi_H) > \hat{T}(\hat{g}_{n+1}^* \chi_H) = S(\beta \chi_H)$, a contradiction since $f_n \leq f_{n+1}$. If $\hat{T}(\hat{g}_{n+1}^* \chi_H) = 0$, then $0 = S(\beta \chi_H)$ and $\hat{g}_{n+1}^* \chi_H = 0$ as a consequence of Lemma 2. However, again since $f_n \leq f_{n+1}$, we have $0 = S(\beta \chi_H) \geq S(\alpha \chi_H) = \hat{T}(\hat{g}_n^* \chi_H)$. It follows from Lemma 2 that $\hat{g}_n^* \chi_H = 0$ which contradicts our assumption that $\hat{g}_n^* \chi_H > \hat{g}_{n+1}^* \chi_H$.

We define

$$\varphi(f) = \vee(\hat{g}_n^*)$$

and conclude from order continuity that $S(f_n) \to S(f)$ and $\hat{T}(\hat{g}_n^*) \to \hat{T}(\varphi f)$ and therefore $S(f) = \hat{T}(\varphi(f))$.

We verify that φ is monotone. Given $f \leq h$, we can express the approximating functions on the same set $\{K_i\}$ so that $f_n \leq h_n$. Then by the same argument as above, $\hat{g}_n^*$ corresponding to f_n is less than or equal to $\hat{q}_n^*$ corresponding to h_n. Thus $\varphi(f) \leq \varphi(h)$.

Given $f \wedge h = 0$, it follows that each f_n in the sequence $\{f_n\}$ convergent to f will be orthogonal to each h. For $f_n = \sum_{i=1}^{m} \alpha_i \chi_{K_i}$, we have $\hat{g}_n^* = \sum_{i=1}^{m} \hat{g}_i^* \chi_{K_i}$ and so also orthogonal to h. Thus $\varphi f \wedge h = 0$.

For $S(f\chi_K)$ for any K, we note that it follows from Theorem 1, that the equalities in (*) are valid with multiplications by χ_K and in turn the limits. □

Examples: We note that there are a variety of nonlinear functionals T for which T is unconstrained, uniformly continuous, and order continuous. Often these are associated with linear operators.

Let L be an order continuous and uniformly continuous linear functional on E. We define $T_3(f) = L(f + (f \wedge n))$ for a fixed $n \in \mathbb{N}$. Arguing as we did for the functional T_1, we have $|(f + (f \wedge n)) - (g + (g \wedge n))| < |f - g|$ and then it follows that T_3 is uniformly continuous. It is also easy to see that T_3 is order continuous. Note that if $T_3(f) > 0$, then $L(f) > 0$ and $T_3(mf) = L(mf + (mf \wedge n)) > mL(f)$ and thus goes to infinity as m goes to infinity. If $0 < \alpha < \beta$, $T_3(\alpha f) = L(\alpha f + (\alpha f \wedge n)) < L(\beta f + (\beta f \wedge n)) = T_3(\beta f)$. Therefore T_3 is nonlinear but order continuous, uniformly continuous and unconstrained.

Let $T_4(f) = L(\sqrt{f})$. Since the square root function is uniformly continuous, it follows that T_4 is uniformly continuous (we could have used any monotone uniformly continuous function). It is easy to see that T_4 is order continuous. It is also a routine verification to see that T_4 is unconstrained.

Now by Theorem 3, all the generalized orthomorphisms on E characterize all the operator absolutely continuous with respect to T_3 or T_4.

We consider special cases which will ensure that $\hat{g}^*$ is in E.

Definition 7. S is *dominated* by T if $S << T$ and there exists an element $l \in E^+$ so that $S(f) \leq T(lf)$ for every $f \in I(e)^+$ (the positive elements in $I(e)$).

Theorem 3. *Let S and T be order continuous functionals on E^+ with T unconstrained and uniformly continuous. If S is dominated by T, then there exists a generalized orthomorphism φ from E^+ to E^+ so that for each $f \in E^+$ and clopen $K \subset X$,*

$$S(f\chi_K) = T(\varphi(f)\chi_K).$$

Proof. We first consider $S(\chi_K)$ for K clopen. For $\hat{g}^*$ as in Theorem 1, we have $T((\hat{g}^* \wedge ne)\chi_K) \leq S(\chi_K) \ \leq T(l\chi_K)$ for each clopen $K \subset X$ where l is proscribed by the dominated property. We will verify that $\hat{g}^* \leq l$. Assume that this is not the case. Let K^* be a non-empty clopen subset of $\{x : (\hat{g}^* \wedge ne)(x) > \alpha l(x)\}$ for some fixed n and $\alpha > 1$. Note that if $T(l\chi_{K^*}) = 0$, then $S(\chi_{K^*}) = 0$ and in turn $\hat{T}(\hat{g}^*\chi_{K^*}) = 0$ which implies $\hat{g}^*\chi_{K^*} = 0$ by Lemma 2, but this is not possible since $\hat{g}^*\chi_K^* > \alpha l\chi_K^*$. We now have (the strict inequality below a consequence of the unconstrained assumption on T)

$$S(\chi_{K^*}) \geq T((\hat{g}^* \wedge ne)\chi_{K^*}) \geq T(\alpha l\chi_{K^*}) > T(l\chi_{K^*}),$$

but $T(l\chi_{K^*}) \geq S(\chi_{K^*})$, a contradiction. Thus $\hat{g}^* \leq \ l$.

Following the pattern in the proof of Theorem 2, we will have for $S(\alpha_i\chi_{K_i})$, the corresponding $\hat{g}_i^*\chi_{K_i} \leq \alpha_i\chi_{K_i}l$. In turn, for f_n, so that we will have $\hat{g}_n^* \leq l$. Thus $\varphi(f) \leq l$, i.e. an element of E^+ as desired. □

We will say that T is a linear functional on E^+ if $T(\alpha f + g) = \alpha T(f) + T(g)$ for $\alpha \geq 0$. Now in analogy to Theorems 2 and 3, given S and T are linear, we have the following formulation without the use of measure theory.

Corollary 1. *Let S and T be order continuous linear functionals on E^+ with T uniformly continuous.*

(i) *If $S << T$, there exists $g \in (C^\infty(X))^+$ so that for every $f \in E^+$,*

$$S(f) = \hat{T}(gf).$$

(ii) *If S is dominated by T, there exists $g \in E^+$ so that*

$$S(f) = T(gf).$$

Proof. We first note that in Theorem 1, $S(\chi_K) = \hat{T}(\hat{g}^*\chi_K)$. Then by linearity, we have $S(\alpha\chi_K) = \hat{T}(\hat{g}^*\alpha\chi_K)$. For any $f \in E^+$, we consider the sequence of vectors f_n as in the proof of Theorem 2. For a fixed n, noting the orthogonal additivity of $\hat{T}$, we have

$$S(f_n) = S(\sum_{i=1}^{m} \alpha_i\chi_{K_i}) = \sum_{i=1}^{m} \hat{T}(\hat{g}^*\alpha_i\chi_{K_i}) = \hat{T}(\sum_{i=1}^{m} \hat{g}^*\alpha_i\chi_{K_i}) = \hat{T}(\hat{g}^* f_n).$$

Now, $S(f_n) \to S(f)$ and $\hat{T}(\hat{g}^* f_n) \to \hat{T}(\hat{g}^* f)$. Thus we have $S(f) = T(\hat{g}^* f)$ for every $f \in E^+$. Setting $g = \hat{g}^*$, we have the desired result for (i). For (ii), Theorem 3 assures that $\hat{g}^* = \varphi(f)$ is in E^+ and we set $g = \hat{g}^*$. □

References

[1] Jürgen Appell and Petr P. Zabrejko, *Nonlinear superposition operators*, Cambridge Tracts in Mathematics, vol. 95, Cambridge University Press, Cambridge, 1990, DOI 10.1017/CBO9780511897450. MR1066204

[2] Nariman Abasov and Marat Pliev, *Disjointness-preserving orthogonally additive operators in vector lattices*, Banach J. Math. Anal. **12** (2018), no. 3, 730–750, DOI 10.1215/17358787-2018-0001. MR3824749

[3] Zafer Ercan and Antony W. Wickstead, *Towards a theory of nonlinear orthomorphisms*, Functional analysis and economic theory (Samos, 1996), Springer, Berlin, 1998, pp. 65–78, DOI 10.1007/978-3-642-72222-6_6. MR1730120

[4] William Feldman, *Operators on Banach lattices and the Radon-Nikodým theorem*, Proc. Amer. Math. Soc. **100** (1987), no. 3, 517–521, DOI 10.2307/2046440. MR891156

[5] William Feldman, *A factorization for orthogonally additive operators on Banach lattices*, J. Math. Anal. Appl. **472** (2019), no. 1, 238–245, DOI 10.1016/j.jmaa.2018.11.021. MR3906371

[6] W. A. J. Luxemburg and A. R. Schep, *A Radon-Nikodým type theorem for positive operators and a dual*, Nederl. Akad. Wetensch. Indag. Math. **40** (1978), no. 3, 357–375. MR507829

[7] J. M. Mazón and S. Segura de León, *Order bounded orthogonally additive operators*, Rev. Roumaine Math. Pures Appl. **35** (1990), no. 4, 329–353. MR1082516

[8] Peter Meyer-Nieberg, *Banach lattices*, Universitext, Springer-Verlag, Berlin, 1991, DOI 10.1007/978-3-642-76724-1. MR1128093

[9] Marat Pliev, *On C-compact orthogonally additive operators*, J. Math. Anal. Appl. **494** (2021), no. 1, Paper No. 124594, 15, DOI 10.1016/j.jmaa.2020.124594. MR4156136

[10] M. Pliev and F. Polat, *The Radon-Nikodým theorem for disjointness preserving orthogonally additive operators*, Operator theory and differential equations, Trends Math., Birkhäuser/Springer, Cham, [2021] ©2021, pp. 155–161, DOI 10.1007/978-3-030-49763-7_13. MR4221149

[11] Helmut H. Schaefer, *Banach lattices and positive operators*, Die Grundlehren der mathematischen Wissenschaften, Band 215, Springer-Verlag, New York-Heidelberg, 1974. MR0423039

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