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A QUADRUPLE INTEGRAL INVOLVING THE PRODUCT OF GENERALIZED PARABOLIC CYLINDER FUNCTIONS $D_v(\beta x)D_u(\alpha z)$: DERIVATION AND EVALUATION

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Abstract. The aim of the present document is to evaluate a quadruple integral involving the product of the generalized Parabolic Cylinder functions $D_v(\beta x)D_u(\alpha z)$ expressed in terms of the Hurwitz-Lerch zeta function. Special cases are evaluated in terms of fundamental constants. All the results in this work are new.

1. Significance statement

Parabolic functions are detailed in the book of Buchholz [6] and are used as basic approximating functions in the theory of contour integrals with a coalescing saddle point and an algebraic singularity, and in the theory of differential equations with two coalescing turning points section (12.16) in [4]. The main applications of Parabolic Cylinder functions in mathematical physics arise when solving the Helmholtz equation section (12.17) [4]. Definite integrals of the product of Parabolic functions in the work of diffraction theory are studied in the work of Barr [7] and Malyshev [9] and their properties studies in the work by Sleeman [8]. In this present work we will expand upon current integrals of the product of Parabolic cylinder functions by deriving a quadruple integral involving these functions and express this integral in terms of the Hurwitz-Lerch zeta function. The goal of this derivation is to provide additional integral formula of these functions where these formulae are applicable.

2. Introduction

In this paper we derive the quadruple definite integral given by

$$\begin{aligned}(2.1)\qquad &\int_{\mathbb{R}_+^4} x^{m-1}z^{-m}t^{-m-u+1}y^{m-v}D_u(z\alpha)D_v(x\beta)\\ &\log^k\left(\frac{axy}{tz}\right)e^{-b\left(t^2+y^2\right)-\frac{\beta^2x^2}{4}-\frac{1}{4}\alpha^2z^2}dxdydzdt\end{aligned}$$

where the parameters $k, a, \alpha, \beta, u, v, m$ are general complex numbers and $Re(b) > 0, Re(v) < Re(m)$. This definite integral will be used to derive special cases in terms of special functions and fundamental constants. The derivations follow the

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method used by us in [1]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw \tag{2.2}$$

where C is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of x, y, z and t, then take a definite quadruple integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (2.2) by another function of x, y, z and t and take the infinite sums of both sides such that the contour integral of both equations are the same.

3. Definite integral of the contour integral

We use the method in [1]. The variable of integration in the contour integral is $r = w + m$. The cut and contour are in the first quadrant of the complex r-plane. The cut approaches the origin from the interior of the first quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form the quadruple integral by replacing y by

$$\log\left(\frac{axy}{tz}\right) \tag{3.1}$$

and multiplying by

$$x^{m-1}z^{-m}t^{-m-u+1}y^{m-v}D_u(z\alpha)D_v(x\beta)e^{-b\left(t^2+y^2\right)-\frac{\beta^2x^2}{4}-\frac{1}{4}\alpha^2z^2} \tag{3.2}$$

then taking the definite integral with respect to $x \in [0,\infty)$, $y \in [0,\infty)$, $z \in [0,\infty)$ and $t \in [0,\infty)$ to obtain

(3.3)
$$\begin{aligned}
&\frac{1}{\Gamma(k+1)}\int_{\mathbb{R}_+^4} x^{m-1}z^{-m}t^{-m-u+1}y^{m-v}D_u(z\alpha)D_v(x\beta) \\
&\quad \log^k\left(\frac{axy}{tz}\right)e^{-b\left(t^2+y^2\right)-\frac{\beta^2x^2}{4}-\frac{1}{4}\alpha^2z^2}dxdydzdt \\
&\quad = \frac{1}{2\pi i}\int_{\mathbb{R}_+^4}\int_C a^w w^{-k-1}x^{m+w-1}z^{-m-w}t^{-m-u-w+1}y^{m-v+w} \\
&\quad D_u(z\alpha)D_v(x\beta)e^{-b\left(t^2+y^2\right)-\frac{\beta^2x^2}{4}-\frac{1}{4}\alpha^2z^2}dwdxdydzdt \\
&\quad = \frac{1}{2\pi i}\int_C\int_{\mathbb{R}_+^4} a^w w^{-k-1}x^{m+w-1}z^{-m-w}t^{-m-u-w+1}y^{m-v+w} \\
&\quad D_u(z\alpha)D_v(x\beta)e^{-b\left(t^2+y^2\right)-\frac{\beta^2x^2}{4}-\frac{1}{4}\alpha^2z^2}dxdydzdtdw \\
&\quad = \frac{1}{2\pi i}\int_C \pi^2a^w w^{-k-1}2^{\frac{1}{2}(u+v-5)}b^{\frac{1}{2}(u+v-3)}\alpha^{m+w-1}\beta^{-m-w}\csc(\pi(m+w))dw
\end{aligned}$$

from equation (3.9.1.3) in [5] and equation (3.326.2) in [2] where $0 < Re(m+w) < -Re(v)$, $|\arg\alpha| < 3\pi/4$, $|\arg\beta| < 3\pi/4$ and using the reflection formula (8.334.3) in [2] for the Gamma function. We are able to switch the order of integration over x, y, z and t using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C} \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty)$.

4. The Hurwitz-Lerch zeta function and infinite sum of the contour integral

In this section we use Equation (2.2) to derive the contour integral representations for the Hurwitz-Lerch zeta function.

4.1. **The Hurwitz-Lerch zeta function.**

The Hurwitz-Lerch zeta function (25.14) in [4] has a series representation given by

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}(v+n)^{-s}z^{n} \tag{4.1}$$

where $|z|<1, v\neq 0,-1,..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{e^{t}-z}dt \tag{4.2}$$

where $Re(v)>0$, and either $|z|\leq 1, z\neq 1, Re(s)>0$, or $z=1, Re(s)>1$.

4.2. **Infinite sum of the contour integral.**

Using equation (2.2) and replacing y by

$$\log(a)+\log(\alpha)-\log(\beta)+i\pi(2y+1) \tag{4.3}$$

then multiplying both sides by

$$-i\pi^2\alpha^{m-1}\beta^{-m}e^{i\pi m(2y+1)}2^{\frac{1}{2}(u+v-5)+1}b^{\frac{1}{2}(u+v-3)} \tag{4.4}$$

taking the infinite sum over $y\in[0,\infty)$ and simplifying in terms of the Hurwitz-Lerch zeta function we obtain

(4.5)

$$\begin{aligned}
&-\frac{1}{\Gamma(k+1)}i^{k+1}\pi^{k+2}e^{i\pi m}\alpha^{m-1}\beta^{-m}b^{\frac{1}{2}(u+v-3)}2^{\frac{1}{2}(2k+u+v-3)}\\
&\Phi\left(e^{2im\pi},-k,\frac{-i\log(a)-i\log(\alpha)+i\log(\beta)+\pi}{2\pi}\right)\\
&=-\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C i\pi^2a^ww^{-k-1}2^{\frac{1}{2}(u+v-3)}b^{\frac{1}{2}(u+v-3)}\alpha^{m+w-1}\beta^{-m-w}e^{i\pi(2y+1)(m+w)}dw\\
&=-\frac{1}{2\pi i}\int_C i\pi^2a^ww^{-k-1}2^{\frac{1}{2}(u+v-3)}b^{\frac{1}{2}(u+v-3)}\alpha^{m+w-1}\beta^{-m-w}\\
&\quad\cdot\sum_{y=0}^{\infty}e^{i\pi(2y+1)(m+w)}dw\\
&=\frac{1}{2\pi i}\int_C \pi^2a^ww^{-k-1}2^{\frac{1}{2}(u+v-5)}b^{\frac{1}{2}(u+v-3)}\alpha^{m+w-1}\beta^{-m-w}\csc(\pi(m+w))dw
\end{aligned}$$

from equation (1.232.3) in [2] where $Im(\pi(m+w))>0$ in order for the sum to converge.

5. Definite integral in terms of the Lerch function

Theorem 5.1. *For all $k,a,\alpha,\beta,u,v,m \in \mathbb{C}, Re(b) > 0, Re(v) < Re(m)$,*

$$\int_{\mathbb{R}_+^4} x^{m-1}z^{-m}t^{-m-u+1}y^{m-v}D_u(z\alpha)D_v(x\beta)\log^k\left(\frac{axy}{tz}\right) e^{-b\left(t^2+y^2\right)-\frac{\beta^2x^2}{4}-\frac{1}{4}\alpha^2z^2}dxdydzdt = -ii^k\pi^{k+2}e^{i\pi m}\alpha^{m-1}\beta^{-m}b^{\frac{1}{2}(u+v-3)}2^{\frac{1}{2}(2k+u+v-3)} \Phi\left(e^{2im\pi},-k,\frac{-i\log(a)-i\log(\alpha)+i\log(\beta)+\pi}{2\pi}\right) \tag{5.1}$$

Proof. The right-hand sides of relations (3.3) and (4.5) are identical; hence, the left-hand sides of the same are identical too. Simplifying with the Gamma function yields the desired conclusion. □

Example 5.2. The degenerate case.

$$\int_{\mathbb{R}_+^4} x^{m-1}z^{-m}t^{-m-u+1}y^{m-v}D_u(z\alpha)D_v(x\beta) e^{-b\left(t^2+y^2\right)-\frac{\beta^2x^2}{4}-\frac{1}{4}\alpha^2z^2}dxdydzdt = \pi^2\alpha^{m-1}\beta^{-m}\csc(\pi m)2^{\frac{1}{2}(u+v-5)}b^{\frac{1}{2}(u+v-3)} \tag{5.2}$$

Proof. Use equation (5.1) and set $k=0$ and simplify using entry (2) in Table below (64:12:7) in [3]. □

Example 5.3. The Hurwitz zeta function $\zeta(s,v)$,

$$\int_{\mathbb{R}_+^4} \frac{t^{\frac{1}{2}-u}y^{\frac{1}{2}-v}D_u(z\alpha)D_v(x\beta)\log^k\left(\frac{axy}{tz}\right)e^{-b\left(t^2+y^2\right)-\frac{\beta^2x^2}{4}-\frac{1}{4}\alpha^2z^2}}{\sqrt{x}\sqrt{z}}dxdydzdt = \frac{1}{\sqrt{\alpha}\sqrt{\beta}}i^k\pi^{k+2}b^{\frac{1}{2}(u+v-3)}2^{\frac{1}{2}(2k+u+v-3)} \left(2^k\zeta\left(-k,\frac{-i\log(a)-i\log(\alpha)+i\log(\beta)+\pi}{4\pi}\right) - 2^k\zeta\left(-k,\frac{1}{2}\left(\frac{-i\log(a)-i\log(\alpha)+i\log(\beta)+\pi}{2\pi}+1\right)\right)\right) \tag{5.3}$$

Proof. Use equation (5.1) set $m=1/2$ and simplify in terms of the Hurwitz zeta function $\zeta(s,v)$ using entry (4) in Table below (64:12:7) in [3]. □

Example 5.4. The digamma function $\psi^{(0)}(x)$,

$$\int_{\mathbb{R}_+^4} \frac{t^{\frac{1}{2}-u}y^{\frac{1}{2}-v}D_u(z\alpha)D_v(x\beta)e^{-b\left(t^2+y^2\right)-\frac{\beta^2x^2}{4}-\frac{1}{4}\alpha^2z^2}}{\sqrt{x}\sqrt{z}\log\left(\frac{axy}{tz}\right)}dxdydzdt = \frac{1}{\sqrt{\alpha}\sqrt{\beta}}i\pi 2^{\frac{1}{2}(u+v-7)}b^{\frac{1}{2}(u+v-3)}\left(\psi^{(0)}\left(\frac{-i\log(a)-i\log(\alpha)+i\log(\beta)+\pi}{4\pi}\right) - \psi^{(0)}\left(\frac{-i\log(a)-i\log(\alpha)+i\log(\beta)+3\pi}{4\pi}\right)\right) \tag{5.4}$$

Proof. Use equation (5.3) and apply l'Hopital's rule as $k \to -1$ and simplify using equation (64:4:1) in [3]. □

6. Invariant index form

In this section we will derive an integral form with the invariance of the indices u and v under the right-hand side of equation (5.1).

Example 6.1.

$$\int_{\mathbb{R}^4_+} \frac{t^{\frac{1}{2}-u} y^{\frac{1}{2}-v} D_u(z) D_v(x) e^{\frac{1}{4}\left(-2t^2-x^2-2y^2-z^2\right)}}{\sqrt{x}\sqrt{z}\left(\log^2\left(\frac{xy}{tz}\right)+\pi^2\right)} dxdydzdt = \frac{\log(2)}{2} \tag{6.1}$$

and

$$\int_{\mathbb{R}^4_+} \frac{t^{\frac{1}{2}-u} y^{\frac{1}{2}-v} D_u(z) D_v(x) e^{-\frac{t^2}{2}-\frac{x^2}{4}-\frac{y^2}{2}-\frac{z^2}{4}} \log\left(\frac{xy}{tz}\right)}{\sqrt{x}\sqrt{z}\left(\log^2\left(\frac{xy}{tz}\right)+\pi^2\right)} dxdydzdt = 0 \tag{6.2}$$

Proof. Use equation (5.4) and set $a = -1, \alpha = \beta = 1, b = 1/2$ then rationalize the denominator and equate real and imaginary parts and simplify. □

7. Discussion

In this paper, we have presented a novel method for deriving a new integral transform involving the product of generalized Parabolic Cylinder functions $D_v(\beta x) D_u(\alpha z)$ along with some interesting definite integrals using contour integration.

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