PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 9, Pages 180–185 (April 20, 2022)
https://doi.org/10.1090/bproc/82

DETECTING MOTIVIC EQUIVALENCES WITH MOTIVIC HOMOLOGY

DAVID HEMMINGER

(Communicated by Julie Bergner)

ABSTRACT. Let k be a field, let R be a commutative ring, and assume the exponential characteristic of k is invertible in R. In this note, we prove that isomorphisms in Voevodsky's triangulated category of motives $\mathcal{DM}(k; R)$ are detected by motivic homology groups of base changes to all separable finitely generated field extensions of k. It then follows from previous conservativity results that these motivic homology groups detect isomorphisms between certain spaces in the pointed motivic homotopy category $\mathcal{H}(k)_*$.

Let k be a field, let R be a commutative ring, and assume the exponential characteristic of k is invertible in R. Let $\mathcal{DM}(k; R)$ denote Voevodsky's triangulated category of motives over k with coefficients in R.

In a failed analogy with topology, motivic homology groups do not detect isomorphisms in $\mathcal{DM}(k; R)$ (see Section 1). However, it is often possible to work in a context partially agnostic to the base field k. In this note, we prove a detection result for those circumstances.

Theorem 1. *Let $\varphi\colon M \to N$ be a morphism in $\mathcal{DM}(k; R)$. Suppose that either*

(a) *For every separable finitely generated field extension F/k, the induced map on motivic homology $\mathrm{H}_*(M_F, R(*)) \to \mathrm{H}_*(N_F, R(*))$ is an isomorphism, or*

(b) *Both M and N are compact, and for every separable finitely generated field extension F/k, the induced map on motivic cohomology $\mathrm{H}^*(N_F, R(*)) \to \mathrm{H}^*(M_F, R(*))$ is an isomorphism.*

Then φ is an isomorphism.

Note that if X is a separated scheme of finite type over k, then the motive $\mathrm{M}(X)$ of X in $\mathcal{DM}(k; R)$ is compact, by [6, Theorem 11.1.13].

The proof is easy: following a suggestion of Bachmann, we reduce to the fact (Proposition 4) that a morphism in $\mathcal{SH}(k)$ which induces an isomorphism on homotopy sheaves evaluated on all separable finitely generated field extensions must be an isomorphism. This follows immediately from Morel's work on unramified presheaves (see [10]).

Together with existing results, Theorem 1 detects isomorphisms between certain spaces in the pointed motivic homotopy category $\mathcal{H}(k)_*$. Wickelgren-Williams

Received by the editors December 15, 2020, and, in revised form, February 4, 2021.
2020 *Mathematics Subject Classification.* Primary 14F42, 14C15.
This work was partially supported by National Science Foundation grant DMS-1701237.

showed that the functor

$$\Sigma^\infty_{S^1}\colon \mathcal{H}(k)_* \to \mathcal{SH}^{S^1}(k)$$

is conservative when restricted to the subcategory of $\mathbb{A}^1$-1-connected spaces [15, Corollary 2.23]. Bachmann proved that the functor

$$\Sigma^\infty_{\mathbb{G}_m}\colon \mathcal{SH}^{S^1}(k)_{\geq 0} \cap \mathcal{SH}^{S^1}(k)(n) \to \mathcal{SH}(k)$$

is conservative when $\mathrm{char}(k) = 0$ and $n = 1$ or k is perfect and $n = 3$ [3, Corollary 4.15], and Feld showed that $\Sigma^\infty_{\mathbb{G}_m}$ is conservative when $n = 2$ and k is an infinite perfect field of characteristic not 2 [7, Theorem 6]. Here $\mathcal{SH}^{S^1}(k)(n) \subseteq \mathcal{SH}^{S^1}(k)$ denotes the full subcategory generated under homotopy colimits by the objects $X_+ \wedge \mathbb{G}_m^{\wedge n}$, for X a smooth scheme over k. By another theorem of Bachmann, the functor

$$\mathrm{M}\colon \mathcal{SH}^c(k) \to \mathcal{DM}(k;\mathbb{Z})$$

is conservative when k is a perfect field of finite 2-étale cohomological dimension, where $\mathcal{SH}^c(k) \subseteq \mathcal{SH}(k)$ denotes the full subcategory of compact objects [2, Theorem 16]. Using Bachmann's results, Totaro showed that if k is a finitely generated field of characteristic zero, $X \in \mathcal{SH}(k)^c$, $\mathrm{M}(X) = 0$ in $\mathcal{DM}(k,\mathbb{Z})$, and $\mathrm{H}_*(X(\mathbb{R}), \mathbb{Z}[1/2]) = 0$ for every embedding of k into $\mathbb{R}$, then $X = 0$ [13, Theorem 6.1].

Finally, $\mathcal{DM}(k;\mathbb{Z}[1/p])$ is equivalent to the category with objects the same as $\mathcal{DM}(k;\mathbb{Z})$ and hom-sets tensored by $\mathbb{Z}[1/p]$, by [6, Proposition 11.1.5]. It follows that if $\mathcal{C}$ is a full subcategory of $\mathcal{H}(k)_*$ such that the composition of functors $\mathcal{C} \subseteq \mathcal{H}(k)_* \to \mathcal{DM}(k;\mathbb{Z})$ is conservative, then the composition

$$\mathcal{H}(k)_*[1/p] \cap \mathcal{C} \subseteq \mathcal{H}(k)_* \to \mathcal{DM}(k;\mathbb{Z}) \to \mathcal{DM}(k;\mathbb{Z}[1/p])$$

is conservative. Altogether, we have Corollary 2.

Corollary 2. *Let $\mathcal{H}(k)_*(n), \mathcal{H}(k)_{*,\geq 1}, \mathcal{H}(k)^c_* \subseteq \mathcal{H}(k)_*$ denote the subcategory generated under homotopy colimits by the objects $X_+ \wedge \mathbb{G}_m^{\wedge n}$, for X a smooth scheme over k; the subcategory of $\mathbb{A}^1$-1-connected spaces; and the subcategory of objects sent to compact objects in $\mathcal{SH}(k)$ under stabilization, respectively. Let k be a perfect field with exponential characteristic p. Let $n = 1$ if $p = 1$, $n = 2$ if $p > 2$ and k is infinite, and $n = 3$ otherwise. Let $\varphi\colon X \to Y$ be a morphism in*

$$\mathcal{H}(k)_*(n) \cap \mathcal{H}(k)_{*,\geq 1} \cap \mathcal{H}(k)^c_* \cap \mathcal{H}(k)_*[1/p].$$

Suppose that either

(a) *k has finite 2-étale cohomological dimension, or*
(b) *k is a finitely generated field of characteristic zero, and for every embedding of k into $\mathbb{R}$, the induced map*

$$\mathrm{H}_*(X(\mathbb{R}), \mathbb{Z}[1/2]) \to \mathrm{H}_*(Y(\mathbb{R}), \mathbb{Z}[1/2])$$

is an isomorphism.

Suppose further that either

(a) *for every finitely generated field F/k, the induced map on motivic homology $\mathrm{H}_*(X_F, \mathbb{Z}(*)) \to \mathrm{H}_*(Y_F, \mathbb{Z}(*))$ is an isomorphism, or*
(b) *for every finitely generated field F/k, the induced map on motivic cohomology $\mathrm{H}^*(Y_F, \mathbb{Z}(*)) \to \mathrm{H}^*(X_F, \mathbb{Z}(*))$ is an isomorphism.*

Then φ is an isomorphism in $\mathcal{H}(k)_*$.

Remark 3. Suppose X is a smooth separated scheme of finite type over k. Then

$$X_+ \wedge S^{m+n,n} \in \mathcal{H}(k)_*(n) \cap \mathcal{H}(k)_{*,\geq 1} \cap \mathcal{H}(k)_*^c,$$

where $m = 0$ if X is $\mathbb{A}^1$-1-connected, $m = 1$ if X is $\mathbb{A}^1$-connected, and $m = 2$ otherwise. Localization away from a prime p is well-behaved on $\mathbb{A}^1$-1-connected spaces; see [1]. Perfect fields of finite 2-étale cohomological dimension include algebraically closed fields, finite fields, and totally imaginary number fields.

1. The derived category of motives

We recall here a few basic facts about Voevodsky's derived category of motives $\mathcal{DM}(k;R)$. A more thorough, but still concise, summary of basic properties of $\mathcal{DM}(k;R)$ is [12, Section 5]. Further references on $\mathcal{DM}(k;R)$ are [14] and [6].

The category $\mathcal{DM}(k;R)$ is defined in [6, Definition 11.1.1]. It is tensor triangulated with tensor unit R, and it has arbitrary (not necessarily finite) direct sums. The motive and the motive with compact support define covariant functors M and M^c from separated schemes of finite type over k to $\mathcal{DM}(k;R)$. We write $R(i)[2i]$ for the *Tate motive* $\mathrm{M}^c(A^i_k)$. The object $R(1)$ is invertible in $\mathcal{DM}(k;R)$; we define $R(-i) = R(i)^*$. For $M \in \mathcal{DM}(k;R)$, we write $\mathrm{M}(i) = M \otimes R(i)$.

For any motive $M \in \mathcal{DM}(k)$, the functor $M \otimes -$ has a right adjoint $\underline{\mathrm{Hom}}(M,-)$. By [12, Lemma 5.5], $\underline{\mathrm{Hom}}(M,N) \cong M^* \otimes N$ and $(M^*)^* \cong M$ whenever M is compact. If X is a smooth scheme over k of pure dimension n, a version of Poincaré duality says that $\mathrm{M}(X)^* \cong \mathrm{M}^c(X)(-n)[-2n]$.

Let $HR \in \mathcal{SH}(k)$ denote the Eilenberg-Maclane spectrum of R. If k is perfect, the category of HR-module spectra in $\mathcal{SH}(k)$ is equivalent to $\mathcal{DM}(k;R)$ by [11, Theorem 1.1] and [8, Theorem 5.8]. There is a functor $\mathcal{SH}(k) \to \mathcal{DM}(k;R)$ that can be viewed as smashing with HR; it is left adjoint to the forgetful functor $H\colon \mathcal{DM}(k;R) \to \mathcal{SH}(k)$. The functor $\mathrm{M}\colon \mathrm{Sm}_k \to \mathcal{DM}(k;R)$ factors through $\mathcal{SH}(k)$ as $(HR \wedge -) \circ \Sigma^\infty$.

For $M \in \mathcal{DM}(k;R)$, the motivic homology and motivic cohomology of M are defined by

$$\mathrm{H}_j(M, R(i)) = \mathrm{Hom}(R(i)[j], M)$$

and

$$\mathrm{H}^j(M, R(i)) = \mathrm{Hom}(M, R(i)[j]).$$

Motivic (co)homology does not detect isomorphisms in $\mathcal{DM}(k;R)$, by the following argument (see also [12, Section 7]). Let $\mathcal{DMT}(k;R)$ denote the localizing subcategory of mixed Tate motives, i.e. the smallest localizing subcategory containing the motives $R(i)$ for all integers i. Let C denote the composition $\mathcal{DM}(k;R) \to \mathcal{DMT}(k;R) \to \mathcal{DM}(k;R)$ of the inclusion $\mathcal{DMT}(k;R) \to \mathcal{DM}(k;R)$ with its right adjoint. For $M \in \mathcal{DM}(k;R)$, the counit morphism $C(M) \to M$ always induces an isomorphism on motivic homology, but is only an isomorphism if $M \in \mathcal{DMT}(k;R)$.

Motivic homology does detect isomorphisms in $\mathcal{DMT}(k;R)$. It is also conjectured (see [4, 1.2(A)]) that for l a prime invertible in k the étale realization $\mathcal{DM}(k;\mathbb{Q}) \to \mathcal{D}(\mathbb{Q}_l)$ is conservative after restricting to the subcategory of compact objects in $\mathcal{DM}(k;\mathbb{Q})$. Note that the étale realization functor $\mathcal{DM}(k;\mathbb{F}_l) \to \mathcal{D}(\mathbb{F}_l)$ is not conservative on compact objects: a primitive lth root of unity defines a nontrivial element of $\mathrm{Hom}(\mathbb{F}_l, \mathbb{F}_l(1)) = \mu_l(k)$ that is inverted under étale realization.

2. Proof of Theorem 1

We reduce the theorem to Proposition 4, which follows immediately from work of Morel on unramified presheaves (see [10]).

Proposition 4. *Suppose that for a morphism of spectra $\psi\colon E_1 \to E_2$ in $\mathcal{SH}(k)$ the induced map $\pi_{*,*}(E_1)(F) \to \pi_{*,*}(E_2)(F)$ is an isomorphism for all separable finitely generated field extensions F/k. Then ψ is an isomorphism in $\mathcal{SH}(k)$.*

Proof. A presheaf of sets S on Sm_k is *unramified* if for any $X \in \mathrm{Sm}_k$, $S(X) \cong \prod S(X_\alpha)$, where the X_α are the irreducible components of X, and for any dense open subscheme $U \subset X$, the restriction map $S(X) \to S(U)$ is injective and moreover an isomorphism if $X - U$ has codimension at least 2 everywhere. For a space $Y \in \mathcal{H}(k)_*$, the unstable homotopy sheaf $\pi_n^{\mathbb{A}^1}(Y)$ is unramified for $n \geq 2$, by [10, Theorem 9, Example 1.3] (see also [10, Remark 17]).

For $E \in \mathcal{SH}(k)$,

$$\pi_{i,j}(E) \cong \pi_2^{\mathbb{A}^1}(\Omega^\infty(S^{2-i,-j} \wedge E)),$$

so it is unramified. In particular, for X a smooth variety over k, $\pi_{i,j}(E)(X)$ injects into $\pi_{i,j}(E)(\mathrm{Spec}(k(X)))$. (If S is a presheaf on Sm_k and $Y \cong \lim_\alpha Y_\alpha$ in the category of schemes over k with $Y_\alpha \in \mathrm{Sm}_k$, then $S(Y) = \mathrm{colim}_\alpha(Y_\alpha)$ by definition.) It follows that the unstable homotopy sheaves of the cone of ψ vanish, so ψ is an isomorphism. $\square$

If $M \in \mathcal{DM}(k; R)$ is compact, then $\mathrm{Hom}(M, R(i)[j]) \cong \mathrm{Hom}(R(-i)[-j], M^*)$ and $(M^*)^* \cong M$, so it suffices to prove the theorem with the assumption (a). Let k^{perf} denote the perfect closure of k. By [5, Proposition 8.1], the pullback functor $\mathcal{DM}(k; R) \to \mathcal{DM}(k^{\mathrm{perf}}; R)$ is an equivalence, so it suffices to prove the theorem when k is perfect. Indeed, if $k^{\mathrm{perf}}(a_1, \dots, a_n)$ is a finitely generated field extension of k^{perf}, let F be the separable closure of k in $k(a_1, \dots, a_n)$. Then F is a finitely generated separable field extension of k and $F^{\mathrm{perf}} = (k^{\mathrm{perf}}(a_1, \dots, a_n))^{\mathrm{perf}}$, so the assumption (a) for the base field k^{perf} follows from the assumption (a) for k.

The functor H is conservative by [8, Lemma 2.4], so it suffices to show that HC is zero, where C is the cone of φ. By the hypotheses of the theorem we have that $\mathrm{H}_*(C_{k(X)}, R(*)) = 0$, so the result follows from Proposition 4 and Lemma 5.

Lemma 5. *Let $M \in \mathcal{DM}(k; R)$, and let X be a smooth variety over k. Then*

$$\pi_{i,j}(HM)(\mathrm{Spec}(k(X))) \cong \mathrm{H}_{i+2n}(M_{k(X)}, R(j+n)),$$

where $n = \dim X$.

Proof. We have that

$$\pi_{i,j}(HM)(\mathrm{Spec}(k(X))) = \mathrm{colim}_S\, \pi_{i,j}(HM)(X - S)$$

by definition, where the colimit ranges over all closed subschemes $S \subsetneq X$. Sheafification does not change values on stalks, so by Poincaré duality and the fact that $\mathrm{M}(X - S)$ is compact,

$$\mathrm{colim}_S\, \pi_{i,j}(HM)(X - S) \cong \mathrm{colim}_S\, \mathrm{H}_{j+2n}(\mathrm{M}^c(X - S) \otimes M, R(i+n)).$$

Thus it suffices to show that

$$\mathrm{colim}_S\, \mathrm{H}_*(\mathrm{M}^c(X - S) \otimes M, R(*)) \cong \mathrm{H}_*(M_{k(X)}, R(*)).$$

The property of M that both sides are isomorphic is preserved under arbitrary direct sums. Indeed, if $f\colon \mathrm{Spec}(k(X)) \to \mathrm{Spec}(k)$ is the map induced by the field extension, then $M_{k(X)} = f^*M$. For any morphism of schemes, the associated pullback functor on derived categories of motives has a right adjoint [6, Theorem B.1]. Since f^* and $\mathrm{M}^c(X-S)\otimes -$ are left adjoints, they commute with arbitrary direct sums. Motivic homology commutes with arbitrary direct sums because its representing objects $R(i)[j]$ are compact.

Moreover, if two motives in an exact triangle satisfy the property, then the third does as well, by the long exact sequence in motivic homology for an exact triangle. Thus it suffices to prove the lemma when $M = \mathrm{M}^c(Y)$, where Y is a smooth projective scheme over k. In that case, $M_{k(X)} = \mathrm{M}^c(Y_{k(X)})$, and the motivic homology groups in the lemma are isomorphic to higher Chow groups by [14, Proposition 4.2.9] and [9, Theorem 5.3.14]. Thus it suffices to show that

$$\operatorname{colim}_S \mathrm{CH}^*(Y \times_k (X-S), *; R) \cong \mathrm{CH}^*(Y_{k(X)}, *; R). \tag{1}$$

In fact, (1) holds at the level of cycles: Let

$$\Delta^n = \mathrm{Spec}\left(k[x_0,\ldots,x_n]/\left(\sum_{i=0}^n x_i = 0\right)\right).$$

For every subset $\{i_0,\ldots,i_m\} \subseteq \{0,\ldots,n\}$, there is an associated *face* $\Delta^m \subseteq \Delta^n$. For a smooth scheme T, let $z^*(T,n)$ denote the subgroup of algebraic cycles in $T\times\Delta^n$ generated by the subvarieties which intersect $T\times\Delta^m$ in the expected dimension for each face $\Delta^m\subseteq\Delta^n$. The higher Chow groups $\mathrm{CH}^*(T,*;R)$ are the homology groups of a complex

$$\ldots \to z^*(T,2)\otimes R \to z^*(T,1)\otimes R \to z^*(T,0)\otimes R \to 0.$$

The isomorphism (1) follows from the fact that

$$\operatorname{colim} z^*(Y\times_k (X-S), n) \cong z^*(Y_{k(X)}, n)$$

for all n. □

Acknowledgments

I thank Burt Totaro for suggesting something like Theorem 1 might be true and for pointing me to [12]. I thank Tom Bachmann for suggesting the use of Morel's work on unramified sheaves, which shortened the proof of Theorem 1 from a prior version. I thank Martin Gallauer for his comments on an earlier draft, including pointing out why the functor $\mathcal{DM}(k;\mathbb{F}_l)\to\mathcal{D}(\mathbb{F}_l)$ is not conservative on compact objects.

References

[1] A. Asok, J. Fasel, and M. J. Hopkins, *Localization and nilpotent spaces in $\mathbb{A}^1$-homotopy theory*, arXiv:1909.05185, 2019.

[2] Tom Bachmann, *On the conservativity of the functor assigning to a motivic spectrum its motive*, Duke Math. J. **167** (2018), no. 8, 1525–1571, DOI 10.1215/00127094-2018-0002. MR3807316

[3] T. Bachmann, The zeroth $\mathbb{P}^1$-stable homotopy sheaf of a motivic space. arXiv:2003.12021, 2020.

[4] Alexander Beilinson, *Remarks on n-motives and correspondences at the generic point*, Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), Int. Press Lect. Ser., vol. 3, Int. Press, Somerville, MA, 2002, pp. 35–46. MR1977583

[5] Denis-Charles Cisinski and Frédéric Déglise, *Integral mixed motives in equal characteristic*, Doc. Math. **Extra vol.: Alexander S. Merkurjev's sixtieth birthday** (2015), 145–194. MR3404379

[6] Denis-Charles Cisinski and Frédéric Déglise, *Triangulated categories of mixed motives*, Springer Monographs in Mathematics, Springer, Cham, [2019] ©2019, DOI 10.1007/978-3-030-33242-6. MR3971240

[7] Niels Feld, *Milnor-Witt homotopy sheaves and Morel generalized transfers*, Adv. Math. **393** (2021), Paper No. 108094, 46, DOI 10.1016/j.aim.2021.108094. MR4339534

[8] Marc Hoyois, Shane Kelly, and Paul Arne Østvær, *The motivic Steenrod algebra in positive characteristic*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 12, 3813–3849, DOI 10.4171/JEMS/754. MR3730515

[9] Shane Kelly, *Voevodsky motives and ldh-descent* (English, with English and French summaries), Astérisque **391** (2017), 125. MR3673293

[10] Fabien Morel, $\mathbb{A}^1$*-algebraic topology over a field*, Lecture Notes in Mathematics, vol. 2052, Springer, Heidelberg, 2012, DOI 10.1007/978-3-642-29514-0. MR2934577

[11] Oliver Röndigs and Paul Arne Østvær, *Modules over motivic cohomology*, Adv. Math. **219** (2008), no. 2, 689–727, DOI 10.1016/j.aim.2008.05.013. MR2435654

[12] Burt Totaro, *The motive of a classifying space*, Geom. Topol. **20** (2016), no. 4, 2079–2133, DOI 10.2140/gt.2016.20.2079. MR3548464

[13] Burt Totaro, *Torus actions, Morse homology, and the Hilbert scheme of points on affine space* (English, with English and French summaries), Épijournal Géom. Algébrique **5** (2021), Art. 12, 14, DOI 10.1109/lcsys.2020.2999988. MR4315780

[14] Vladimir Voevodsky, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 188–238. MR1764202

[15] Kirsten Wickelgren and Ben Williams, *The simplicial EHP sequence in* $\mathbb{A}^1$*-algebraic topology*, Geom. Topol. **23** (2019), no. 4, 1691–1777, DOI 10.2140/gt.2019.23.1691. MR3981318

Department of Mathematics, UCLA, Box 951555, Los Angeles, California 90095-1555
Email address: dhemminger22@gmail.com