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THE FUNDAMENTAL SOLUTION TO $\Box_b$ ON QUADRIC MANIFOLDS – PART 1. GENERAL FORMULAS

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Abstract. This paper is the first of a three part series in which we explore geometric and analytic properties of the Kohn Laplacian and its inverse on general quadric submanifolds of $\mathbb{C}^n \times \mathbb{C}^m$. In this paper, we present a streamlined calculation for a general integral formula for the complex Green operator N and the projection onto the nullspace of $\Box_b$. The main application of our formulas is the critical case of codimension two quadrics in $\mathbb{C}^4$ where we discuss the known solvability and hypoellipticity criteria of Peloso and Ricci [J. Funct. Anal. 203 (2003), pp. 321–355] We also provide examples to show that our formulas yield explicit calculations in some well-known cases: the Heisenberg group and a Cartesian product of Heisenberg groups.

1. Introduction

The goal of this paper is to present an explicit integral formula for the complex Green operator and the projection onto the null space of the Kohn Laplacian on quadric submanifolds of $\mathbb{C}^n \times \mathbb{C}^m$. Our result generalizes the formula of [BR13] from the specific case of codimension 2 quadrics in $\mathbb{C}^4$ to the general case of arbitrary n and m, and we also prove a formula for the complex Green operator when it is only a relative inverse of the Kohn Laplacian and not a full inverse. Additionally, the new proof is significantly simpler and uses our calculation of the $\Box_b$-heat kernel on general quadrics [BR11]. We then provide several applications of our formula to the case of codimension 2 quadrics in $\mathbb{C}^4$ and provide context for how this case fits into the solvability/hypoellipticity framework of Peloso and Ricci [PR03]. We also provide a calculation of the complex Green operator in several instances when it is only a relative fundamental solution: $\Box_b$ on the Heisenberg group on functions and $(0, n)$-forms, and on the Cartesian product of Heisenberg groups at the top degree. We conclude with a computation of the Szegö projection on the Cartesian product of Heisenberg groups.

This paper is the first of a series where we explore the geometry and analysis of the Kohn Laplacian $\Box_b$ and its (relative) inverse, the complex Green operator, on quadric submanifolds in $\mathbb{C}^n \times \mathbb{C}^m$. The $\Box_b$-equation, $\Box_b u = f$, governs the behavior of boundary values of holomorphic functions, and the $\Box_b$-operator is a naturally occurring, nonconstant coefficient, nonelliptic operator. Solving the

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equation has been tantalizing mathematicians for the better part of fifty years, and while much is known for solvability/regularity in L^p and other spaces (especially L^2) for hypersurface type CR manifolds, there has been much less work done to determine the structure of the complex Green operator, denoted by N, especially on non-hypersurface type CR manifolds. The problem is that the techniques used to solve the equation are functional analytic in nature and therefore nonconstructive. Consequently, to have any hope of finding an explicit solution, we need additional structure on the CR manifold. From our perspective, the gold standard for results in this area is the calculation by Folland and Stein [FS74] in which they find a beautiful, closed form expression for N on the Heisenberg group.

In the decades following [FS74], mathematicians developed machinery to solve the $\Box_b$-heat equation (or the heat equation associated to the sub-Laplacian) on certain Lie groups. From the formulas for the heat equations, in principle, it is only a matter of integrating the time variable out to recover the formula for N. The first results for the heat equation were for the sub-Laplacian on the Heisenberg group by Hulanicki [Hul76] and Gaveau [Gav77]. More results followed in a similar vein, and the vast majority rely on the group Fourier transform and Hermite functions [PR03, BR09, BR11, YZ08, CCT06, BGG96, BGG00, Eld09]. The problem with these techniques is that the formulas that they generate for the heat kernel are only given up to a partial Fourier transform that is uncomputable in practice. Consequently, any information giving precise size estimates or asymptotics, let alone a formula in the spirit of Folland and Stein, is absent.

A quadric submanifold $M \subset \mathbb{C}^n \times \mathbb{C}^m$ is a CR manifold of the form

$$M = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \operatorname{Im} w = \phi(z, z)\}, \tag{1}$$

where $\phi : \mathbb{C}^n \times \mathbb{C}^n \mapsto \mathbb{C}^m$ is a sesquilinear form (i.e., $\phi(z, z') = \overline{\phi(z', z)}$). The fundamental solution to $\Box_b$ or the sub-Laplacian on quadrics has been studied by many authors, including [BG88, BGG96, BR09, BR13, CCT06, FS74, PR03]. See also Part II of the series [BR20], in which we find useable sufficient conditions for a map T between quadrics to be a $\Box_b$-preserving Lie group isomorphism as well as establish a framework for which appropriate derivatives of the complex Green operator are continuous in L^p and L^p-Sobolev spaces (and hence are hypoelliptic). We apply the general results to codimension 2 quadrics in $\mathbb{C}^4$.

There are two higher codimension papers that need mentioning. First, in [NRS01], Nagel et al. analyze $\Box_b$ and its geometry in the special class of decoupled quadrics where $\operatorname{Im} w_j = \sum_{k=1}^n a_k^j |z_k|^2$. However, many of the interesting cases do not fall into this category. Second, Raich and Tinker compute the Szegö kernel for the polynomial model

$$M = \big\{(z, w) \in \mathbb{C} \times \mathbb{C}^n : \operatorname{Im} w = (a_1, \dots, a_{n-1}, 1)p(\operatorname{Re} z)\big\},$$

where $p : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\lim_{|x|\to\infty} \frac{p(x)}{|x|} = \infty$, and the constants $a_1, \dots, a_{n-1}$ are nonzero [RT15]. The authors write an explicit formula for the Szegö kernel based on an integral formula of Nagel [Nag86] and show that there are significant blowups off of the diagonal. Raich and Tinker evaluate all of the integrals in the case $p(x) = x^2$. That is, the CR manifold M is a quadric, but this case is very special because the tangent space (at each point) has only one complex direction, so every degree is either top or bottom. These are often the exceptional cases and can give misleading intuition.

Associated to each quadric is the *Levi form* $\phi(z,z')$ and for each $\lambda \in \mathbb{R}^m$, the *directional Levi form in the direction* λ is $\phi^\lambda(z,z') = \phi(z,z')\cdot\lambda$ where $\cdot$ is the usual dot product without conjugation. For each λ, there is an $n\times n$ matrix A^λ so that

$$\phi^\lambda(z,z') = z^* A^\lambda z'$$

and we identify the eigenvalues of ϕ^λ with the eigenvalues of the matrix A^λ. Here, the $*$ designates the Hermitian transpose.

The outline of the remainder of the paper is as follows: In Section 2, we record the main result for general quadrics and provide additional context. In Section 3, we discuss the CR geometry and Lie group structure of a general quadric. In Section 4, we prove the main result, and we devote Section 5 to explicit examples.

2. Results and discussion

Under the projection $\pi : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n \times \mathbb{R}^m$ given by $\pi(z, t+is) = (z,t)$, we may identify a quadric M with $\mathbb{C}^n\times\mathbb{R}^m$. The projection induces both a CR structure and Lie group structure on $\mathbb{C}^n\times\mathbb{R}^m$, and we denote this Lie group by G (or G_M). Thus the projection is a CR isomorphism and we refer to the pushforwards and pullbacks of objects from M to G without changing the notation.

2.1. The main result – general formulas for the solution of $\Box_b$ on quadrics.

To state the main result, we need to introduce some notation. For each $\lambda\in\mathbb{R}^m\setminus\{0\}$, let $\mu_1^\lambda,\dots,\mu_n^\lambda$ be the eigenvalues of ϕ^λ (or equivalently, the Hermitian symmetric matrix, A^λ) and $v_1^\lambda,\dots,v_n^\lambda$ be an orthonormal set of eigenvectors. This means

$$\phi^\lambda(v_j^\lambda, v_k^\lambda) = \delta_{jk}\mu_j^\lambda. \tag{2}$$

Let $\alpha = \lambda/|\lambda|$, then $\mu_j^\lambda = |\lambda|\mu_j^\alpha$ and $v_j^\lambda = v_j^\alpha$. If $z\in\mathbb{C}^n$ is expressed in terms of the unit eigenvectors of ϕ^λ, then $z_j^\lambda = z_j^\alpha$ is given by

$$z^\alpha = Z(z,\alpha) = (U^\alpha)^*\cdot z,$$

where U^α is the matrix whose columns are the eigenvectors, v_j^α, $1\le j\le n$, and $\cdot$ represents matrix multiplication with z written as a column vector. Note that the corresponding orthonormal basis of $(0,1)$-covectors for this basis is

$$d\bar Z_j(z,\alpha), 1\le j\le n, \text{ where } d\bar Z(z,\alpha) = (U^\alpha)^T\cdot d\bar z,$$

where $d\bar z$ is written as a column vector of $(0,1)$-forms and the superscript T stands for transpose. It is a fact that the eigenvalues, eigenvectors and hence $z^\alpha = Z(z,\alpha)$ depend smoothly on $z\in\mathbb{C}^n$. However, while the dependence of the eigenvalues is continuous (in fact Lipschitz) in α, the eigenvectors may only be functions of bounded variation (SBV) in α [Rai11, Theorem 9.6].

Let $\mathcal{I}_q = \{L = (\ell_1,\dots,\ell_q) : 1\le \ell_1 < \ell_2 < \cdots < \ell_q \le n\}$. For each $K\in\mathcal{I}_q$, we will need to express $d\bar z^K$, in terms of $d\bar Z(z,\alpha)^L$ for $L\in\mathcal{I}_q$. We have

$$d\bar z^K = \sum_{L\in\mathcal{I}_q} C_{K,L}(\alpha)\, d\bar Z(z,\alpha)^L, \tag{3}$$

where $C_{K,L}(\alpha)$ are the appropriate $q\times q$ minor determinants of $\bar U^\alpha$. Note that if $q=n$, then the above sum only has one term and $C_{K,K}(\alpha)=1$. Additionally, when $q=0$, $\mathcal{I}_0=\emptyset$ and the sum (3) does not appear.

Denote by $\nu(\lambda) = \nu(\alpha)$ the number of nonzero eigenvalues of ϕ^λ. For each q-tuple $L \in \mathcal{I}_q$, set

$$\Gamma_L = \{\alpha \in S^{m-1} : \mu_\ell^\alpha > 0 \text{ for all } \ell \in L \text{ and } \mu_\ell^\alpha < 0 \text{ for all } \ell \notin L\}$$

and

$$\varepsilon_{j,L}^\alpha = \begin{cases} \operatorname{sgn}(\mu_j^\lambda) & \text{if } j \in L, \\ -\operatorname{sgn}(\mu_j^\lambda) & \text{if } j \notin L. \end{cases}$$

Remark 2.1. If $\nu = \max_{\lambda \in \mathbb{R}^m} \nu(\lambda)$, then $\{\lambda \in \mathbb{R}^m \setminus \{0\} : \nu(\lambda) = \nu\}$ is a Zariski open set and hence carries full Lebesgue measure. In particular, if one of the sets Γ_L is nonempty, then $\nu = n$. When $\nu < n$, we arrange our eigenvalues so that $\mu_{\nu+1}^\lambda = \cdots \mu_n^\lambda = 0$ and write $z' = (z_1, \dots, z_\nu)$ and $z'' = (z_{\nu+1}, \dots, z_n)$.

Definition 2.2. Given an index in $K \in \mathcal{I}_q$, we say that a current $N_K = \sum_{K' \in \mathcal{I}_q} \tilde{N}_{K'}(z,t) d\bar{z}^{K'}$ is a *fundamental solution to* $\Box_b$ on forms spanned by $d\bar{z}^K$ if $\Box_b N_K = \delta_0(z,t) d\bar{z}^K$.

N_K acts on smooth forms with compact support by componentwise convolution with respect to the group structure on G, that is, if $f = f_0 d\bar{z}^K$, then $N_K * f = \sum_{K' \in \mathcal{I}_q} \tilde{N}_{K'} * f_0 d\bar{z}^{K'}$. Thus if $f = f_0 d\bar{z}^K$ is a smooth form with compact support, then $\Box_b\{N_K * f\} = f$. In cases where $\Box_b$ has a nontrivial kernel, we let S_K be the projection (Szegö) operator onto this kernel and we say that N_K is a *relative fundamental solution* if $\Box_b\{N_K * f\} = f - S_K(f)$ holds for all compactly supported forms spanned by $d\bar{z}^K$. On quadrics, $\Box_b$ never has closed range in L^2 so the complex Green operator cannot be continuous in L^2. As a consequence, we can only discuss *a* relative inverse and not *the* relative inverse. However, a relative inverse is called *canonical* if its output is orthogonal to $\ker \Box_b$ whenever it belongs to L^2.

We can now state our main result.

Theorem 2.3. *Suppose M is a quadric CR submanifold of $\mathbb{C}^{n+m}$ given by* (1) *with associated projection G. Assume the associated heat kernels, $\tilde{H}_L(s, z^\alpha, \hat{\lambda})$ in (10) for all $L \in I_q$ are jointly integrable in the variables $s > 0$ and $\lambda \in \mathbb{R}^m$. Fix $K \in \mathcal{I}_q$.*

(1) *If $|\Gamma_L| = 0$ for all $L \in \mathcal{I}_q$, then the fundamental solution to $\Box_b$ on forms spanned by $d\bar{z}^K$ is given by*

(4)

$$N_K(z,t) = \frac{4^n}{2(2\pi)^{m+n}} \sum_{L \in \mathcal{I}_q} \int_{\alpha \in S^{m-1}} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L \int_{r=0}^1 \frac{1}{|\log r|^{n-\nu(\alpha)}} \prod_{j=1}^{\nu(\alpha)} \frac{r^{\frac{1}{2}(1-\varepsilon_{j,L}^\alpha)|\mu_j^\alpha|}|\mu_j^\alpha|}{1 - r^{|\mu_j^\alpha|}} \frac{(n+m-2)!}{(A_\alpha(r,z) - i\alpha \cdot t)^{n+m-1}} \frac{dr\,d\alpha}{r},$$

where

$$A_\alpha(r,z) = \frac{2}{|\log r|}|z''^\alpha|^2 + \sum_{j=1}^{\nu(\alpha)} |\mu_j^\alpha| \left(\frac{1 + r^{|\mu_j^\alpha|}}{1 - r^{|\mu_j^\alpha|}} \right) |z_j^\alpha|^2.$$

(2) *If $|\Gamma_L| > 0$ for at least one $L \in \mathcal{I}_q$, then orthogonal projection onto the* $\ker \Box_b$ *applied to forms spanned by $d\bar{z}^K$ is given by convolution with the*

$(0,q)$-form:

(5)
$$S_K(z,t) = \frac{4^n(n+m-1)!}{(2\pi)^{m+n}} \sum_{L\in\mathcal{I}_q} \int_{\alpha\in\Gamma_L} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L \frac{\prod_{j=1}^n |\mu_j^\alpha|}{(\sum_{j=1}^n |\mu_j^\alpha||z_j^\alpha|^2 - i\alpha\cdot t)^{n+m}} d\alpha.$$

In the case $K=\emptyset$, $S_\emptyset(z,t)$ is the Szegö kernel.

(3) *If $|\Gamma_L|>0$ for at least one $L\in\mathcal{I}_q$, then the canonical relative fundamental solution to $\Box_b$ given by $\int_0^\infty e^{-s\Box_b}(I-S_q)ds$ applied to forms spanned by $d\bar{z}^K$ is given by*

(6)
$$\begin{aligned} N_K(z,t) = \frac{4^n(n+m-2)!}{2(2\pi)^{m+n}} \sum_{L\in\mathcal{I}_q} \Bigg(&\int_{\alpha\notin\Gamma_L} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L \\ &\times \int_{r=0}^1 \prod_{j=1}^n \frac{r^{\frac12(1-\varepsilon_{j,L}^\alpha)|\mu_j^\alpha|}|\mu_j^\alpha|}{1-r^{|\mu_j^\alpha|}} \frac{1}{(A(r,z)-i\alpha\cdot t)^{n+m-1}} \frac{dr d\alpha}{r} \\ &+ \int_{\alpha\in\Gamma_L} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L \\ &\times \int_{r=0}^1 \Big[\Big(\prod_{j=1}^n \frac{|\mu_j^\alpha|}{1-r^{|\mu_j^\alpha|}}\Big) \frac{1}{(A_\alpha(r,z)-i\alpha\cdot t)^{n+m-1}} \\ &\qquad - \frac{\prod_{j=1}^n |\mu_j^\alpha|}{(A_\alpha(0,z)-i\alpha\cdot t)^{n+m-1}}\Big] \frac{dr d\alpha}{r} \Bigg), \end{aligned}$$

where

$$A_\alpha(r,z) = \sum_{j=1}^n |\mu_j^\alpha| \left(\frac{1+r^{|\mu_j^\alpha|}}{1-r^{|\mu_j^\alpha|}}\right) |z_j^\alpha|^2.$$

In all cases, the integrals converge absolutely.

Remark 2.4. In many of the most important cases, the functions $C_{K,L}(\alpha) = \delta_{KL}$ and formulas from the theorem simplify. There are several cases when this simplification occurs. The first is when $q=0$ or $q=n$. The second is when the orthonormal basis $\{v_j^\lambda\}$ is independent of λ. This independence happens both when $m=1$ (the hypersurface type case) and in the sum of squares case considered by Nagel, Ricci, and Stein [NRS01], discussed in Section 1.

Remark 2.5. It is a straightforward exercise to recover the classical complex Green operator on the Heisenberg group from (4) [BR13]. Additionally, [BR13, Theorem 2] is now a simple and immediate application of (4).

2.2. **Solvability, hypoellipticity, and ϕ^λ.** In [PR03], Peloso and Ricci say that

(1) $\Box_b$ is *solvable* if given any smooth $(0,q)$-form ψ on G with compact support, there exists a $(0,q)$-current u on G so that $\Box_b u = \psi$;
(2) $\Box_b$ is *hypoelliptic* if given any $(0,q)$-current ψ on G, ψ is smooth on any open set on which $\Box_b\psi$ is smooth.

Peloso and Ricci are able to characterize solvability and hypoellipticity of $\Box_b$.

Theorem 2.6 ([PR03]). *Let $n^+(\lambda)$, resp., $n^-(\lambda)$, be the number of positive, resp., negative eigenvalues of ϕ^λ. Then*

(1) *$\Box_b$ is solvable on $(0,q)$-forms if and only if there does not exist $\lambda \in \mathbb{R}^m \setminus \{0\}$ for which $n^+(\lambda) = q$ and $n^-(\lambda) = n - q$.*
(2) *$\Box_b$ is hypoelliptic on $(0,q)$-forms if and only if there does not exist $\lambda \in \mathbb{R}^m \setminus \{0\}$ for which $n^+(\lambda) \le q$ and $n^-(\lambda) \le n - q$.*

Remark 2.7. The condition $|\Gamma_L| > 0$ is equivalent to the nontriviality of $S_K(z,t)$ and is easy to check. By combining the solvability criteria of [PR03] (Theorem 2.6) and the formula for the $\Box_b$-heat kernel from [BR11] (Theorem 3.1), it must be the case that $\nu = n$ (see (10)) as $S_L(z,t) = \lim_{s\to\infty} H_L(s,z,t)$ and solvability is equivalent $S_L(z,t) = 0$. The latter statement follows from the fact the condition in part (1) of Theorem 2.6 is an open condition, that is, when solvability fails, Γ_L is a (union of) cones, at least one of which will be open and hence has nonzero measure.

3. The Kohn Laplacian on quadrics

For a discussion of the group theoretic properties of G, please see [BR11] or [PR03]. By definition, the operator $\Box_b$ is defined on $(0,q)$ forms as $\Box_b = \bar\partial_b\bar\partial_b^* + \bar\partial_b^*\bar\partial_b$ without reference to any particular coordinate system. However in order to do computations, we need formulas for $\Box_b$ with respect to carefully chosen coordinates.

For $v \in \mathbb{R}^{2n} \approx \mathbb{C}^n$, let ∂_v be the real vector field given by the directional derivative in the direction of v. Then the right invariant vector field at an arbitrary $g = (z,w) \in M$ corresponding to v is given by

$$X_v(g) = \partial_v + 2\,\mathrm{Im}\,\phi(v,z)\cdot D_t = \partial_v - 2\,\mathrm{Im}\,\phi(z,v)\cdot D_t.$$

Let Jv be the vector in $\mathbb{R}^{2n}$ which corresponds to iv in $\mathbb{C}^n$ (where $i = \sqrt{-1}$). The CR structure on G is then spanned by vectors of the form:

$$Z_v(g) = (1/2)(X_v - iX_{Jv}) = (1/2)(\partial_v - i\partial_{Jv}) - i\overline{\phi(z,v)}\cdot D_t \tag{7}$$

and

$$\bar Z_v(g) = (1/2)(X_v + iX_{Jv}) = (1/2)(\partial_v + i\partial_{Jv}) + i\phi(z,v)\cdot D_t. \tag{8}$$

Let $v_1,\dots,v_n$ be any orthonormal basis for $\mathbb{C}^n$. Let $X_j = X_{v_j}$, $Y_j = X_{Jv_j}$, and let $Z_j = (1/2)(X_j - iY_j)$, $\bar Z_j = (1/2)(X_j + iY_j)$ be the right invariant CR vector fields defined above (which are also the left invariant vector fields for the group structure with ϕ replaced by $-\phi$). A $(0,q)$-form can be expressed as $\sum_{K\in\mathcal{I}_q}\phi_K d\bar z^K$. An explicit formula for $\Box_b$ on quadrics is written down by Peloso and Ricci [PR03] (see also [BR11]) which takes the following form: if $\phi = \sum_K \phi_K d\bar z^K$ is a $(0,q)$-form, then

$$\Box_b\Big(\sum_{K\in\mathcal{I}_q}\phi_K d\bar z^K\Big) = \sum_{K,L\in\mathcal{I}_q}\Box^v_{LK}\phi_K d\bar z^L, \tag{9}$$

where

$$\Box^v_{LL} = (1/2)\sum_{\ell=1}^n\left(Z_\ell\bar Z_\ell + \bar Z_\ell Z_\ell\right) + (1/2)\left(\sum_{\ell\in L}[Z_\ell,\bar Z_\ell] - \sum_{k\notin L}[Z_k,\bar Z_k]\right).$$

If $L \ne K$, then $\Box^v_{LK}$ is zero unless $|L\cap K| = q-1$, in which case

$$\Box^v_{LK} = (-1)^{d_{kl}}[Z_k,\bar Z_\ell],$$

where $k \in K$ is the unique element not in L and $\ell \in L$ is the unique element not in K and d_{kl} is the number of indices between k and ℓ. The notation $\Box^v_{LK}$ indicates the dependency of this differential operator on the particular orthonormal basis $v = (v_1, \dots, v_n)$ chosen and the resulting basis (i.e., $Z_1, \dots, Z_n$) and the associated dual basis of $(0, 1)$-forms (i.e., $d\bar{z}_1, \dots, d\bar{z}_n$).

Note that if $|L \cap K| = q - 1$, then $\Box^v_{LK}$ is quite simple since it is a linear combination over $\mathbb{C}$ of t_j derivatives. In the next section, we will use the coordinates $z^\alpha = Z(z, \alpha)$ derived from the basis $v^\alpha_1, \dots, v^\alpha_n$ used in Section 2, and we will see that we can ignore the $\Box^{v^\alpha}_{LK}$ when $L \neq K$.

3.1. **Fourier transform of $\Box_b$.** Since the quadric defining equations are independent of $t \in \mathbb{R}^m$, we can use the Fourier transform in the t-variables:

$$\hat{f}(\lambda) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} f(t) e^{-i\lambda \cdot t} dt.$$

In the case that f is a function of (z, t), we use the notation $f(z, \hat{\lambda})$ to denote the partial Fourier transform of f in the t-variables. We transform $\Box_b$ via the Fourier transform and consider the fundamental solution to the heat operator in the transformed variables. We then use the z^α coordinates relative to the basis v^α_j chosen in Section 2 for the z-variable in $f(z^\alpha, \hat{\lambda})$ with $\alpha = \frac{\lambda}{|\lambda|}$. Thus, λ plays two roles - first as the Fourier transform variable and second, as the label for the coordinates relative to the basis v^α_j which diagonalizes ϕ^λ. Also note that the operation of Fourier transform in t and the operation of expressing z in terms of the z^α coordinates are interchangeable (i.e., these operations commute).

For a general orthonormal basis $v = \{v_1, \dots, v_n\}$, let $\Box^{v,\hat{\lambda}}_{LL}$ be the partial Fourier transform in t of the sub-Laplacian $\Box^v_{LL}$. When $v = v^\alpha$, we have (from [BR11]):

$$\Box^{v^\alpha,\hat{\lambda}}_{LL} = -\frac{1}{4}\Delta + 2i \sum_{k=1}^n \mu^\lambda_k \operatorname{Im}\{z^\alpha_k \partial_{z^\alpha_k}\} + \sum_{k=1}^n (\mu^\lambda_k)^2 |z^\alpha_k|^2 - \left(\sum_{k \in L} \mu^\lambda_k - \sum_{k \notin L} \mu^\lambda_k \right),$$

where Δ is the ordinary Laplacian in $z = z^\alpha$ coordinates.

Also note that

$$\begin{aligned} \Box^{v^\alpha}_{LK} &= (-1)^{d_{k\ell}} [Z_{v^\alpha_k}, \bar{Z}_{v^\alpha_\ell}] \\ &= 2i(-1)^{d_{k\ell}} \operatorname{Re} \phi(v^\alpha_k, v^\alpha_\ell) \cdot D_t \text{ using (7) and (8).} \end{aligned}$$

Using (2), we conclude that the Fourier transform of $\Box^{v^\alpha}_{LK}$ is

$$\Box^{v^\alpha,\hat{\lambda}}_{LK} = -2(-1)^{d_{k\ell}} \operatorname{Re} \phi(v^\alpha_k, v^\alpha_\ell) \cdot \lambda = 0$$

when $L \neq K$ (i.e., when $\ell \neq k$). The significance of this calculation is that the partial Fourier transform of $\Box_{LK}$ (expressed in global coordinates) is incorporated into the operators $\Box^{v^\alpha,\hat{\lambda}}_{LL}$.

Next, we recall the heat kernel and Szegö kernel for the $\Box^{v^\alpha,\hat{\lambda}}_{LL}$ heat equation. Let $\tilde{H}_L(s, z^\alpha, \hat{\lambda})$ be the "heat kernel", i.e., the solution to the following boundary value problem:

$$\left[\frac{\partial}{\partial s} + \Box_{LL}^{v^\alpha,\hat{\lambda}}\right]\{\tilde{H}_L(s, z^\alpha, \hat{\lambda})\} = 0 \text{ for } s > 0,$$
$$\tilde{H}_L(s = 0, z^\alpha, \hat{\lambda}) = (2\pi)^{-m/2}\delta_0(z^\alpha),$$
$$= (2\pi)^{-m/2}\delta_0(z),$$

where δ_0 is the Dirac-delta function centered at the origin in the z variables. Let $\tilde{S}_L(z^\alpha, \hat{\lambda})$ be the Szegö kernel which represents orthogonal projection of $L^2(\mathbb{C}^n)$ onto the kernel of $\Box_{LL}^{v^\alpha,\hat{\lambda}}$. Note that the tilde over the $\tilde{H}_L$ and $\tilde{S}_L$ indicates that these terms are functions rather than differential forms. By contrast, N_K and S_K in Theorem 2.3 do not have tildes and they are differential $(0, q)$-forms.

Theorem 3.1 ([BR11])**.** *Let $L \in \mathcal{I}_q$ be a given multiindex of length q and fix a nonzero $\lambda \in \mathbb{R}^m$ and $\alpha = \frac{\lambda}{|\lambda|}$. Then*

(1) *The heat kernel which solves the above boundary value problem is*

(10)
$$\tilde{H}_L(s, z^\alpha, \hat{\lambda}) = \frac{2^{n-\nu(\alpha)}}{(2\pi)^{m/2+n}s^{n-\nu(\alpha)}}e^{-\frac{|z''\{\alpha\}|^2}{s}}\prod_{j=1}^{\nu(\alpha)}\frac{2e^{s\varepsilon^\alpha_{j,L}|\mu_j^\lambda|}|\mu_j^\lambda|}{\sinh(s|\mu_j^\lambda|)}e^{-|\mu_j^\lambda|\coth(s|\mu_j^\lambda|)|z_j^\alpha|^2}.$$

(2) *If $\alpha \in \Gamma_L$, then the projection onto* $\ker \Box_{LL}^{v^\alpha,\hat{\lambda}}$ *is given by*

$$\tilde{S}_L(z^\alpha, \hat{\lambda}) = \lim_{s\to\infty}\tilde{H}_L(s, z^\alpha, \hat{\lambda}) = \frac{4^n}{(2\pi)^{n+m/2}}\prod_{j=1}^{n}|\mu_j^\lambda|e^{-|\mu_j^\lambda||z_j^\alpha|^2}, \tag{11}$$

otherwise $\tilde{S}_L(z^\alpha, \hat{\lambda}) = 0$.

(3) *The connection between the fundamental solution to the heat equation and the canonical relative fundamental solution to $\Box_{LL}^{v^\alpha,\hat{\lambda}}$, denoted by $\tilde{N}_L(z^\alpha, \hat{\lambda})$, is given as follows:*

$$\tilde{N}_L(z^\alpha, \hat{\lambda}) = \int_0^\infty \left[\tilde{H}_L(s, z^\alpha, \hat{\lambda}) - \tilde{S}_L(z^\alpha, \hat{\lambda})\right] ds. \tag{12}$$

In particular,

$$\Box_{LL}^{v^\alpha,\hat{\lambda}}\{\tilde{N}_L(z^\alpha, \hat{\lambda})\} = (2\pi)^{-m/2}(\delta_0(z) - \tilde{S}_L(z^\alpha, \hat{\lambda})), \tag{13}$$

Both of the kernels $\tilde{H}_L(s, \cdot, \hat{\lambda})$ and $\tilde{S}_L(\cdot, \hat{\lambda})$ act on $L^2(\mathbb{C}^n)$ via a twisted convolution, $*_\lambda$, where $(f *_\lambda g)(z) = \int_{w\in\mathbb{C}^n} f(w)g(z - w)e^{-2i\lambda\cdot \operatorname{Im}\phi(z,w)}dw$, as defined in [BR11, Section 5.4], but this plays no role here.

Let $\mathcal{F}_\lambda^{-1}$ denote the inverse Fourier transform in λ - that is, if $\tilde{f}(z, \lambda)$ is an integrable function of $\lambda \in \mathbb{R}^m$, then

$$\mathcal{F}_\lambda^{-1}(\tilde{f}(z, \lambda))(t) := \frac{1}{2^{m/2}}\int_{\lambda\in\mathbb{R}^m}\tilde{f}(z, \lambda)e^{i\lambda\cdot t}d\lambda.$$

Now we can formulate our relative solution to $\Box_b$ and Szegö kernel in terms of the inverse Fourier transform.

Proposition 3.2. *For a given index $K \in \mathcal{I}_q$, the relative fundamental solution to $\Box_b$ applied to a form spanned by $d\bar{z}^K$ given by $\int_0^\infty e^{-s\Box_b}(I - S_K)ds$ is*

$$N_K(z,t) = \mathcal{F}_\lambda^{-1}\left\{\sum_{L\in\mathcal{I}_q} C_{K,L}(\alpha)\tilde{N}_L(z^\alpha,\hat{\lambda})d\bar{Z}(z,\alpha)^L\right\}(t). \tag{14}$$

Moreover, the orthogonal projection onto the $\ker\Box_b$ *applied to forms spanned by $d\bar{z}^K$ is given by convolution with the $(0,q)$-form*

$$S_K(z,t) = \mathcal{F}_\lambda^{-1}\left\{(2\pi)^{-m/2}\sum_{L\in\mathcal{I}_q} C_{K,L}(\alpha)\tilde{S}_L(z^\alpha,\hat{\lambda})d\bar{Z}(z,\alpha)^L\right\}(t). \tag{15}$$

Proof. With the definitions of N_K and S_K given by (14) and (15), respectively, we shall show $\Box_b N_K = I - S_K$. On the transform side, we have

$$\begin{aligned}
\Box_b^{v^\alpha,\hat{\lambda}}\left\{N_K(z^\alpha,\hat{\lambda})\right\} &= \sum_{L\in\mathcal{I}_q} C_{K,L}(\alpha)\Box_b^{v^\alpha,\hat{\lambda}}\left\{\tilde{N}_L(z^\alpha,\hat{\lambda})d\bar{Z}(z,\alpha)^L\right\}\\
&= \sum_{L\in\mathcal{I}_q} C_{K,L}(\alpha)\Box_{LL}^{\hat{\lambda}}\{\tilde{N}_L(z^\alpha,\hat{\lambda})\}d\bar{Z}(z,\alpha)^L\\
&= (2\pi)^{-m/2}\sum_{L\in\mathcal{I}_q} C_{K,L}(\alpha)\big[\delta_0(z^\alpha) - \tilde{S}_L(z^\alpha,\hat{\lambda})\big]d\bar{Z}(z,\alpha)^L\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{from (13)}\\
&= (2\pi)^{-m/2}\delta_0(z)\otimes 1_\lambda d\bar{z}^K\\
&\quad - (2\pi)^{-m/2}\sum_{L\in\mathcal{I}_q} C_{K,L}(\alpha)\tilde{S}_L(z^\alpha,\hat{\lambda})d\bar{Z}(z,\alpha)^L \text{ from (3)},
\end{aligned}$$

where the function 1_λ is the constant function which is 1 in the λ coordinates. Now take the inverse Fourier transform (in λ) of both sides. The left side becomes $\Box_b N_K$ and then use the fact that $F_\lambda^{-1}\{(2\pi)^{-m/2}1_\lambda\}(t) = \delta_0(t)$ and we obtain

$$\begin{aligned}
\Box_b\{N_K(z,t)\} &= \delta_0(z)\delta_0(t)d\bar{z}^K - \mathcal{F}_\lambda^{-1}\left\{(2\pi)^{-m/2}\sum_{L\in\mathcal{I}_q} C_{K,L}(\alpha)\tilde{S}_L(z^\alpha,\hat{\lambda})d\bar{Z}(z,\alpha)^L\right\}\\
&= \delta_0(z)\delta_0(t)d\bar{z}^K - S_K(z,t) \text{ using (15)}
\end{aligned}$$

as desired. ∎

4. A new derivation of the integral formula – Proof of Theorem 2.3

Proof of Theorem 2.3. We first assume the Szegö kernel is zero, that is, $|\Gamma_L| = \emptyset$ for all $L \in \mathcal{I}_q$. Consequently, it follows from (12) that $\tilde{N}_L(z^\alpha,\hat{\lambda}) = \int_0^\infty \tilde{H}_L(s,z^\alpha,\hat{\lambda})ds$. To prepare for the calculation of $\tilde{N}_L(z,t)$, we use polar coordinates and write $\lambda = \alpha\tau$ where α belongs to the unit sphere S^{m-1} and $\tau > 0$. We observe

$$\frac{2e^{s\varepsilon_{j,L}^\alpha|\mu_j^\lambda|}|\mu_j^\lambda|}{\sinh(s|\mu_j^\lambda|)} = \frac{4e^{s\tau\varepsilon_{j,L}^\alpha|\mu_j^\alpha|}|\mu_j^\alpha|\tau}{e^{s|\mu_j^\alpha|\tau} - e^{-s|\mu_j^\alpha|\tau}} = \frac{4e^{s\tau(\varepsilon_{j,L}^\alpha-1)|\mu_j^\alpha|}|\mu_j^\alpha|\tau}{1 - e^{-2s|\mu_j^\alpha|\tau}}$$

and

$$\coth(s|\mu_j^\lambda|) = \frac{e^{s\tau|\mu_j^\alpha|} + e^{-s\tau|\mu_j^\alpha|}}{e^{s\tau|\mu_j^\alpha|} - e^{-s\tau|\mu_j^\alpha|}} = \frac{1 + e^{-2s\tau|\mu_j^\alpha|}}{1 - e^{-2s\tau|\mu_j^\alpha|}}.$$

We now recover $N_K(z,t)$ from (14) by computing the inverse Fourier transform using polar coordinates ($\lambda = \alpha\tau, \alpha \in S^{m-1}, \tau > 0$).

$$\begin{aligned} &N_K(z,t)\\ &= (2\pi)^{-m/2} \sum_{L\in\mathcal{I}_q} \int_{s=0}^{\infty} \int_{\lambda\in\mathbb{R}^m} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L \tilde{H}_L(s,z,\hat{\lambda}) e^{it\cdot\lambda} d\lambda ds\\ &= (2\pi)^{-m/2} \sum_{L\in\mathcal{I}_q} \int_{s=0}^{\infty} \int_{\tau=0}^{\infty} \int_{\alpha\in S^{m-1}} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L \tilde{H}_L(s,z,\widehat{\alpha\tau}) e^{it\cdot\alpha\tau} \tau^{m-1} d\alpha d\tau ds, \end{aligned}$$

where $d\alpha$ is the surface volume form on the unit sphere in $\mathbb{R}^m$. Since $\tilde{H}_L(s, z^\alpha, \hat{\lambda})$ is assumed to be jointly integrable in the variables $s > 0$ and λ, we can integrate the s, τ, α variables in any order.

Let $r = e^{-2s\tau}$ in the s-integral and so $ds = -dr/(2\tau r)$ and the oriented r-limits of integration become 1 to 0. We obtain

$$\begin{aligned} N_K(z,t) = \frac{4^n}{2(2\pi)^{m+n}} &\sum_{L\in\mathcal{I}_q} \int_{r=0}^{1} \int_{\alpha\in S^{m-1}} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L\\ \int_{\tau=0}^{\infty} &\frac{1}{|\log r|^{n-\nu(\alpha)}} \prod_{j=1}^{\nu(\alpha)} \frac{r^{\frac{1}{2}(1-\varepsilon^\alpha_{j,L})|\mu^\alpha_j|}|\mu^\alpha_j|}{1-r^{|\mu^\alpha_j|}} e^{-\tau(A_\alpha(r,z)-it\cdot\alpha)} \tau^{n+m-2} d\tau d\alpha \frac{dr}{r}, \end{aligned}$$

where

$$A_\alpha(r,z) = \frac{2}{|\log r|} |z''^\alpha|^2 + \sum_{j=1}^{\nu(\alpha)} |\mu^\alpha_j| \left(\frac{1+r^{|\mu^\alpha_j|}}{1-r^{|\mu^\alpha_j|}} \right) |z^\alpha_j|^2.$$

We now perform the τ-integral by using the following formula:

$$\int_{\tau=0}^{\infty} \tau^p e^{-a\tau} d\tau = \frac{p!}{a^{p+1}} \text{ for } \operatorname{Re} a > 0$$

which concludes the proof for (4).

Repeating this argument for the Szegö kernel using (15), we have

$$\begin{aligned} S_K(z,t) &= \frac{4^n}{(2\pi)^{m+n}} \sum_{L\in\mathcal{I}_q} \int_{\alpha\in\Gamma_L} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L\\ &\quad \int_0^{\infty} \Big(\prod_{j=1}^{n} |\mu^\alpha_j| \Big) \tau^{n+m-1} e^{-\tau(\sum_{j=1}^n |\mu^\alpha_j||z^\alpha_j|^2 - i\alpha\cdot t)} d\tau d\alpha\\ &= \frac{4^n(n+m-1)!}{(2\pi)^{m+n}} \sum_{L\in\mathcal{I}_q} \int_{\alpha\in\Gamma_L} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L \frac{\prod_{j=1}^n |\mu^\alpha_j|}{(\sum_{j=1}^n |\mu^\alpha_j||z^\alpha_j|^2 - i\alpha\cdot t)^{n+m}} d\alpha \end{aligned}$$

which concludes the proof for (5).

Finally, if $S_L(z,t) \neq 0$, then using (14) and (12)

$$
\begin{aligned}
&N_K(z,t)\\
&= \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{s=0}^{\infty} \sum_{L\in\mathcal{I}_q}\\
&\Big(\int_{\alpha\in\Gamma_L} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L \int_{\tau=0}^{\infty} \Big(\tilde{H}_L(s,z,\widehat{\tau\alpha}) - \tilde{S}_L(z,\widehat{\tau\alpha})\Big) e^{i\tau(\alpha\cdot t)} \tau^{m-1} d\tau d\alpha\Big)\, ds\\
&+ \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{s=0}^{\infty} \sum_{L\in\mathcal{I}_q} \Big(\int_{\alpha\in\Gamma_L} \int_{\alpha\notin\Gamma_L} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L \int_{\tau=0}^{\infty} \tilde{H}_L(s,z,\widehat{\tau\alpha}) e^{i\tau(\alpha\cdot t)} \tau^{m-1} d\tau d\alpha\Big)\, ds\\
&= I_K + II_K.
\end{aligned}
$$

The second set of integrals is virtually identical to what we computed earlier and we get

$$
\begin{aligned}
II_K = \frac{4^n (n+m-2)!}{2(2\pi)^{m+n}} \int_{r=0}^{1} \sum_{L\in\mathcal{I}_q} \Big(\int_{\alpha\notin\Gamma_L} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L\\
\prod_{j=1}^{n} \frac{r^{\frac{1}{2}(1-\varepsilon^\alpha_{j,L})|\mu^\alpha_j|} |\mu^\alpha_j|}{1 - r^{|\mu^\alpha_j|}} \frac{d\alpha}{(A(r,z) - i\alpha\cdot t)^{n+m-1}}\Big) \frac{dr}{r},
\end{aligned}
$$

where

$$
A_\alpha(r,z) = \sum_{j=1}^{n} |\mu^\alpha_j| \left(\frac{1 + r^{|\mu^\alpha_j|}}{1 - r^{|\mu^\alpha_j|}}\right) |z^\alpha_j|^2.
$$

For the first set of integrals, we observe that

$$
\begin{aligned}
I_K &= \frac{4^n}{2(2\pi)^{m+n}} \sum_{L\in\mathcal{I}_q} \Big(\int_{r=0}^{1} \int_{\alpha\in\Gamma_L} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L \int_{\tau=0}^{\infty}\\
&\Big[\Big(\prod_{j=1}^{n} \frac{|\mu^\alpha_j|}{1 - r^{|\mu^\alpha_j|}}\Big) e^{-\tau(A_\alpha(r,z) - it\cdot\alpha)} - \prod_{j=1}^{n} |\mu^\alpha_j| e^{\tau(A_\alpha(0,z) - it\cdot\alpha)}\Big] \tau^{n+m-2} d\tau d\alpha \frac{dr}{r}\Big)\\
&= \frac{4^n (n+m-2)!}{2(2\pi)^{m+n}} \sum_{L\in\mathcal{I}_q} \Big(\int_{r=0}^{1} \int_{\alpha\in\Gamma_L} C_{K,L}(\alpha) d\bar{Z}(z,\alpha)^L\\
&\Big[\Big(\prod_{j=1}^{n} \frac{|\mu^\alpha_j|}{1 - r^{|\mu^\alpha_j|}}\Big) \frac{1}{(A_\alpha(r,z) - it\cdot\alpha)^{n+m-1}} - \frac{\prod_{j=1}^{n} |\mu^\alpha_j|}{(A_\alpha(0,z) - it\cdot\alpha)^{n+m-1}}\Big] d\alpha \frac{dr}{r}\Big).
\end{aligned}
$$

This completes the proof of (6).

That the convergence of the resulting integrals is absolute follows from a straightforward Taylor expansion argument around $r = 0$ and $r = 1$, the only possible points where the integrand appears to blow up. □

5. Examples

We analyze three examples in this section, all of which fall into the cases discussed in Remark 2.4 so the formulas from Theorem 2.3 are slightly simpler. We discuss codimension 2 quadrics in $\mathbb{C}^4$ when $q = 0, 2$, the Heisenberg group (so $m = 1$), and the product the Heisenberg groups (so we fall into the sum of squares case).

5.1. **Codimension 2 quadrics in $\mathbb{C}^4$.** When $n = m = 2$, we wrote down the formulas for N in the case of three canonical examples [BR13]:

- M_1 where $\phi(z, z) = (|z_1|^2, |z_2|^2)^T$
- M_2 where $\phi(z, z) = (2\,\mathrm{Re}(z_1\bar{z}_2), |z_1|^2 - |z_2|^2)^T$
- M_3 where $\phi(z, z) = (2|z_1|^2, 2\,\mathrm{Re}(z_1\bar{z}_2))^T$

These examples are canonical in the sense that any quadric in $\mathbb{C}^2 \times \mathbb{C}^2$ whose Levi form has image which is *not* contained in a one-dimensional cone is biholomorphic to one of these three examples (see [Bog91]). Additionally, these three examples perfectly demonstrate the three possibilities for solvability/hypoellipticity of $\Box_b$ on quadrics.

The quadric M_1 is simply a Cartesian product of Heisenberg groups and both solvability and hypoellipticity are impossible for any degree. In this case,

$$A^\lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

so the eigenvalues of A^λ are λ_1 and λ_2, so

$$\{(n^+(\lambda), n^-(\lambda))\} = \{(2, 0), (1, 0), (1, 1), (0, 1), (0, 2)\}.$$

For M_2, it follows from Peloso and Ricci [PR03] that solvability and hypoellipticity occur for $(0, q)$ forms if and only if $q = 0$ or $q = 2$. In this case,

$$A^\lambda = \begin{pmatrix} \lambda_2 & \lambda_1 \\ \lambda_1 & -\lambda_2 \end{pmatrix}$$

which gives us eigenvalues $\pm|\lambda|$, so that for all $\lambda \in \mathbb{R}^2 \setminus \{0, 0\}$, $(n^+(\lambda), n^-(\lambda)) = (1, 1)$. Additionally, we showed that the complex Green operator is given by (group) convolution with respect to the kernel

$$N_2(z, t) = C(|z|^4 + |t|^2)^{-3/2},$$

where C is a constant [BR13, Theorem 3].

For M_3, $\Box_b$ is solvable if and only if $q = 0$ or 2 and is never hypoelliptic. In this case,

$$A^\lambda = \begin{pmatrix} 2\lambda_1 & \lambda_2 \\ \lambda_2 & 0 \end{pmatrix}$$

so that the eigenvalues are $\lambda_1 \pm |\lambda|$. Thus $\{(n^+(\lambda), n^-(\lambda))\} = \{(1, 0), (1, 1), (0, 1)\}$ with the degenerate values occurring when $\lambda_2 = 0$. In Corollary 5.1, we give a more useful formula for N on M_3 [BR13]. The analysis of the operator is extremely complicated and delicate and is the subject of a later work in the series [BR]. We must mention the paper of Nagel, Ricci, and Stein which analyzes L^p estimates on a class of higher codimension quadrics in $\mathbb{C}^n \times \mathbb{C}^m$ which depend only on $|z_j|^2$, $1 \le j \le m$ [NRS01]. However, their result applies to neither M_2 nor M_3 for these quadrics cannot be described in this manner.

5.2. **Example M_3.** Let $q = 0$. As defined in Section 1,

$$M_3 = \{(z, w) \in \mathbb{C}^2 \times \mathbb{C}^2 : \mathrm{Im}\, w_1 = 2|z_1|^2, \mathrm{Im}\, w_2 = 2\,\mathrm{Re}(z_1\bar{z}_2)\}.$$

Here, $m = n = 2$, and for $\alpha = (\cos\theta, \sin\theta)$, we easily compute $\mu_1^\alpha = 1 + \cos\theta$, $\mu_2^\alpha = \cos\theta - 1$. The function ϕ satisfies both A_0 and A_2, though we will concentrate on the case $q = 0$ (the case for $q = 2$ is similar). Since $L = \{0\}$, $\varepsilon_j^\alpha = -\mathrm{sgn}(\mu_j^\alpha)$

and so $\varepsilon_1^\alpha = -1$ and $\varepsilon_2^\alpha = +1$ (except when $\theta = 0$ or 1 which is a set of measure zero). We obtain

(16)

$$N(z,t) = (2\pi)^{-4} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 4^2 \frac{r^{\cos\theta}\sigma_1(\theta)\sigma_2(\theta)}{(1-r^{\sigma_1(\theta)})(1-r^{\sigma_2(\theta)})} \times \frac{dr d\theta}{(-i\alpha(\theta)\cdot t + \sigma_1(\theta)E_1(r,\theta)|z_1^\theta|^2 + \sigma_2(\theta)E_2(r,\theta)|z_2^\theta|^2)^3},$$

where

$$\alpha(\theta) = (\cos\theta, \sin\theta), \sigma_1(\theta) = 1+\cos\theta, \sigma_2(\theta) = 1-\cos\theta, E_j(r,\theta) = \frac{1+r^{\sigma_j(\theta)}}{1-r^{\sigma_j(\theta)}}.$$

We first wrote this formula in [BR13]. We wish to express it in a more useful and computable form which we will use in [BR].

We let $t = (t_1, t_2)$ which gives $\alpha(\theta)\cdot t = t_1\cos\theta + t_2\sin\theta$. We also let

$$x = r^{\sigma_1}, \text{ so } dx = \sigma_1 r^{\sigma_1 - 1} dr \text{ and } \sigma = \frac{\sigma_2}{\sigma_1} = \frac{1-\cos\theta}{1+\cos\theta}, d\theta = \frac{d\sigma}{(\sigma+1)\sqrt{\sigma}}$$

and obtain

$$\cos\theta = \frac{1-\sigma}{1+\sigma} \text{ and } \sin\theta = \frac{\pm 2\sqrt{\sigma}}{1+\sigma},$$

where $\pm$ is $+$ for $\theta \in [0,\pi]$ and $-$ for $\theta \in (\pi, 2\pi]$. Also the interval $0 \le \theta \le \pi$ corresponds to the oriented σ interval $[0,\infty)$ and the interval $\pi \le \theta \le 2\pi$ corresponds to $(\infty, 0]$. In Theorem 2.3, the point z is expressed in terms of the eigenvectors of ϕ^λ. To this end, we set

$$z_1\{\sqrt{\sigma}\} = \frac{1}{\sqrt{1+\sigma}}\big(z_1 + \sqrt{\sigma} z_2\big),$$
$$z_2\{\sqrt{\sigma}\} = -\frac{1}{\sqrt{1+\sigma}}\big(\sqrt{\sigma} z_1 - z_2\big),$$

and

$$\tilde{z}_1\{\sqrt{\sigma}\} = -z_1\{\sqrt{\sigma}\} = \frac{1}{\sqrt{1+\sigma}}\big(-z_1 + \sqrt{\sigma} z_2\big),$$
$$\tilde{z}_2\{\sqrt{\sigma}\} = -z_2\{\sqrt{\sigma}\} = -\frac{1}{\sqrt{1+\sigma}}\big(\sqrt{\sigma} z_1 + z_2\big).$$

We then obtain Corollary 5.1 to Theorem 2.3:

Corollary 5.1. *The fundamental solution to $\Box_b$ for M_3 on functions is given by convolution with the kernel*

$$N(z,t) = 2(2\pi)^{-4} \int_{\sigma=0}^{\infty} \int_{x=0}^{1} \frac{\sqrt{\sigma}(\sigma+1)}{(1-x)(1-x^\sigma)} \frac{dx d\sigma}{\left[-i\left(t_1\frac{1-\sigma}{2} + t_2\sqrt{\sigma}\right) + \left(\frac{1+x}{1-x}\right)|z_1\{\sqrt{\sigma}\}|^2 + \sigma\left(\frac{1+x^\sigma}{1-x^\sigma}\right)|z_2\{\sqrt{\sigma}\}|^2\right]^3}$$
$$+ 2(2\pi)^{-4} \int_{\sigma=0}^{\infty} \int_{x=0}^{1} \frac{\sqrt{\sigma}(\sigma+1)}{(1-x)(1-x^\sigma)} \frac{dx d\sigma}{\left[-i\left(t_1\frac{1-\sigma}{2} - t_2\sqrt{\sigma}\right) + \left(\frac{1+x}{1-x}\right)|\tilde{z}_1\{\sqrt{\sigma}\}|^2 + \sigma\left(\frac{1+x^\sigma}{1-x^\sigma}\right)|\tilde{z}_2\{\sqrt{\sigma}\}|^2\right]^3}.$$

This formula is the launching point for [BR].

5.3. **The Heisenberg group.** Denote the Heisenberg group $\mathcal{H}^n \cong \mathbb{R}^{2n} \times \mathbb{R}$. The Kohn Laplacian $\Box_b$ has a nontrivial kernel in the case that $L = \emptyset$ or $L = \{1, \dots, n\}$. The calculation for these two cases is identical and we prove the details in the case $L = \{1, \dots, n\}$. A derivation of a related formula from the classical methods appears in [Ste93, pp.615-617]. We set

$$\log\Big(\frac{|z|^2 - it}{|z|^2 + it}\Big) = \log(|z|^2 - it) - \log(|z|^2 + it) \tag{17}$$

for all $z \in \mathbb{C}^n$ and $t \in \mathbb{R}$ and assume that the logarithm is defined via the principal branch.

Theorem 5.2. *On the Heisenberg group $\mathcal{H}^n$,*

(1) *The relative fundamental solution $e^{-s\Box_b}(I - S_0)$ to $\Box_b = \bar\partial_b^*\bar\partial_b$ on functions is given by the integration kernel*

$$N_\emptyset(z,t) = \frac{2^{n-2}(n-1)!}{\pi^{n+1}} \frac{1}{(|z|^2 + it)^n}\Big[\log\Big(\frac{|z|^2 + it}{|z|^2 - it}\Big) - \sum_{j=1}^{n-1}\frac{1}{j}\Big].$$

(2) *The relative fundamental solution $e^{-s\Box_b}(I - S_n)$ to $\Box_b = \bar\partial_b\bar\partial_b^*$ on $(0,n)$-forms is given by the integration kernel*

$$N_{\{1,\dots,n\}}(z,t) = \frac{2^{n-2}(n-1)!}{\pi^{n+1}} \frac{1}{(|z|^2 - it)^n}\Big[\log\Big(\frac{|z|^2 - it}{|z|^2 + it}\Big) - \sum_{j=1}^{n-1}\frac{1}{j}\Big].$$

Remark 5.3.

(1) Up to a function in $\ker\Box_b$, our formula appears to be the complex conjugate of the formula in [Ste93, Chapter XIII, Equation (51)]. This is a consequence of the fact that our computations are taken with respect to right invariant vector fields and not left invariant vector fields.
(2) For a discussion regarding the consequences of the existence of a relative fundamental solution, we again refer the reader to [Ste93, Chapter XIII, Section 4.2]. It is easy to see that the convolution N with a Schwartz function will be an object in L^2 and hence orthogonal to $\ker\Box_b$.

Proof. Since $L = \{1, \dots, n\}$, the Szegö kernel $S(z,\hat\lambda) = S_L(z,\hat\lambda)$ has support $\operatorname{supp} S_L(z,\hat\lambda) = [0,\infty)$ which means (suppressing L)

$$N(z,t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty\int_0^\infty \big(H(s,z,\hat\lambda) - S(z,\hat\lambda)\big)e^{it\lambda}ds\,d\lambda + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0\int_0^\infty H(s,z,\hat\lambda)e^{it\lambda}ds\,d\lambda.$$

Equation (6) yields

(18)

$$N(z,t) = \frac{4^n(n-1)!}{2(2\pi)^{n+1}}\int_0^1 \frac{1}{r}\Big[\frac{1}{(1-r)^n}\frac{1}{(\frac{1+r}{1-r}|z|^2 - it)^n} - \frac{1}{(|z|^2 - it)^n}\Big] + \frac{r^{n-1}}{(1-r)^n}\frac{1}{(\frac{1+r}{1-r}|z|^2 + it)^n}dr.$$

Set $a = \frac{|z|^2 - it}{|z|^2 + it}$ and for $\delta > 0$, $a_\delta = \frac{|z|^2 + \delta - it}{|z|^2 + \delta + it}$ (so $|a| = |a_\delta| = 1$). The reason that we introduce a_δ is that a logarithm appears in the integral, and $\log a$ is not well defined with the principal branch if $|z|^2 = 0$. By introducing δ, it is immediate that for any a_δ

$$\log a_\delta = \log(|z|^2 + \delta - it) - \log(|z|^2 + \delta + it)$$

and by sending $\delta \to 0$, we obtain $\log a$ as in (17). Ignoring the constants, we compute

$$\begin{aligned}
I_\delta &= \int_0^1 \frac{1}{r}\Big[\frac{1}{((1+r)(|z|^2+\delta)it(1-r))^n} - \frac{1}{(|z|^2+\delta-it)^n}\Big] \\
&\qquad + \frac{r^{n-1}}{((1+r)(|z|^2+\delta)+it(1-r))^n}dr \\
&= \int_0^1 \frac{1}{r}\Big[\frac{1}{((|z|^2+\delta+it)r+|z|^2+\delta-it)^n} - \frac{1}{(|z|^2+\delta-it)^n}\Big] \\
&\qquad + \frac{r^{n-1}}{((|z|^2+\delta-it)r+(|z|^2+\delta+it))^n}dr \\
&= \frac{1}{(|z|^2+\delta+it)^n}\Big[\int_0^1 \Big(\frac{1}{(r+a_\delta)^n} - \frac{1}{a_\delta^n}\Big)\frac{dr}{r} + \int_0^1 \frac{r^{n-1}}{(a_\delta r+1)^n}dr\Big].
\end{aligned}$$

For the second integral, we change variables $r = \frac{1}{s}$ and compute

$$\int_0^1 \frac{s^{n-1}}{(a_\delta s+1)^n}ds = \int_1^\infty \frac{1}{(r+a_\delta)^n}\frac{dr}{r}.$$

Thus,

$$(|z|^2+\delta+it)^n I_\delta = \lim_{\epsilon\to 0}\Big[\int_\epsilon^\infty \frac{1}{(r+a_\delta)^n}\frac{dr}{r} + \frac{1}{a_\delta^n}\log\epsilon\Big].$$

A geometric series argument shows that

$$\frac{1}{r(r+a_\delta)^n} = \frac{1}{a_\delta^n r} - \frac{1}{a_\delta^n(r+a_\delta)} - \sum_{j=1}^{n-1}\frac{1}{a_\delta^{n-j}(r+a_\delta)^{j+1}}.$$

Therefore

$$\begin{aligned}
&(|z|^2+\delta+it)^n I_\delta \\
&= \lim_{\epsilon\to 0}\Big[\int_\epsilon^\infty \frac{1}{a_\delta^n r} - \frac{1}{a_\delta^n(r+a_\delta)}dr - \sum_{j=1}^{n-1}\int_\epsilon^\infty \frac{1}{a_\delta^{n-j}(r+a_\delta)^{j+1}}dr + \frac{1}{a_\delta^n}\log\epsilon\Big] \\
&= \lim_{\epsilon\to 0}\Big[\frac{\log(a_\delta+\epsilon)}{a_\delta^n} - \sum_{j=1}^{n-1}\frac{1}{ja_\delta^{n-j}(a_\delta+\epsilon)^j}\Big] = \frac{1}{a_\delta^n}\Big(\log a_\delta - \sum_{j=1}^{n-1}\frac{1}{j}\Big).
\end{aligned}$$

Thus, if we set $N_\delta(z,t)$ to equal the right hand side of (18) except with $|z|^2$ replaced by $|z|^2+\delta$, then $\frac{|z|^2+\delta-it}{|z|^2+\delta-it}$ stays away from the branch cut and

$$\begin{aligned}
&N_\delta(z,t) \\
&= \frac{2^{n-2}(n-1)!}{\pi^{n+1}}\frac{1}{(|z|^2+\delta-it)^n}\Big[\log\Big(\frac{|z|^2+\delta-it}{|z|^2+\delta+it}\Big) - \sum_{j=1}^{n-1}\frac{1}{j}\Big] \\
&= \frac{2^{n-2}(n-1)!}{\pi^{n+1}}\frac{1}{(|z|^2+\delta-it)^n}\Big[\log(|z|^2+\delta-it) - \log(|z|^2+\delta+it) - \sum_{j=1}^{n-1}\frac{1}{j}\Big].
\end{aligned}$$

This function is continuous in δ, thus we may send $\delta \to 0$ and obtain the theorem. $\Box$

5.4. **The Cartesian product of Heisenberg groups.** In contrast to the explicit computability of the Heisenberg group case, if

$$M = \{(z, w) \in \mathbb{C}^2 \times \mathbb{C}^2 : \operatorname{Im} w_j = |z_j|^2\},$$

$L = \{1, 2\}$, and $\alpha = (\cos\theta, \sin\theta)$, then $\Gamma^{\alpha}_{\{1,2\}}$ is the first quadrant and from Theorem 2.3, we have

$$\begin{aligned}
N_{\{1,2\}}(z,t) = \frac{1}{\pi^4}\int_{r=0}^{1}\int_{\frac{\pi}{2}}^{\pi} &|\cos\theta\sin\theta| \frac{r^{|\cos\theta|}}{(1-r^{|\cos\theta|})(1-r^{|\sin\theta|})} \\
&\times \frac{1}{(A_\alpha(r) - i(t_1\cos\theta + t_2\sin\theta))^{n+m-1}} d\theta \frac{dr}{r} \\
+ \frac{1}{\pi^4}\int_{r=0}^{1}\int_{\pi}^{\frac{3\pi}{2}} &|\cos\theta\sin\theta| \frac{r^{|\cos\theta+\sin\theta|}}{(1-r^{|\cos\theta|})(1-r^{|\sin\theta|})} \\
&\times \frac{1}{(A_\alpha(r) - i(t_1\cos\theta + t_2\sin\theta))^{n+m-1}} d\theta \frac{dr}{r} \\
+ \frac{1}{\pi^4}\int_{r=0}^{1}\int_{\frac{3\pi}{2}}^{2\pi} &|\cos\theta\sin\theta| \frac{r^{|\sin\theta|}}{(1-r^{|\cos\theta|})(1-r^{|\sin\theta|})} \\
&\times \frac{1}{(A_\alpha(r) - i(t_1\cos\theta + t_2\sin\theta))^{n+m-1}} d\theta \frac{dr}{r} \\
+ \frac{1}{\pi^4}\int_{r=0}^{1}\int_{0}^{\frac{\pi}{2}} &\cos\theta\sin\theta \Big[\frac{1}{(1-r^{\cos\theta})(1-r^{\sin\theta})} \\
&\times \frac{1}{(A_\alpha(r) - i(t_1\cos\theta + t_2\sin\theta))^{n+m-1}} \\
&- \frac{1}{(A_\alpha(0) - i(t_1\cos\theta + t_2\sin\theta))^{n+m-1}} \Big] d\theta \frac{dr}{r}
\end{aligned}$$

where

$$A_\alpha(r) = |\cos\theta| \left(\frac{1 + r^{|\cos\theta|}}{1 - r^{|\cos\theta|}}\right)|z_1|^2 + |\sin\theta| \left(\frac{1 + r^{|\sin\theta|}}{1 - r^{|\sin\theta|}}\right)|z_2|^2.$$

On the other hand, using (5), we compute the Szegö kernel

$$\begin{aligned}
S_\emptyset(z,t) &= \frac{6}{\pi^4}\int_{\pi}^{\frac{3\pi}{2}} \frac{\cos\theta\sin\theta}{((|z_1|^2 + it_1)\cos\theta + (|z_2|^2 + it_2)\sin\theta)^4} d\theta \\
&= \frac{1}{\pi^4(|z_1|^2 + it_1)^2(|z_2|^2 + it_2)^2}.
\end{aligned}$$

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