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THE TWO-SIDED POMPEIU PROBLEM FOR DISCRETE GROUPS

PETER A. LINNELL AND MICHAEL J. PULS

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ABSTRACT. We consider a two-sided Pompeiu type problem for a discrete group G. We give necessary and sufficient conditions for a finite subset K of G to have the $\mathcal{F}(G)$-Pompeiu property. Using group von Neumann algebra techniques, we give necessary and sufficient conditions for G to be an $\ell^2(G)$-Pompeiu group.

1. INTRODUCTION

Let $\mathbb{C}$ be the complex numbers, $\mathbb{R}$ the real numbers, $\mathbb{Z}$ the integers and $\mathbb{N}$ the natural numbers. Let $2 \leq n \in \mathbb{N}$ and let K be a compact subset of $\mathbb{R}^n$ with positive Lebesgue measure. The Pompeiu problem asks the following: Is $f = 0$ the only continuous function on $\mathbb{R}^n$ that satisfies

$$\int_{\sigma(K)} f\,dx = 0 \tag{1.1}$$

for all rigid motions σ? If the answer to the question is yes, then K is said to have the Pompeiu property. It is known that disks of positive radius do not have the Pompeiu property, see [20, Section 6] and the references therein for the details. The question now becomes: Are disks the only compact subsets of positive measure in $\mathbb{R}^n$ that do not have the Pompeiu property? This question is still open. It was stated in [19] that all polytopes in $\mathbb{R}^n, n \geq 2$, have the Pompeiu property, a proof of this result was recently given in [13, Corollary 1.3]. Since disks are invariant under rotations, and other sets in $\mathbb{R}^n$ are not, a reasonable question to ask is what would happen if a group of translations replaced the group of rigid motions. Thus it would be interesting to study the Pompeiu problem for groups that are being acted on by translations, see [1, 3, 15, 17, 18, 20] and the references therein for more information about various variations of the Pompeiu problem.

In [18] the following version of the Pompeiu problem was studied: Let G be a unimodular group with Haar measure μ. Suppose K is a relatively compact subset of G with positive measure. Is $f = 0$ the only function in $L^1(G)$ that satisfies

$$\int_{gKh} f(x)\,d\mu = 0 \tag{1.2}$$

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for all $g, h \in G$? This question was studied further in [3]. If G is abelian, then this problem reduces to a one-sided translation Pompeiu type problem. The purpose of this note is to investigate (1.2) in the discrete group setting and function spaces other than $\ell^1(G)$.

For the rest of this paper, G will always be a discrete group and $\mathcal{C}$ will denote a class of complex-valued functions on G that contain the zero function. Let f be a complex-valued function on G. We shall represent f as a formal sum $\sum_{g \in G} a_g g$, where $a_g \in \mathbb{C}$ and $f(g) = a_g$. Define $\mathcal{F}(G)$ to be the set of all functions on G, and let $\ell^p(G)$ be the functions in $\mathcal{F}(G)$ that satisfy $\sum_{g \in G} |a_g|^p < \infty$. The group ring $\mathbb{C}G$ consists of those functions where $a_g = 0$ for all but finitely many g. The group ring can also be thought of as the functions on G with compact support. Let K be a finite subset of G. In this paper we will consider the following discrete version of (1.2): When is $f = 0$ the only function in $\mathcal{C}$ that satisfies

$$\sum_{x \in gKh} f(x) = 0 \tag{1.3}$$

for all $g, h \in G$?

The following related Pompeiu type problem for discrete groups was investigated in [16]: When is $f = 0$ the only function in $\mathcal{C}$ that satisfies

$$\sum_{x \in gK} f(x) = 0 \tag{1.4}$$

for all $g \in G$?

It is not difficult to see that $f = 0$ is the only function in $\mathcal{C}$ that satisfies (1.3) if it is the only function in $\mathcal{C}$ satisfying (1.4). Clearly, (1.3) and (1.4) coincide if G is abelian.

We shall say that a finite subset K of G is a *$\mathcal{C}$-Pompeiu set* if $f = 0$ is the only function in $\mathcal{C}$ that satisfies (1.3). A *$\mathcal{C}$-Pompeiu group* is a group for which every nonempty finite subset is a $\mathcal{C}$-Pompeiu set.

The identity element of G will be denoted by 1. If $S \subseteq G$, then we will write χ_S to indicate the characteristic function on $S, \chi_S(g) = 1$ if $g \in S$ and $\chi_S(g) = 0$ if $g \notin S$. If S consists of one element g, then χ_g will be the usual point mass concentrated at g. For $g \in G$, the left translation of f by g is given by $L_g f(x) = f(gx)$ and the right translation of f by g is denoted by $R_g f(x) = f(xg^{-1})$, where $x \in G$.

One of our main results is:

Theorem 1.1. *Let G be a discrete group and suppose K is a finite subset of G. Let I be the ideal in $\mathbb{C}G$ that is generated by χ_K. Then K is an $\mathcal{F}(G)$-Pompeiu set if and only if $I = \mathbb{C}G$.*

We will show that as a consequence of this result, algebraically closed groups and universal groups are examples of $\mathcal{F}(G)$-Pompeiu groups. This contrasts sharply with the one-sided translation Pompeiu type problem studied in [16], since there are nonzero functions in $\ell^1(G)$ -in fact $\mathbb{C}G$- for the above groups that satisfy (1.4).

We also study the case when $\mathcal{C} = \ell^2(G)$. Suppose K is a nontrivial subgroup of G. Denote by X the closed subspace of $\ell^2(G)$ generated by χ_K that is invariant under left and right-translations by elements of G. Let Y be the closed subspace of $\ell^2(G)$ generated by χ_K that is invariant under left-translations by elements of G. Clearly $Y \subseteq X \subseteq \ell^2(G)$. We shall see that if $X = \ell^2(G)$, then K is an $\ell^2(G)$-Pompeiu set. We now assume that the subgroup K is also finite. It follows from the paragraph after Proposition 2.3 in [16] that $Y \neq \ell^2(G)$, but it could still be the case $X = \ell^2(G)$ which means that K is an $\ell^2(G)$-Pompeiu set. This illuminates the difference between (1.3) and (1.4) in that it is possible there exists a nonzero function in $\ell^2(G)$ that satisfies (1.4) but the zero function is the only function in $\ell^2(G)$ that satisfies (1.3). However, this situation changes if K is also a normal subgroup of G since Y will then be invariant under right-translations by elements of G in addition to being invariant under left-translations. In other words $Y = X$, which says that K is not an $\ell^2(G)$-Pompeiu set when K is a nontrivial finite normal subgroup of G. We will prove Theorem 1.2, which is a generalization of [3, Corollary 2.7], that shows the existence of nontrivial finite normal subgroups of G actually characterizes $\ell^2(G)$-Pompeiu groups.

Theorem 1.2. *Let G be a discrete group. Then G is an $\ell^2(G)$-Pompeiu group if and only if G does not contain a nontrivial finite normal subgroup.*

Let $A(G)$ denote the Fourier algebra of G. We shall see that the following result is a consequence of the proof of Theorem 1.2.

Theorem 1.3. *Let G be a discrete group. Then G is an $A(G)$-Pompeiu group if and only if G does not contain a nontrivial finite normal subgroup.*

It appears that the first investigation of a Pompeiu type problem in the discrete setting was the paper [21]. For discrete groups the Pompeiu problem with respect to left translations was studied in [16]. Pompeiu type problems for finite subsets of the plane were examined in [7]. An interesting connection between the Fuglede conjecture and the Pompeiu problem for finite abelian groups was established in [8].

In Section 2 we give some preliminary material and preliminary results, including giving an equivalent condition to (1.3) in terms of convolution equations. We prove Theorem 1.1 in Section 3 and we also give examples of groups that are $\mathcal{F}(G)$-Pompeiu groups. In Section 4 we discuss group von Neumann algebras and prove Theorem 1.2.

2. Preliminaries

In this section we give some necessary background and prove some preliminary results. Let $f = \sum_{g\in G} a_g g$ and let $h = \sum_{g\in G} b_g g$ be functions on G. The convolution of f and h is given by

$$f * h = \sum_{g,x\in G} a_g b_x gx = \sum_{g\in G}\left(\sum_{x\in G} a_{gx^{-1}} b_x\right) g.$$

Sometimes we will write $f * h(g) = \sum_{x\in G} f(gx^{-1})h(x)$ for when the function $f * h$ is evaluated at g. With respect to pointwise addition and convolution, $\mathbb{C}G$ is a ring. Also if $f \in \ell^1(G)$ and $h \in \ell^p(G)$, then $f * h \in \ell^p(G)$. However if both f and h are in $\ell^2(G)$ it might be the case $f * h$ is not in $\ell^2(G)$.

Recall that L_g and R_g denote the left and right translations of a function f by $g \in G$. Note that $L_g f = \chi_{g^{-1}} * f$ and $R_g f = f * \chi_g$. For a function f, $\tilde{f}$ will denote the function $\tilde{f}(x) = f(x^{-1})$, where $x \in G$ and $\bar{f}$ will indicate the complex conjugate of f. The following simple lemma gives a useful characterization of (1.3) in terms of convolution equations.

Lemma 2.1. *Let K be a finite subset of G, and let f be a complex-valued function on G. Then f satisfies* (1.3) *if and only if $\widetilde{\chi_K} * L_g f = 0$ for all $g \in G$.*

Proof. Let $g, h \in G$, then

$$\begin{aligned} \sum_{x \in gKh} f(x) &= \sum_{x \in Kh} L_g f(x) \\ &= \sum_{k \in K} L_g f(kh) \\ &= \sum_{x \in G} \chi_K(x) L_g f(xh) \\ &= \widetilde{\chi_K} * L_g f(h). \end{aligned}$$

Thus $\sum_{x \in gKh} f(x) = 0$ for all $g, h \in G$ if and only if $\widetilde{\chi_K} * L_g f = 0$ for all $g \in G$. □

A similar calculation shows that (1.3) is also equivalent to $R_g f * \widetilde{\chi_K} = 0$ for all $g \in G$. Observe that $\widetilde{\chi_K} * L_g f = \widetilde{\chi_K} * \chi_{g^{-1}} * f$ and $R_g f * \widetilde{\chi_K} = f * \chi_g * \widetilde{\chi_K}$.

Remark 2.2. It was shown in [16, Proposition 2.3] that the single convolution equation $\chi_K * \tilde{f} = 0$ is equivalent to (1.4).

If X is a set, then $|X|$ will indicate the cardinality of X. Let R be a ring and recall that the center of R is the set of all elements in R that commute under multiplication with all elements of R. An *idempotent* in R is an element e that satisfies $e^2 = e$. A central idempotent for R is an idempotent contained in the center of R. In [16] it was shown that if G contains a nonidentity element of finite order, then there is a finite subset K of G and a nonzero function in $\mathbb{C}G$ that satisfies (1.4). We shall see that this is not the case for (1.3). What is true though is

Proposition 2.3. *Suppose K is a nontrivial finite normal subgroup of G, then K is not a $\mathcal{C}$-Pompeiu set for any class of functions that contains $\mathbb{C}G$.*

Proof. Write $\chi_K = \sum_{k \in K} k$ for the characteristic function on K. Also $\widetilde{\chi_K} = \chi_K$ since K is a subgroup of G. Now,

$$\chi_K * \chi_K = \sum_{k \in K} k|K| = |K|\chi_K,$$

thus $\chi_K * (\chi_K - |K|) = 0$. For $g \in G, \chi_K * \chi_g = \chi_g * \chi_K$ if and only if $gK = Kg$. Consequently, χ_K is in the center of $\mathbb{C}G$ since K is normal in G. Let $g \in G$, then

$$\begin{aligned} \chi_K * L_g(\chi_K - |K|) &= \chi_K * \chi_{g^{-1}} * (\chi_K - |K|) \\ &= \chi_{g^{-1}} * \chi_K * (\chi_K - |K|) \\ &= 0. \end{aligned}$$

Hence, $\widetilde{\chi_K} * L_g(\chi_K - |K|) = 0$ for all $g \in G$. Lemma 2.1 yields that K is not a $\mathcal{C}$-Pompeiu set for any $\mathcal{C}$ containing $\mathbb{C}G$. □

Remark 2.4. In the above proof, $\frac{\chi_K}{|K|}$ is a central idempotent in $\mathbb{C}G$. We shall see that central idempotents play a critical role in the proof of Theorem 1.2.

3. Theorem 1.1

In this section we prove Theorem 1.1 and give some examples of groups that are $\mathcal{F}(G)$-Pompeiu groups.

Let $f = \sum_{g\in G} a_g g \in \mathbb{C}G$ and $h = \sum_{g\in G} b_g g \in \mathcal{F}(G)$. Define a map $\langle\cdot,\cdot\rangle\colon \mathbb{C}G \times \mathcal{F}(G) \to \mathbb{C}$ by

$$\langle f, h\rangle = \sum_{g\in G} a_g \overline{b_g}.$$

For a fixed $h \in \mathcal{F}(G)$, $\langle\cdot, h\rangle$ is a linear functional on $\mathbb{C}G$. Now suppose T is a linear functional on $\mathbb{C}G$. Define $h(g) = \overline{T(g)}$ for each $g \in G$. Thus each linear functional on $\mathbb{C}G$ defines an element of $\mathcal{F}(G)$. Hence, the vector space dual of $\mathbb{C}G$ can be identified with $\mathcal{F}(G)$.

3.1. **Proof of Theorem 1.1.** We now prove Theorem 1.1. Let K be a finite subset of G and let I be the ideal in $\mathbb{C}G$ generated by χ_K. We begin by showing that if K is an $\mathcal{F}(G)$-Pompeiu set, then $I = \mathbb{C}G$. Now assume that K is an $\mathcal{F}(G)$-Pompeiu set and $I \neq \mathbb{C}G$. Because I is a subspace of $\mathbb{C}G$, there is a nonzero $f \in \mathcal{F}(G)$ for which $\langle \alpha, f\rangle = 0$ for all $\alpha \in I$.

Fix $g \in G$ and let $h \in G$. Since I is an ideal, $R_h L_g \chi_K \in I$, which means $\langle R_h L_g \chi_K, f\rangle = 0$. Now

$$\begin{aligned}\langle R_h L_g \chi_K, f\rangle &= \sum_{y\in G} R_h L_g \chi_K(y)\overline{f(y)} \\ &= \sum_{y\in G} \chi_K(gyh^{-1})\overline{f(y)} \\ &= \sum_{y\in G} \chi_K(yh^{-1})\overline{f(g^{-1}y)} \\ &= (\widetilde{\chi_K} * L_{g^{-1}}\overline{f})(h).\end{aligned}$$

Thus $(\widetilde{\chi_K} * L_{g^{-1}}\overline{f})(h) = 0$ for all $h \in G$. Consequently, $\widetilde{\chi_K} * L_{g^{-1}}\overline{f} = 0$ for all $g \in G$. Thus $\overline{f}$ is a nonzero function that satisfies (1.3), contradicting our assumption that K is an $\mathcal{F}(G)$-Pompeiu set. Hence, $I = \mathbb{C}G$.

Conversely, assume $I = \mathbb{C}G$. We will finish the proof of the theorem by showing that $f = 0$ is the only function that satisfies (1.3). Set $\widetilde{I} = \{\widetilde{\alpha} \mid \alpha \in I\}$. Now $\widetilde{I}$ is generated by $\widetilde{\chi_K}$ and $\widetilde{I} = \mathbb{C}G$ since $I = \mathbb{C}G$. Assume that $f \in \mathcal{F}(G)$ satisfies (1.3). Then by Lemma 2.1, $\widetilde{\chi_K} * L_g f = R_g f * \widetilde{\chi_K} = 0$ for all $g \in G$. We now obtain $f * \widetilde{I} = 0 = \widetilde{I} * f$ since $\widetilde{\chi_K} * L_g f = \widetilde{\chi_K} * \chi_{g^{-1}} * f$ and $R_g f * \widetilde{\chi_K} = f * \chi_g * \widetilde{\chi_K}$. Now $\chi_1 \in \widetilde{I}$ and $0 = \chi_1 * f = f$, thus K is an $\mathcal{F}(G)$-Pompeiu set and the theorem is proved.

3.2. **Examples.** Before we give examples of $\mathcal{F}(G)$-Pompeiu groups we need to define the augmentation ideal of a group ring. Define a map from $\mathbb{C}G$ into $\mathbb{C}$ by

$$\varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g.$$

The map ε is a ring homomorphism onto $\mathbb{C}$. The augmentation ideal of $\mathbb{C}G$, which we will denote by $\omega(\mathbb{C}G)$, is the kernel of ε. For information about $\omega(\mathbb{C}G)$ see [14, Chapter 3]. If K is a nonempty finite subset of G, then $\chi_K \notin \omega(\mathbb{C}G)$ due to $\varepsilon(\chi_K) = |K|$. The main result of [2] showed that the only nontrivial ideal in $\mathbb{C}G$ for algebraically closed groups and universal groups is $\omega(\mathbb{C}G)$. Thus the ideal generated by χ_K in these groups must be all of $\mathbb{C}G$. Therefore, algebraically closed groups and universal groups are $\mathcal{F}(G)$-Pompeiu groups.

Remark 3.1. These groups have elements of finite order, which implies that there exists a nontrivial finite subset K in G and a nonzero function in $\mathbb{C}G$ that satisfies (1.4). See [16, Section 2] for the details.

4. Theorem 1.2

In this section we will prove Theorem 1.2. We begin by discussing group von Neumann algebras. See Dixmier's book [5] for a general discussion of von Neumann algebras, and for more detailed explanations of group von Neumann algebras see [11, Section 8] and [12, Section 1.1.1]. Recall that $\ell^2(G)$ is the set of all formal sums $\sum_{g\in G} a_g g$ for which $\sum_{g\in G} |a_g|^2 < \infty$. Furthermore, $\ell^2(G)$ is a Hilbert space with Hilbert bases $\{g \mid g \in G\}$. For $f = \sum_{g\in G} a_g g \in \ell^2(G)$ and $h = \sum_{g\in G} b_g g \in \ell^2(G)$, the inner product $\langle f, h\rangle$ is defined to be $\sum_{g\in G} a_g \overline{b_g}$. If $f \in \mathbb{C}G$ and $h \in \ell^2(G)$, then $f * h \in \ell^2(G)$. In fact, multiplication on the left by f is a continuous linear operator on $\ell^2(G)$. Thus we can consider $\mathbb{C}G$ to be a subring of $\mathcal{B}(\ell^2(G))$, the set of bounded operators on $\ell^2(G)$. Denote by $\mathcal{N}(G)$ the weak closure of $\mathbb{C}G$ in $\mathcal{B}(\ell^2(G))$. The space $\mathcal{N}(G)$ is known as the *group von Neumann algebra* of G. For $T \in \mathcal{B}(\ell^2(G))$ the following are standard facts.

(i) $T \in \mathcal{N}(G)$ if and only if there exists a net (T_n) in $\mathbb{C}G$ such that $\lim_{n\to\infty}\langle T_n u, v\rangle \to \langle Tu, v\rangle$ for all $u, v \in \ell^2(G)$.
(ii) $T \in \mathcal{N}(G)$ if and only if $(Tf) * \chi_g = T(f * \chi_g)$ for all $g \in G$.

Another way of expressing (ii) is that $T \in \mathcal{N}(G)$ if and only if T is a right $\mathbb{C}G$-map. Using (ii) we can see that if $T \in \mathcal{N}(G)$ and $T\chi_1 = 0$, then $T\chi_g = 0$ for all $g \in G$ and hence $Tf = 0$ for all $f \in \mathbb{C}G$. It follows that $T = 0$ and so the map defined by $T \mapsto T\chi_1$ is injective. Therefore the map $T \mapsto T\chi_1$ allows us to identify $\mathcal{N}(G)$ with a subspace of $\ell^2(G)$. Thus algebraically we have

$$\mathbb{C}G \subseteq \mathcal{N}(G) \subseteq \ell^2(G).$$

It is not difficult to show that if $f \in \ell^2(G)$, then $f \in \mathcal{N}(G)$ if and only if $f * h \in \ell^2(G)$ for all $h \in \ell^2(G)$. For $f = \sum_{g\in G} a_g g \in \ell^2(G)$, define $f^* = \sum_{g\in G} \overline{a_g} g^{-1} \in \ell^2(G)$. Then for $f \in \mathcal{N}(G)$ we have $\langle f * u, v\rangle = \langle u, f^* * v\rangle$ for all $u, v \in \ell^2(G)$; thus f^* is the adjoint operator of f.

Two elements x, y in G are said to be *conjugate* in G if there exists a $g \in G$ for which $g^{-1}xg = y$. Recall that the conjugation action of G on itself is an equivalence relation. Suppose C is a finite conjugacy class of G. Let $c = \sum_{x\in C} x$, then $c \in \mathbb{C}G$. The group ring elements c are known as *finite class sums*. For $x \in G$, denote by C_x the class containing x. We will need the following

Lemma 4.1. *Let $\mathcal{S}$ be the set of all finite class sums of G. Each element in the center of $\mathcal{N}(G)$, $\mathcal{Z}(\mathcal{N}(G))$, is a formal sum of elements in $\mathcal{S}$.*

Proof. Let $f = \sum_{g\in G} a_g g \in \mathcal{N}(G)$. Then $f \in \mathcal{Z}(\mathcal{N}(G))$ if and only if $\chi_{g^{-1}} * f * \chi_g = f$ for all $g \in G$. Since

$$\chi_{g^{-1}} * f * \chi_g = \sum_{x\in G} a_x(g^{-1}xg) = \sum_{y\in G} a_{gyg^{-1}} y,$$

we see immediately that $a_{gyg^{-1}} = a_y$ because $(\chi_{g^{-1}} * f * \chi_g)(y) = f(y)$. Thus f is constant on C_y. If $f(y) \neq 0$ on C_y, then C_y is finite due to $f \in \ell^2(G)$. The class sums have disjoints supports, thus if $f \in \mathcal{Z}(\mathcal{N}(G))$ then it is a formal sum of finite class sums in $\mathcal{S}$. □

The *finite conjugate subgroup* of G is defined by

$$\Delta(G) = \{g \in G \mid g \text{ has a finite number of conjugates}\}.$$

The following immediate consequence of Lemma 4.1 will be crucial in our proof of Theorem 4.3.

Corollary 4.2. *The center of $\mathcal{N}(G)$ is contained in the center of $\mathcal{N}(\Delta G)$.*

Proof. Every finite class sum contained in $\mathbb{C}G$ is contained in $\mathbb{C}(\Delta G)$. □

We will prove Theorem 1.2 by reducing to the $\mathcal{N}(G)$ case, which we now prove.

Theorem 4.3. *If G is a group with no nontrivial finite normal subgroups, then G is an $\mathcal{N}(G)$-Pompeiu group.*

Proof. It follows from [14, Lemma 4.1.5(iii)] that $\Delta(G)$ is a torsion-free abelian group since G has no nontrivial finite normal subgroups. Let K be a finite subset of G and let I be the weakly closed ideal in $\mathcal{N}(G)$ generated by $\widetilde{\chi_K}$. Now suppose there exists a nonzero $f \in \mathcal{N}(G)$ that satisfies $\widetilde{\chi_K} * L_g f = 0$ for all $g \in G$. So f belongs to the annihilator ideal $I^\perp$ of I in $\mathcal{N}(G)$. Thus $\mathcal{N}(G) = I \bigoplus I^\perp$, where $\bigoplus$ denotes the direct sum. Thus there exists a nonzero central idempotent e in $\mathcal{N}(G)$ for which $e * \mathcal{N}(G) = I^\perp$. Because $\chi_1 \in \mathcal{N}(G), e \in I^\perp$ and it follows that $I * e = 0$. By Corollary 4.2, e also belongs to the center of $\mathcal{N}(\Delta G)$. Let T be a right transversal for $\Delta(G)$ in G. Write $\widetilde{\chi_K} = \sum_{t\in T} \chi_{Kt} * t$, where $\chi_{Kt} \in \mathbb{C}(\Delta G)$. Now

$$\begin{aligned} 0 = \widetilde{\chi_K} * e &= (\sum_{t\in T} \chi_{Kt} * t) * e \\ &= \sum_{t \in T} (\chi_{Kt} * e) * t. \end{aligned}$$

Thus $\chi_{Kt} * e = 0$ for each $t \in T$. Because $\widetilde{\chi_K} \neq 0$ there exists a t' in T for which $\chi_{Kt'} \neq 0$. This contradicts the fact that $\chi_{Kt'} * \beta \neq 0$ for all $0 \neq \beta \in \ell^2(\Delta G)$, which was proved in [4]. Hence, there is no nonzero $f \in \mathcal{N}(G)$ that satisfies $\widetilde{\chi_K} * L_g f = 0$ for all $g \in G$. Therefore, every finite subset of G is an $\mathcal{N}(G)$-Pompeiu set and G is an $\mathcal{N}(G)$-Pompeiu group. □

4.1. **Proof of Theorem 1.2.** We now prove Theorem 1.2. We start with a definition. A nonzero divisor in a ring R is an element s such that $sr \neq 0 \neq rs$ for all $r \in R \setminus 0$.

Proposition 2.3 says that if G contains a nontrivial finite normal subgroup, then it cannot be an $\ell^2(G)$-Pompeiu group.

Conversely, assume there exists a nonempty finite subset K of G and a nonzero $f \in \ell^2(G)$ that satisfies $\widetilde{\chi_K} * L_g f = 0$ for all $g \in G$. By [10, Lemma 7] there exists a nonzero divisor $\theta \in \mathcal{N}(G)$ such that $f * \theta \in \mathcal{N}(G)$. Suppose $f * \theta = 0$. Then $\theta^* * f^* = 0$. If $e \in \mathcal{B}(\ell^2(G))$ is the projection from $\ell^2(G)$ onto $\overline{f^* * \mathbb{C}G}$, then $e \in \mathcal{N}(G)$ by [9, Lemma 5], $e \neq 0$ because $f^* \neq 0$ and $\theta^* * e = 0$. Hence $e * \theta = 0$, contradicting the fact θ is a nonzero divisor in $\mathcal{N}(G)$, so $f * \theta \neq 0$. It follows from Theorem 4.3 that there exists a $g \in G$ such that $\widetilde{\chi_K} * L_g(f * \theta) \neq 0$ since $0 \neq f * \theta \in \mathcal{N}(G)$. But $\widetilde{\chi_K} * L_g(f * \theta) = (\widetilde{\chi_K} * g^{-1} * f) * \theta = 0$ because we are assuming $\widetilde{\chi_K} * L_g f = 0$ for all $g \in G$, a contradiction. Hence, there does not exist a nonzero $f \in \ell^2(G)$ and a nonempty finite subset K of G for which $\widetilde{\chi_K} * L_g f = 0$ for all $g \in G$. Therefore it follows from Lemma 2.1 that G is an $\ell^2(G)$-Pompeiu group, as desired.

Remark 4.4. A key ingredient in the proof of Theorem 1.2 was [10, Lemma 7], which basically says that if $f \in \ell^2(G)$ then there exists a $\theta \in \mathcal{N}(G)$ for which $f * \theta \in \mathcal{N}(G)$. This allowed us to reduce from the $\ell^2(G)$-case to the $\mathcal{N}(G)$-case. What makes this possible is the existence of a ring $\mathcal{U}(G)$ that is the classical right quotient ring for $\mathcal{N}(G)$. This means that every element in $\mathcal{N}(G)$ is either a zero divisor or invertible in $\mathcal{U}(G)$. Another important fact is that algebraically

$$\mathbb{C}G \subseteq \mathcal{N}(G) \subseteq \ell^2(G) \subseteq \mathcal{U}(G),$$

see [11, Section 8] for the details.

4.2. **Proof of Theorem 1.3.** Let $A(G)$ denote the Fourier algebra of G. Then every element of $A(G)$ can be written in the form $f_1 * f_2$ with $f_1, f_2 \in \ell^2(G)$ [6, Theorem 2.4.3]. Thus

$$\ell^2(G) \subseteq A(G) \subset \mathcal{U}(G).$$

Now applying the argument used in the proof of Theorem 1.2 establishes Theorem 1.3.

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Department of Mathematics, Virginia Tech, Blacksburg, Virginia 24061-1026

Department of Mathematics, John Jay College-CUNY, 524 West 59th Street, New York, New York 10019

Email address: mpuls@jjay.cuny.edu