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NONEXISTENCE AND PARAMETER RANGE ESTIMATES FOR CONVOLUTION DIFFERENTIAL EQUATIONS

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(Communicated by Wenxian Shen)

This paper is dedicated to the memory of my brother Ben Goodrich (8 November 1988–25 February 2022), who was taken from this life much too soon

Abstract. We consider nonlocal differential equations with convolution coefficients of the form

$$-M\Big(\big(a * u^q\big)(1)\Big)u''(t) = \lambda f\big(t, u(t)\big), t \in (0, 1),$$

and we demonstrate an explicit range of λ for which this problem, subject to given boundary data, will not admit a nontrivial positive solution; if $a \equiv 1$, then the model case

$$-M\Big(\|u\|^q_{L^q(0,1)}\Big)u''(t) = \lambda f\big(t, u(t)\big), t \in (0, 1)$$

is obtained. The range of λ is calculable in terms of initial data, and our results allow for a variety of kernels, a, to be utilized, including, for example, those leading to a fractional integral coefficient of Riemann-Liouville type. Two examples are provided in order to illustrate the application of the result.

1. Introduction

For sufficiently regular functions a and u define by $t \mapsto (a * u)(t)$, $t \geq 0$, the finite convolution

$$(a * u)(t) := \int_0^t a(t - s)u(s)\ ds.$$

In this brief note we consider the convolution-type nonlocal differential equation

$$-M\Big(\big(a * u^q\big)(1)\Big)u''(t) = \lambda f\big(t, u(t)\big), t \in (0, 1), \tag{1.1}$$

where $\lambda > 0$ and $q \geq 1$ are parameters and both M and f are continuous functions. We demonstrate that, subject to given boundary data, problem (1.1) will *not* admit a positive solution when λ is sufficiently large. The lower bound on λ is explicitly calculable in terms of initial data, and so, a specific range of λ can be provided. Note that if the kernel a satisfies $a(x) \equiv 1$, then problem (1.1) reduces to the model case

$$-M\Big(\|u\|^q_{L^q(0,1)}\Big)u''(t) = \lambda f\big(t, u(t)\big), t \in (0, 1). \tag{1.2}$$

One motivation for studying the much more general convolution-type problem (1.1) is because this includes as a special case fractional integral nonlocalities of Riemann-Liouville type. Indeed, put $b(t) := \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$ for $0 < \alpha < 1$ and one has that

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$(b * u^q)(1)$ is the α-th order fractional Riemann-Liouville integral of u^q at $t = 1$—see, for example, [6, 25, 26, 28, 41–43, 47, 48, 51] for additional details on the fractional calculus and, in particular, how convolution operators arise naturally in the study of such operators. Our results also apply to a wide variety of boundary data, and Examples 2.3 and 2.4 provide examples in the case of Dirichlet boundary conditions.

Our main result, Theorem 2.1, demonstrates that the integral operator $T : \mathscr{C}\big([0,1]\big) \to \mathscr{C}\big([0,1]\big)$ defined by

$$(Tu)(t) := \lambda \int_0^1 \Big(M\big((a * u^q)(1)\big)\Big)^{-1} G(t,s) f\big(s, u(s)\big)\, ds \tag{1.3}$$

has no nontrivial fixed points under certain conditions, where the function $G : [0,1] \times [0,1] \to [0,+\infty)$ is determined by the boundary conditions to which we wish to subject (1.1). Since a lack of fixed points of T implies a lack of solution of (1.1) when equipped with the boundary data encoded by G, in this way we are able to consider a variety of boundary conditions simultaneously.

The study of nonlocal differential equations is quite extensive. The model case (1.2) is a commonly studied case in the one-dimensional setting (or the analogous problem in the PDEs setting)—see, for example, Alves and Covei [2], Corrêa [10], Corrêa, Menezes, and Ferreira [11], do Ó, Lorca, Sánchez, and Ubilla [13], Goodrich [17, 18], Infante [32], Stańczy [45], Wang, Wang, and An [46], Yan and Ma [49], and Yan and Wang [50]. Another commonly studied model case is

$$-M\Big(\|u'\|^q_{L^q(0,1)}\Big)u''(t) = \lambda f\big(t, u(t)\big), t \in (0,1), \tag{1.4}$$

which is an example of a one-dimensional Kirchhoff-type problem; various analogous problems in the PDEs setting are also frequently studied—see, for example, Afrouzi, Chung, and Shakeri [1], Azzouz and Bensedik [4], Boulaaras [7], Boulaaras and Guefaifia [8], Chung [9], Goodrich [19, 23], and Infante [30, 31]. Kirchhoff-type equations, in particular, arise from steady-state (i.e., time independent) solutions of the nonlocal wave-type PDE $u_{tt} - M\left(\int_\Omega |Du|^2\, d\boldsymbol{s}\right)(\Delta u)(\boldsymbol{x}) = f\big(\boldsymbol{x}, u(\boldsymbol{x})\big), \boldsymbol{x} \in \Omega \subset \mathbb{R}^n$, which was studied by Kirchhoff in the late 1800s—see, for instance, the paper by Graef, Heidarkhani, and Kong [29] for additional discussion. More generally, nonlocal differential equations have been extensively studied, in part, due to their application in diverse modeling such as beam deflection [33] and chemical reactor theory [38]—see [5, 15, 16, 34–36, 39, 40] for additional details.

Recently Goodrich together with Lizama [20–22, 24, 27] has introduced a new methodology for treating problems such as (1.2) and (1.4). This methodology relies on the nonstandard cone

$$\mathscr{K} := \big\{u \in \mathscr{C}\big([0,1]\big) \;:\; u \geq 0, (a * u)(1) \geq C_0\|u\|_\infty\big\}, \tag{1.5}$$

where C_0 is a positive constant defined later in Section 2, and the associated open set

$$\widehat{V}_\rho := \big\{u \in \mathscr{K} \;:\; (a * u^q)(1) < \rho\big\}. \tag{1.6}$$

Note that (1.5) demands that the functional $u \mapsto (a * u)(1)$ be coercive with coercivity constant C_0. The key topological fact is that when $u \in \partial\widehat{V}_\rho$ it follows that $(a * u^q)(1) = \rho$, which gives us direct control over the argument of M in (1.1). In particular, when studying existence of positive solutions to (1.1) this allows us to consider the case in which M is allowed to vanish and change sign, infinitely often; really, it need only be the case that $M(t) > 0$ on a set of positive but, nonetheless, small measure. This is very different than most competing methodologies, in which $M(t) > 0$ is demanded generally for all $t \geq 0$. Even regarding the very recent papers by Ambrosetti and Arcoya [3], Delgado, Morales-Rodrigo, Santos Júnior, and Suárez [12], and Santos Júnior and Siciliano [44], which are rich in good mathematical ideas and insights, our new methodology avoids some of the restrictions seen there.

In spite of the wide literature there are few *nonexistence* results. In fact, we are not aware of any results of this type for the very general nonlocal equation (1.1). Our goal in this paper is to make an effort to begin to fill this gap. The methodology that we use to produce our nonexistence result is noteworthy because we *directly* use the coercivity condition in (1.5) and the open set in (1.6) in order to deduce the nonexistence result. This is unusual because typically when deducing nonexistence for a one-dimensional boundary value problem it is more standard to deduce a contradiction involving $\|\cdot\|_\infty$ (cf., Infante and Pietramala [37, Theorem 4.1]). We take a very different tactic, avoiding completely this type of "norm-wise" contradiction. Instead we directly use $\widehat{V}_\rho$ together with the coercivity condition in $\mathscr{K}$ in order to demonstrate that for each $\rho > 0$ there can be no $u \in \partial\widehat{V}_\rho$ such that (1.3) admits a positive fixed point. Then as any nontrivial and, thus, positive fixed point of (1.3) must live in $\bigcup_{0<\rho<+\infty} \partial\widehat{V}_\rho$, the desired result follows (note that this uses the fact—see Section 2—that $a(t) > 0$, a.e. $t \in [0,1]$).

This unusual approach allows us to take advantage of the fact that whenever $u \in \partial\widehat{V}_\rho$ it follows that $(a * u^q)(1) = \rho$, which gives us more direct control over the integral operator T in (1.3). We believe this novel methodology most likely can be extended to other classes of nonlocal boundary problems such as the ones mentioned earlier in this section.

2. Main result

Let T be the operator defined in (1.3) in Section 1. Throughout the remainder of the note we denote by $\|\cdot\|_\infty$ the maximum norm on $[0,1]$, with which we equip the space $\mathscr{C}\big([0,1]\big)$. Furthermore, with abuse of notation we denote by $\mathbf{1}$ the constant map $\mathbf{1}(x) \equiv 1$. Finally, we state some general restrictions, which we impose on the functions a, f, G, and M in definition of the operator T. We note, in passing, that although we state the domain of a as $[0,1]$, because a need only be L^1, it is allowable that a be defined, for example, only on $(0,1)$. The kernel $a(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$ described in Section 1, for instance, is defined only for $t > 0$, but this is of no concern in what follows. Note that condition (H1.1) implies that f satisfies "standard growth" from below.

H1: The functions $M : [0,\infty) \to \mathbb{R}$, $f : [0,1] \times [0,\infty) \to [0,\infty)$, and $a : [0,1] \to [0,\infty)$ satisfy the following properties.

(1) Both M and f are continuous. Moreover, f satisfies the inequality

$$f(t,u) \geq c_1 u^r, t \in [0,1], u > 0,$$

where $c_1 > 0$ is a constant and $r > q$.

(2) $a \in L^1\big((0,1)\big)$

(3) $a(t) > 0$, a.e. $t \in [0,1]$

H2: The function $G : [0,1] \times [0,1] \to [0,\infty)$ satisfies the following properties.

(1) It is continuous.

(2) Putting $\mathscr{G}(s) := \max_{t\in[0,1]} G(t,s)$, $0 \le s \le 1$, the set $S_0 := \big\{s \in [0,1] \ : \ \mathscr{G}(s) \neq 0\big\} \subseteq [0,1]$ has full measure and

$$C_0 := \inf_{s\in S_0} \frac{1}{\mathscr{G}(s)}\big(a * G(\cdot,s)\big)(1) = \inf_{s\in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 a(1-t)G(t,s)\ dt.$$

is finite and positive.

(3) With $a^{-\frac{r}{q-r}} \in L^1\big((0,1)\big)$ the quantity

$$G_0 := \sup_{t\in S_0} \left(\left(a^{-\frac{r}{q-r}} * \big(G(t,\cdot)\big)^{\frac{q}{q-r}}\right)(1)\right)^{\frac{q-r}{q}}$$

is well defined and satisfies $0 < G_0 < \infty$, where r is the number from condition (H1).

We now present our nonexistence result.

Theorem 2.1. *Assume that each of conditions (H1) and (H2) is true. If*

$$\lambda > \sup_{\rho>0\ :\ A(\rho)>0} \frac{\rho^{\frac{1-r}{q}} M(\rho)}{c_1 C_0 G_0 \big((a*\mathbf{1})(1)\big)^{\frac{1-q}{q}}},$$

then the integral operator T cannot have a positive fixed point.

Proof. For contradiction assume that the operator T has a nontrivial positive fixed point—namely, that $(Tu_0)(t) = u_0(t)$ for each $t \in [0,1]$ and with $u_0 \in \mathscr{C}\big([0,1]\big)$ such that both $\|u_0\| > 0$ and $u_0(t) \ge 0$ for all $t \in [0,1]$. Since $\|u_0\| > 0$, there exists a number $\rho > 0$ such that $u_0 \in \partial\widehat{V}_\rho$—that is, since $a(t) > 0$, a.e. $t \in [0,1]$, it holds that

$$\big(a * u_0^q\big)(1) = \rho. \tag{2.1}$$

We will consider three cases.

(A) $M(\rho) > 0$
(B) $M(\rho) = 0$
(C) $M(\rho) < 0$

Obviously, for a given $\rho > 0$, cases (A), (B), and (C) are exhaustive. Our goal is to show that for each $\rho > 0$ each of these cases leads to a contradiction under the assumptions of the theorem, and so, T cannot have a nontrivial fixed point, as claimed.

So, let us first consider case (A)—i.e., we will assume that $M(\rho) > 0$. A simple calculation (see, for example, either [17, Lemma 2.3] or [22, Lemma 2.3]) demonstrates that for any $u \in \mathscr{C}\big([0,1]\big)$ the operator T satisfies the coercivity inequality

$$(a * Tu)(1) \ge C_0 \|Tu\|_\infty, \tag{2.2}$$

which is simply a consequence of the definition of C_0 in condition (H3.2). Then using inequality (2.2) together with the fact that u_0 is a fixed point of T we calculate

$$
\begin{aligned}
(2.3)\qquad \big(a * u_0^q\big)(1) = \big(a * Tu_0^q\big)(1) &= \int_0^1 a(1-s)\big((Tu_0)(s)\big)^q\, ds\\
&= \int_0^1 \big(a(1-s)\big)^{1-q}\big(a(1-s)(Tu_0)(s)\big)^q\, ds\\
&\geq \left(\int_0^1 a(1-s)\, ds\right)^{1-q}\left(\int_0^1 a(1-s)(Tu_0)(s)\, ds\right)^q\\
&= \big((a*\mathbf{1})(1)\big)^{1-q}\big((a*Tu_0)(1)\big)^q\\
&\geq C_0^q\|Tu_0\|_\infty^q\big((a*\mathbf{1})(1)\big)^{1-q},
\end{aligned}
$$

where we have used the reverse Hölder inequality to obtain the first inequality.

Next, using that we are in case (A) together with identity (2.1), observe that

$$
\begin{aligned}
\|Tu_0\|_\infty &= \max_{t\in[0,1]} \lambda \int_0^1 \underbrace{\Big(M\big((a*u^q)(1)\big)\Big)^{-1}}_{>0} G(t,s)f\big(s,u_0(s)\big)\, ds\\
(2.4)\qquad &= \sup_{t\in S_0} \lambda\int_0^1 \big(M(\rho)\big)^{-1}G(t,s)f\big(s,u_0(s)\big)\, ds\\
&= \left(\frac{\lambda}{M(\rho)}\right)\left(\sup_{t\in S_0}\int_0^1 G(t,s)f\big(s,u_0(s)\big)\, ds\right),
\end{aligned}
$$

from which it follows that

$$
(2.5)\qquad \|Tu_0\|_\infty^q = \left(\frac{\lambda}{M(\rho)}\right)^q\left(\sup_{t\in S_0}\int_0^1 G(t,s)f\big(s,u_0(s)\big)\, ds\right)^q.
$$

Note that to switch to the supremum in (2.4) we use the fact that Tu is continuous on $[0,1]$ by virtue of condition (H2.1) together with the fact that S_0 has full measure.

We next work on estimating the second factor appearing in identity (2.5). To this end recall that $f(t,u) \geq c_1u^r$, for all $t \in [0,1]$ and $u \geq 0$, and where $r > q$. Then, again recalling from condition (H2.3) that $[0,1] \setminus S_0$ is Lebesgue null, we estimate

(2.6)

$$\begin{aligned}
&\int_0^1 G(t,s)f\big(s,u_0(s)\big)\ ds\\
&\quad\geq \int_0^1 G(t,s)\cdot c_1\big(u_0(s)\big)^r\ ds\\
&\quad= c_1\int_{S_0} G(t,s)\big(a(1-s)\big)^{-\frac{r}{q}}\big(a(1-s)\big)^{\frac{r}{q}}\big(u_0(s)\big)^r\ ds\\
&\quad\geq c_1\left(\int_{S_0}\big(a(1-s)\big)^{-\frac{r}{q-r}}\big(G(t,s)\big)^{\frac{q}{q-r}}\ ds\right)^{\frac{q-r}{q}}\left(\int_{S_0} a(1-s)\big(u_0(s)\big)^q\ ds\right)^{\frac{r}{q}}\\
&\quad= c_1\left(\int_{S_0}\big(a(1-s)\big)^{-\frac{r}{q-r}}\big(G(t,s)\big)^{\frac{q}{q-r}}\ ds\right)^{\frac{q-r}{q}}\big((a*u_0^q)(1)\big)^{\frac{r}{q}}\\
&\quad= c_1\left(\int_{S_0}\big(a(1-s)\big)^{-\frac{r}{q-r}}\big(G(t,s)\big)^{\frac{q}{q-r}}\ ds\right)^{\frac{q-r}{q}}\cdot\rho^{\frac{r}{q}}\\
&\quad= c_1\rho^{\frac{r}{q}}\left(\left(a^{-\frac{r}{q-r}}*\big(G(t,\cdot)\big)^{\frac{q}{q-r}}\right)(1)\right)^{\frac{q-r}{q}},
\end{aligned}$$

where we have used the reverse Hölder inequality, again keeping in mind that $r > q$, together with identity (2.1) again. Consequently, putting (2.6) into (2.5) we see that

(2.7)

$$\begin{aligned}
&\|Tu_0\|_\infty^q\\
&\quad= \left(\frac{\lambda}{M(\rho)}\right)^q\left(\sup_{t\in[0,1]}\int_0^1 G(t,s)f\big(s,u_0(s)\big)\ ds\right)^q\\
&\quad\geq \left(\frac{\lambda}{M(\rho)}\right)^q\left(\sup_{t\in S_0} c_1\left(\int_{S_0}\big(a(1-s)\big)^{-\frac{r}{q-r}}\big(G(t,s)\big)^{\frac{q}{q-r}}\ ds\right)^{\frac{q-r}{q}}\cdot\rho^{\frac{r}{q}}\right)^q\\
&\quad= \rho^r\left(\frac{c_1\lambda}{M(\rho)}\right)^q\left(\underbrace{\sup_{t\in S_0}\left(\left(a^{-\frac{r}{q-r}}*\big(G(t,\cdot)\big)^{\frac{q}{q-r}}\right)(1)\right)^{\frac{q-r}{q}}}_{=G_0}\right)^q = \rho^r\left(\frac{c_1\lambda G_0}{M(\rho)}\right)^q.
\end{aligned}$$

Therefore, upon combining estimates (2.3) and (2.7) we deduce that

(2.8)

$$\rho = (a*u_0^q)(1) \geq C_0^q\|Tu_0\|_\infty^q\big((a*\mathbf{1})(1)\big)^{1-q} \geq C_0^q\rho^r\left(\frac{c_1\lambda G_0}{M(\rho)}\right)^q\big((a*\mathbf{1})(1)\big)^{1-q}.$$

But recall that by assumption it holds that

$$\lambda > \sup_{\rho>0\ :\ A(\rho)>0}\frac{\rho^{\frac{1-r}{q}}M(\rho)}{c_1C_0G_0\big((a*\mathbf{1})(1)\big)^{\frac{1-q}{q}}}.$$

Therefore, from inequality (2.8) together with the lower bound on λ we deduce that

$$\rho \geq C_0^q\rho^r\left(\frac{c_1\lambda G_0}{M(\rho)}\right)^q\big((a*\mathbf{1})(1)\big)^{1-q} > \rho,$$

and so, we arrive at a contradiction. In other words, it must be the case that $u_0 \neq Tu_0$. Consequently, the operator T cannot have a fixed point u_0 satisfying $u_0 \in \partial\widehat{V}_\rho$. In fact, since in the definition of λ the supremum is taken over all ρ such that $A(\rho) > 0$, we conclude that T cannot have a fixed point in the set

$$\mathscr{X}_1 := \Big\{u \in \mathscr{C}\big([0,1]\big) \ : \ M\big((a * u^q)(1)\big) > 0\Big\}.$$

All in all, therefore, we conclude that in case (A) the operator T cannot have a positive fixed point.

Next we consider case (B). If $M(\rho) = 0$, then the operator T itself is not well defined. However, this case can be safely excluded from consideration because if $M(\rho) = 0$, then differential equation itself degenerates to

$$0 = f\big(t, u_0(t)\big), t \in (0,1). \tag{2.9}$$

But by the restriction on f in the statement of the theorem we know that $f(t,u) \geq c_1 u^r > 0$ whenever $u > 0$. Hence, $t \mapsto f\big(t, u(t)\big)$ cannot be identically zero if $t \mapsto u(t)$ itself is not zero identically, and so, it follows that T cannot have a positive fixed point in this case or else identity (2.9) would contradict the assumption on f.

Finally, we consider case (C)—i.e., the case $M(\rho) < 0$. Supposing that T did have a fixed point $u_0 \in \mathscr{C}\big([0,1]\big)$, if u_0 was a positive fixed point so that $u_0(t) \geq 0$, $t \in [0,1]$, then we would calculate

$$0 \leq u_0(t) = \lambda \underbrace{\int_0^1 \underbrace{\big(M(\rho)\big)^{-1}}_{<0} \underbrace{G(t,s) f\big(s, u_0(s)\big)}_{>0} \ ds}_{<0} < 0,$$

which is evidently a contradiction. Consequently, $Tu_0 \not\equiv u_0$ whenever $u \in \partial\widehat{V}_\rho$ with $M(\rho) < 0$.

In summary, for any $u_0 \in \mathscr{C}\big([0,1]\big)$ satisfying both $\|u_0\|_\infty > 0$ and $u_0(t) \geq 0$, $t \in [0,1]$, in each of cases (A), (B), and (C) the function u_0 cannot have be a fixed point of the operator T. Therefore, we conclude that T has no nontrivial fixed points under the hypotheses of the theorem. And this completes the proof. ☐

Remark 2.2. Notice that in the *local* case Theorem 2.1 is consistent with a known result. In particular, suppose that

$$M(\rho) \equiv 1$$

so that problem (1.1) reduces to

$$-u''(t) = \lambda f\big(t, u(t)\big), t \in (0,1).$$

Then the condition in the statement of Theorem 2.1 becomes

$$\lambda > \sup_{\rho > 0} \frac{\rho^{\frac{1-r}{q}}}{c_1 C_0 G_0 ((a * \mathbf{1})(1))^{\frac{1-q}{q}}} = +\infty.$$

And this means that the nonexistence theorem does *not* apply. But this is exactly what we would expect. Indeed, condition (H1.1) is compatible with the configuration (uniformly in t)

$$\lim_{u \to 0^+} \frac{f(t,u)}{u} = 0 \text{ and } \lim_{u \to \infty} \frac{f(t,u)}{u} = \infty.$$

But this configuration, which occurs when f is superlinear, yields *existence* of solution in the *local* case—see, for example, the landmark paper by Erbe and Wang [14, Theorem 1 part (i), p. 744]. Thus, the conclusion of Theorem 2.1 is consistent with the known result in case $M(\rho) \equiv 1$.

To conclude this note we provide an application of Theorem 2.1 to problem (1.1) in the case of Dirichlet boundary conditions. We do this first in case $a \equiv \mathbf{1}$ and then in case $a(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, $t \in (0,1]$.

Example 2.3. Suppose that $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ satisfies condition (H1) with $r = 8$ and $c_1 = 1$—i.e., $f(t,u) \geq u^8, 0 \leq t \leq 1, u \geq 0$, and that the function $M : [0,+\infty) \to \mathbb{R}$ satisfies

$$M(\rho) := \begin{cases} -\rho\cos\rho, & 0 \leq \rho < \frac{\pi}{2} \\ \left(\rho - \frac{\pi}{2}\right)\sin\rho, & \frac{\pi}{2} \leq \rho < +\infty \end{cases}.$$

Let us consider the following boundary value problem, in which we have selected $a \equiv \mathbf{1}$.

$$\begin{aligned} -M\Big(\|u\|_{L^4(0,1)}^4\Big)u''(t) &= \lambda f\big(t,u(t)\big), 0 < t < 1 \\ u(0) &= 0 \\ u(1) &= 0. \end{aligned} \tag{2.10}$$

In other words, in problem (2.10) the nonlocal coefficient is $M\big((\mathbf{1} * u^4)(1)\big)$. Notice that this corresponds to the differential equation (1.1) equipped with Dirichlet boundary conditions and with the kernel a selected to be the function $\mathbf{1}$. Moreover, we have selected

$$q := 4 < 8 =: r.$$

Given the boundary conditions in (2.10) it is known that the associated Green's function is

$$G(t,s) := \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1 \end{cases}.$$

Then with G selected as above it follows that a fixed point of T is a solution of the differential equation and conversely. Consequently, Theorem 2.1 can be used to exhibit nonexistence of a positive solution to problem (2.10).

Now, for $0 < t < 1$ we calculate

$$\begin{aligned} \int_0^1 \big(G(t,s)\big)^{\frac{q}{q-r}}\ ds &= \int_0^1 \big(G(t,s)\big)^{-\frac{1}{2}}\ ds \\ &= \int_0^t \big(s(1-t)\big)^{-\frac{1}{2}}\ ds + \int_t^1 \big(t(1-s)\big)^{-\frac{1}{2}}\ ds = \frac{2}{\sqrt{t}\sqrt{1-t}}. \end{aligned}$$

But then

$$\sup_{t\in(0,1)} \left(\int_0^1 \big(G(t,s)\big)^{\frac{q}{q-r}}\ ds\right)^{\frac{q-r}{q}} = \sup_{t\in(0,1)} \left(\frac{2}{\sqrt{t}\sqrt{1-t}}\right)^{-2} = \frac{1}{16}.$$

Note that here we selected $S_0 := (0,1)$, which does have full measure. Thus, $G_0 = \frac{1}{16}$. It can also be shown that in this case (see, for example, [17, Example

2.7]) $C_0 = \frac{1}{2}$. Consequently,

$$\sup_{\rho>0\,:\,A(\rho)>0} \frac{\rho^{\frac{1-r}{q}} M(\rho)}{c_1 C_0 G_0\big((a*\mathbf{1})(1)\big)^{\frac{1-q}{q}}} = \sup_{\rho>0\,:\,A(\rho)>0} 32\rho^{-\frac{7}{4}} M(\rho) \approx 5.468,$$

where we have estimated the supremum to three decimal places of accuracy. So, we conclude that problem (2.10) does not have a positive solution (i.e., the associated operator T does not have a positive fixed point) for (to three decimal places of accuracy)

$$\lambda > 5.468.$$

Note that our result applies even though $\liminf_{\rho\to+\infty} M(\rho) = -\infty$. As mentioned in Section 1, this is unusual.

Example 2.4. Although we chose $a \equiv \mathbf{1}$ in Example 2.3, this was purely for the sake of convenience so as to illustrate the application of the result in a cleaner setting. So, in this example let us consider problem (1.1) subjected again to Dirichlet boundary conditions but with a not identically $\mathbf{1}$. Indeed, for $t > 0$ consider the kernel $a(t) := \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, $\alpha > 0$, which was mentioned in Section 1 as playing an important role in the theory of the Riemann-Liouville fractional integral. In consideration of the previous example, since

$$\big(a(1-s)\big)^{-\frac{r}{q-r}} = \left(\frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}\right)^2 = \frac{1}{\big(\Gamma(\alpha)\big)^2}(1-s)^{2(\alpha-1)},$$

we see that

$$\big(a(1-s)\big)^{-\frac{r}{q-r}} \big(G(t,s)\big)^{\frac{q}{q-r}} = \frac{1}{\big(\Gamma(\alpha)\big)^2} \begin{cases} t^{-\frac{1}{2}}(1-s)^{2\alpha-\frac{5}{2}}, & 0 \le t \le s \le 1 \\ s^{-\frac{1}{2}}(1-s)^{2(\alpha-1)}(1-t)^{-\frac{1}{2}}, & 0 \le s \le t \le 1 \end{cases},$$

from which it follows, for $0 < t < 1$, that

$$\begin{aligned} &\int_0^1 \big(a(1-s)\big)^{-\frac{r}{q-r}} \big(G(t,s)\big)^{\frac{q}{q-r}}\, ds \\ &\quad = \frac{1}{(4\alpha-3)\sqrt{t}\sqrt{1-t}\,\big(\Gamma(\alpha)\big)^2}\left[2(1-t)^{2\alpha-1} + 2t(4\alpha-3)\,{}_2F_1\left(\frac{1}{2}, 2-2\alpha; \frac{3}{2}; t\right)\right], \end{aligned}$$

provided that $\alpha > \frac{3}{4}$ (so that the integral converges). Note that ${}_2F_1\left(\frac{1}{2}, 1-2\alpha; \frac{3}{2}; t\right)$ is the hypergeometric function. It can then be deduced that

$$\sup_{t\in(0,1)} \int_0^1 \big(a(1-s)\big)^{-\frac{r}{q-r}} \big(G(t,s)\big)^{\frac{q}{q-r}}\, ds$$

is positive and finite. In other words, in the case of Dirichlet boundary conditions the result is applicable with an α-th order Riemann-Liouville fractional integral coefficient provided that $\alpha > \frac{3}{4}$.

Consequently, the result applies to physically meaningful settings in which $a \not\equiv \mathbf{1}$. Note that this result covers the case when the argument of M is $\|u\|_{L^q(0,1)}^q$. Indeed, when $\alpha = 1$ we note that $(b * u^q)(1) = (\mathbf{1} * u^q)(1) = \|u\|_{L^q(0,1)}^q$. In a certain sense, then, the restriction $\alpha > \frac{3}{4}$ is the sort of restriction one might *a priori* guess since it asserts that if the operator $u \mapsto (b * u^q)(1)$ is "too" fractional (i.e., in a certain sense possesses too strong of a singular nonlocal kernel), then the result may not apply.

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