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ON A GENERALIZATION OF THE HÖRMANDER CONDITION

SOICHIRO SUZUKI

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ABSTRACT. We consider a natural generalization of the classical Hörmander condition in the Calderón–Zygmund theory. Recently the author [J. Fourier Anal. Appl. 27 (2021)] proved the L^p boundedness of singular integral operators under the L^1 mean Hörmander condition, which was originally introduced by Grafakos and Stockdale [Bull. Hellenic Math. Soc. 63 (2019), pp. 54–63]. In this paper, we show that the L^1 mean condition actually coincides with the classical one. On the other hand, we introduce a new variant of the Hörmander condition, which is strictly weaker than the classical one but still enough for the L^p boundedness. Moreover, it still works in the non-doubling setting with a little modification.

1. Introduction

Our aim is to generalize Theorem A, which is one of the most famous results in the classical Calderón–Zygmund theory of singular integral operators.

Theorem A ([1, Lemma 2, Theorem 1], [2, (1.24)], [7, Theorem 2.2])**.** *Let T be a singular integral operator associated with a kernel K. Suppose that T is bounded on $L^{p_0}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ and its kernel K satisfies the Hörmander condition:*

$$(1.1)\qquad [K]_{H_\infty} := \sup_{B\subset\mathbb{R}^d} \sup_{y\in B} \int_{x\in\mathbb{R}^d\setminus 2B} |K(x,y) - K(x,c(B))|\,dx < \infty,$$

where the supremum $\sup_{B\subset\mathbb{R}^d}$ is taken over all balls $B \subset \mathbb{R}^d$, $c(B)$ is the center of B, $2B$ denotes the ball with the same center as B and whose radius is twice as long. Then T is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ and from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, thus on $L^p(\mathbb{R}^d)$ for any $1 < p < p_0$.

In 2019, Grafakos and Stockdale [6] introduced an L^q mean Hörmander condition
(1.2)

$$[K]_{H_q} := \sup_{B\subset\mathbb{R}^d} \left(\frac{1}{|B|} \int_{y\in B} \left(\int_{x\in\mathbb{R}^d\setminus 2B} |K(x,y) - K(x,c(B))|\,dx \right)^q dy \right)^{1/q} < \infty$$

in order to establish a "limited-range" version of Theorem A. The author [11] improved Theorem A by assuming the L^1 mean Hörmander condition using an idea inspired by Fefferman [4, THEOREM 2'].

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Theorem B ([11, Theorem 1, Theorem 3])**.** *Let T be a singular integral operator associated with a kernel K. Suppose that T is bounded on $L^{p_0}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ and its kernel K satisfies the L^1 mean Hörmander condition:*

$$[K]_{H_1} = \sup_{B\subset\mathbb{R}^d} \frac{1}{|B|}\int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B} |K(x,y)-K(x,c(B))|\,dx\,dy < \infty. \tag{1.3}$$

Then T is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ and from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, thus on $L^p(\mathbb{R}^d)$ for any $1 < p < p_0$.

In this paper, we show that the L^1 mean Hörmander condition (1.3) is the same as the classical one (1.1) and therefore Theorems A and B are equivalent.

Theorem 1. *The inequality*

$$[K]_{H_1} \le [K]_{H_\infty} \lesssim [K]_{H_1} \tag{1.4}$$

holds for any $K \in L^1_{\mathrm{loc}}((\mathbb{R}^d\times\mathbb{R}^d)\setminus\Delta)$, where Δ denotes the diagonal set $\{(x,x) : x\in\mathbb{R}^d\}$.

Now we introduce a new variant of the Hörmander condition:

$$[K]_{H_*} := \sup_{B\subset\mathbb{R}^d} \frac{1}{|B|}\int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B} \left| K(x,y) - \frac{1}{|B|}\int_{z\in B} K(x,z)\,dz \right| dx\,dy < \infty, \tag{1.5}$$

which is a natural generalization of the L^1 mean Hörmander condition in terms of BMO. Recall that a function $f\in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is in $\mathrm{BMO}(\mathbb{R}^d)$ if

$$\|f\|_{\mathrm{BMO}} := \sup_{B\subset\mathbb{R}^d}\frac{1}{|B|}\int_{y\in B}\left| f(y) - \frac{1}{|B|}\int_{z\in B} f(z)\,dz\right| dy < \infty. \tag{1.6}$$

We call (1.5) a BMO Hörmander condition. Note that we can easily see $[K]_{H_*} \le 2[K]_{H_\infty}$, which is analogous to $\|f\|_{\mathrm{BMO}} \le 2\|f\|_\infty$. We will show the following:

Theorem 2. *Let T be a singular integral operator associated with a kernel K. Suppose that T is bounded on $L^{p_0}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ and its kernel K satisfies $[K]_{H_*} < \infty$. Then T is bounded from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, thus on $L^p(\mathbb{R}^d)$ for any $1 < p < p_0$. On the other hand, T is not bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ in general. In particular, the* BMO *Hörmander condition is strictly weaker than the classical one.*

Moreover, Theorem 2 still holds in the non-doubling setting with an appropriate modification. Let μ be a Radon measure on $\mathbb{R}^d$ which satisfies the polynomial growth condition: there exists a constant $C_\mu > 0$ and $0 < n \le d$ such that

$$\mu(B(c,r)) \le C_\mu r^n \tag{1.7}$$

for any balls $B(c,r)$. In this case, the following generalization of Theorem A is known.

Theorem C ([9, Theorem 6.1], [12, Theorem 4.2])**.** *Let T be a singular integral operator associated with a kernel K. Suppose that T is bounded on $L^{p_0}(\mu)$ for some $1 < p_0 < \infty$ and its kernel K satisfies the Hörmander condition with respect to μ:*

$$[K]_{H_\infty} := \sup_{B\subset\mathbb{R}^d}\sup_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B} |K(x,y)-K(x,c(B))|\,d\mu(x) < \infty, \tag{1.8}$$

and $|K(x,y)| \lesssim |x-y|^{-n}$. *Then* T *is bounded from* $L^1(\mu)$ *to* $L^{1,\infty}(\mu)$ *and from* $H^1_{\rm atb}(\mu)$ *to* $L^1(\mu)$*, thus on* $L^p(\mu)$ *for any* $1 < p < p_0$*, where* $H^1_{\rm atb}(\mu)$ *is the atomic block Hardy space introduced by Tolsa* [12]*.*

The $L^1(\mu) \to L^{1,\infty}(\mu)$ boundedness was proved by Nazarov, Treil and Volberg [9], and the $H^1_{\rm atb}(\mu) \to L^1(\mu)$ boundedness by Tolsa [12]. Also Tolsa [13] gave another proof of the $L^1(\mu) \to L^{1,\infty}(\mu)$ boundedness. We will give a natural generalization of Theorem C in the sense of Theorem 2 (see Theorem 3 in Section 4 for details). To establish the theorem, we modify the BMO Hörmander condition into an RBMO version, where RBMO is the Regularized Bounded Mean Oscillation space, which is also introduced by Tolsa [12].

This paper is organized as follows. In Section 2, we discuss the equivalence of the L^1 mean Hörmander condition and the classical one (Theorem 1). In Section 3, we prove the H^1-L^1 estimate and the failure of the L^1-$L^{1,\infty}$ estimate under the BMO Hörmander condition (Theorem 2). In Section 4, we show that Theorem 2 still works in the non-doubling setting with a little modification (Theorem 3).

2. The equivalence of $[K]_{H_1}$ and $[K]_{H_\infty}$

In this section, we prove the equivalence of $[K]_{H_1}$ and $[K]_{H_\infty}$. The argument here is based on the idea suggested by Professor Akihiko Miyachi. At first, we prepare an elementary fact.

Proposition 1. *The inequality*

$$\frac{1}{2}[K]_{H_\infty} \le \sup_{B\subset\mathbb{R}^d} \sup_{y\in B} \int_{x\in\mathbb{R}^d\setminus 3B} |K(x,y)-K(x,c(B))|\,dx \le [K]_{H_\infty} \tag{2.1}$$

holds for any $K \in L^1_{\rm loc}((\mathbb{R}^d\times\mathbb{R}^d)\setminus\Delta)$*.*

Proof of Proposition 1. We consider the first inequality. Fix a ball $B = B(c,2r) \subset \mathbb{R}^d$, $y \in B$ and write $z := (y+c)/2$. Since $B(z,3r) \subset 2B$ and $c, y \in B(z,r)$, we have

$$\begin{aligned}
&\int_{x\in\mathbb{R}^d\setminus 2B} |K(x,y)-K(x,c)|\,dx \\
&\quad\le \int_{x\in\mathbb{R}^d\setminus 2B} |K(x,y)-K(x,z)|\,dx + \int_{x\in\mathbb{R}^d\setminus 2B} |K(x,z)-K(x,c)|\,dx \\
&\quad\le \int_{x\in\mathbb{R}^d\setminus B(z,3r)} |K(x,y)-K(x,z)|\,dx + \int_{x\in\mathbb{R}^d\setminus B(z,3r)} |K(x,c)-K(x,z)|\,dx \\
&\quad\le 2 \sup_{B\subset\mathbb{R}^d}\sup_{y\in B}\int_{x\in\mathbb{R}^d\setminus 3B} |K(x,y)-K(x,c(B))|\,dx.
\end{aligned}$$

The second inequality is trivial. □

Proof of Theorem 1. Fix $r > 0$ and $c, y \in \mathbb{R}^d$ with $|y-c| \le r$. We write

$$\begin{aligned}
I(c,y) &:= \int_{x\in\mathbb{R}^d\setminus B(c,2r)} |K(x,y)-K(x,c)|\,dx,\\
J(c,y) &:= \int_{x\in\mathbb{R}^d\setminus B(c,3r)} |K(x,y)-K(x,c)|\,dx,\\
A &:= B(c,r)\cap B(y,r).
\end{aligned}$$

Since $B(y,2r) \cup B(c,2r) \subset B(c,3r)$, we obtain

$$\begin{aligned} J(c,y) &= \int_{x\in\mathbb{R}^d\setminus B(c,3r)} |K(x,y)-K(x,c)|\,dx \\ &\le \int_{x\in\mathbb{R}^d\setminus B(c,3r)} |K(x,y)-K(x,z)|\,dx + \int_{x\in\mathbb{R}^d\setminus B(c,3r)} |K(x,z)-K(x,c)|\,dx \\ &\le \int_{x\in\mathbb{R}^d\setminus B(y,2r)} |K(x,z)-K(x,y)|\,dx + \int_{x\in\mathbb{R}^d\setminus B(c,2r)} |K(x,z)-K(x,c)|\,dx \\ &= I(y,z) + I(c,z) \end{aligned}$$

for any $z \in A$. Therefore, letting $J = J(c,y)$, we have

$$A \subset \{\, z\in A \,:\, I(c,z) \ge J/2 \,\} \cup \{\, z\in A \,:\, I(y,z) \ge J/2 \,\},$$

which implies at least one of the following:

$$|A|/2 \le |\{\, z\in A \,:\, I(c,z)\ge J/2 \,\}|, \tag{2.2}$$

$$|A|/2 \le |\{\, z\in A \,:\, I(y,z)\ge J/2 \,\}|. \tag{2.3}$$

By the symmetry between c and y, we assume that (2.2) holds without loss of generality. Now we have

$$\begin{aligned} |A|J/4 &\le \int_{z\in A: I(c,z)\ge J/2} J/2\,dz \le \int_{z\in A} I(c,z)\,dz \le \int_{z\in B(c,r)} I(c,z)\,dz \\ &\le |B(c,r)|[K]_{H_1}. \end{aligned}$$

Since

$$|A| \ge |B((c+y)/2, r/2)| = 2^{-d}|B(c,r)|, \tag{2.4}$$

we get $J \le 2^{d+2}[K]_{H_1}$. By Proposition 1, we conclude that the inequality

$$[K]_{H_\infty} \le 2^{d+3}[K]_{H_1} \tag{2.5}$$

holds. □

Remark. Let μ be a Radon measure on $\mathbb{R}^d$ which satisfies the doubling property: there exists a constant $C_\mu > 0$ such that

$$\mu(B(c,2r)) \le C_\mu \mu(B(c,r)) \tag{2.6}$$

for any balls $B(c,r)$, and consider $[K]_{H_\infty}$ and $[K]_{H_1}$ with respect to μ:

$$[K]_{H_\infty} := \sup_{B\subset\mathbb{R}^d} \sup_{y\in B} \int_{x\in\mathbb{R}^d\setminus 2B} |K(x,y)-K(x,c(B))|\,d\mu(x),$$

$$[K]_{H_1} := \sup_{B\subset\mathbb{R}^d} \frac{1}{\mu(B)} \int_{y\in B} \int_{x\in\mathbb{R}^d\setminus 2B} |K(x,y)-K(x,c(B))|\,d\mu(x)\,d\mu(y).$$

We can show the inequality $[K]_{H_\infty} \lesssim [K]_{H_1}$ still holds by the same argument. To see this, note that μ satisfies

$$\mu(A) \ge \mu(B((c+y)/2, r/2)) \gtrsim \mu(B((c+y)/2, 3r/2)) \ge \mu(B(c,r)),$$

which can be a replacement of (2.4).

3. A BMO Hörmander condition

In this section, we study the BMO Hörmander condition,

(1.5)
$$[K]_{H_*} := \sup_{B\subset\mathbb{R}^d} \frac{1}{|B|}\int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B}\left|K(x,y)-\frac{1}{|B|}\int_{z\in B}K(x,z)\,dz\right|dx\,dy<\infty,$$

which is a natural generalization of the L^1 mean Hörmander condition (1.3) in the sense of BMO. At first we observe that $[K]_{H_*}$ satisfies an elemental property, which is analogous to that of the BMO norm (see [3, Proposition 6.5], [5, Proposition 3.1.2] for example).

Proposition 2. *The following are equivalent:*

(i) $[K]_{H_*}<\infty$.

(ii) *There exists a collection of functions* $\{m_B\}_B$ *such that*

$$\sup_{B\subset\mathbb{R}^d}\frac{1}{|B|}\int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B}|K(x,y)-m_B(x)|\,dx<\infty. \tag{3.1}$$

Proof of Proposition 2. (i)⇒(ii) is obvious: consider $m_B(x)=\frac{1}{|B|}\int_{z\in B}K(x,z)\,dz$. (ii)⇒(i) is also easy to check:

$$\begin{aligned}
&\frac{1}{|B|}\int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B}\left|K(x,y)-\frac{1}{|B|}\int_{z\in B}K(x,z)\,dz\right|dx\,dy\\
&\quad\le\frac{1}{|B|}\int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B}|K(x,y)-m_B(x)|\,dx\,dy\\
&\qquad+\frac{1}{|B|}\int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B}\left|m_B(x)-\frac{1}{|B|}\int_{z\in B}K(x,z)\,dz\right|dx\,dy\\
&\le\frac{1}{|B|}\int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B}|K(x,y)-m_B(x)|\,dx\,dy\\
&\qquad+\frac{1}{|B|}\int_{y\in B}\left(\frac{1}{|B|}\int_{z\in B}\int_{x\in\mathbb{R}^d\setminus 2B}|K(x,z)-m_B(x)|\,dx\,dz\right)dy\\
&=\frac{2}{|B|}\int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B}|K(x,y)-m_B(x)|\,dx\,dy.
\end{aligned}$$

□

We write

$$[K]_{H_{**}}:=\inf_{\{m_B\}_B}\sup_{B\subset\mathbb{R}^d}\frac{1}{|B|}\int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B}|K(x,y)-m_B(x)|\,dx\,dy, \tag{3.2}$$

where the infimum is taken over all collections $\{m_B\}_B$. By the proof of Proposition 2, we have

$$[K]_{H_{**}}\le[K]_{H_*}\le 2[K]_{H_{**}}. \tag{3.3}$$

Also note that $[K]_{H_{**}}\le[K]_{H_1}$: consider $m_B(x)=K(x,c(B))$. Therefore, we have

$$[K]_{H_*}\le 2[K]_{H_1}, \tag{3.4}$$

it means that the classical Hörmander condition implies the BMO version. We will show later that the converse is not true.

Now we are going to discuss the boundedness of singular integral operators under the BMO Hörmander condition.

Proposition 3. *Let T be a singular integral operator associated with a kernel K. Suppose that T is bounded on $L^{p_0}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ and its kernel K satisfies $[K]_{H_{**}} < \infty$. Then T is bounded from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ with a constant proportional to $\|T\|_{L^{p_0}\to L^{p_0}} + [K]_{H_{**}}$.*

Proof of Proposition 3. Let $a \in H^1(\mathbb{R}^d)$ be an atom, that is,

$$\operatorname{supp} a \subset B, \quad \|a\|_\infty \le |B|^{-1}, \quad \int_B a = 0$$

for some ball $B \subset \mathbb{R}^d$. Since T is bounded on $L^{p_0}(\mathbb{R}^d)$, it is enough to show that

$$\|Ta\|_1 \lesssim \|T\|_{L^{p_0}\to L^{p_0}} + [K]_{H_{**}}.$$

We decompose $\|Ta\|_1$ as

$$\|Ta\|_1 = \|Ta\|_{L^1(2B)} + \|Ta\|_{L^1(\mathbb{R}^d\setminus 2B)}$$

and prove

$$\|Ta\|_{L^1(2B)} \lesssim \|T\|_{L^{p_0}\to L^{p_0}}, \tag{3.5}$$

$$\|Ta\|_{L^1(\mathbb{R}^d\setminus 2B)} \lesssim [K]_{H_{**}}. \tag{3.6}$$

(3.5). By the Hölder inequality and the L^{p_0} boundedness of T, we have

$$\begin{aligned}\|Ta\|_{L^1(2B)} &\le |2B|^{1/p_0'}\|Ta\|_{p_0}\\ &\le 2^{d/p_0'}\|T\|_{L^{p_0}\to L^{p_0}}|B|^{1/p_0'}\|a\|_{p_0}\\ &\le 2^{d/p_0'}\|T\|_{L^{p_0}\to L^{p_0}}|B|\|a\|_\infty\\ &\le 2^{d/p_0'}\|T\|_{L^{p_0}\to L^{p_0}}.\end{aligned}$$

(3.6). Since

$$\begin{aligned}\|Ta\|_{L^1(\mathbb{R}^d\setminus 2B)} &= \int_{x\in\mathbb{R}^d\setminus 2B}\left|\int_{y\in B} K(x,y)a(y)\,dy\right|dx\\ &= \int_{x\in\mathbb{R}^d\setminus 2B}\left|\int_{y\in B} K(x,y)a(y)\,dy - m_B(x)\int_{y\in B} a(y)\,dy\right|dx\\ &\le \|a\|_\infty \int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B}|K(x,y)-m_B(x)|\,dx\,dy\\ &\le \frac{1}{|B|}\int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B}|K(x,y)-m_B(x)|\,dx\,dy\end{aligned}$$

for any collections $\{m_B\}_B$, we have $\|Ta\|_{L^1(\mathbb{R}^d\setminus 2B)} \le [K]_{H_{**}}$. □

We can see that the assumption $[K]_{H_{**}} < \infty$ is reasonable (thus $[K]_{H_*} < \infty$ is) for the H^1-L^1 estimate. On the other hand, it is not enough for the L^1-$L^{1,\infty}$ estimate. It follows from a simple example.

Proposition 4. *Let* $\varphi \in (L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)) \setminus \{0\}$, $\psi \in (L^2(\mathbb{R}^d) \cap \mathrm{BMO}(\mathbb{R}^d)) \setminus L^\infty(\mathbb{R}^d)$ *and* $K(x, y) := \varphi(x)\psi(y)$. *Then*

$$T\colon f \mapsto \int_{\mathbb{R}^d} K(\cdot, y) f(y)\, dy \text{ is bounded on } L^2(\mathbb{R}^d), \tag{3.7}$$

$$[K]_{H_*} < \infty, \tag{3.8}$$

$$T \text{ is not bounded from } L^1(\mathbb{R}^d) \text{ to } L^{1,\infty}(\mathbb{R}^d). \tag{3.9}$$

Proof of Proposition 4.

(3.7). It is obvious that $\|T\|_{L^2 \to L^2} \le \|\varphi\|_2 \|\psi\|_2$.

(3.8). We have

$$\begin{aligned}
&\frac{1}{|B|} \int_{y \in B} \int_{x \in \mathbb{R}^d \setminus 2B} \left| K(x, y) - \frac{1}{|B|} \int_{z \in B} K(x, z)\, dz \right| dx\, dy \\
&= \left(\int_{x \in \mathbb{R}^d \setminus 2B} |\varphi(x)|\, dx \right) \cdot \left(\frac{1}{|B|} \int_{y \in B} \left| \psi(y) - \frac{1}{|B|} \int_{z \in B} \psi(z)\, dz \right| dy \right) \\
&\le \|\varphi\|_1 \|\psi\|_{\mathrm{BMO}}
\end{aligned}$$

for any ball $B \subset \mathbb{R}^d$, thus $[K]_{H_*} \le \|\varphi\|_1 \|\psi\|_{\mathrm{BMO}}$.

(3.9). For each $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, Tf is given by

$$Tf(x) = \int_{\mathbb{R}^d} \varphi(x)\psi(y) f(y)\, dy = \varphi(x) \int_{\mathbb{R}^d} \psi(y) f(y)\, dy,$$

hence

$$\|Tf\|_{1,\infty} = \|\varphi\|_{1,\infty} \left| \int_{\mathbb{R}^d} \psi(y) f(y)\, dy \right|.$$

Since $\psi \notin L^\infty(\mathbb{R}^d)$, there exists a sequence of measurable sets $\{A_j\}_{j \in \mathbb{N}}$ such that

$$0 < |A_j| < \infty, \quad A_j \subset \{\, y \in \mathbb{R}^d : |\psi(y)| > j \,\}.$$

Define $f_j \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by

$$f_j := \frac{\chi_{A_j}}{|A_j|} \cdot \frac{\overline{\psi}}{|\psi|},$$

then f_j satisfies $\|f_j\|_1 = 1$ and

$$\left| \int_{\mathbb{R}^d} \psi(y) f_j(y)\, dy \right| = \frac{1}{|A_j|} \int_{A_j} |\psi(y)|\, dy \ge j$$

for each j, thus T is not bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. □

4. The H^1_{atb}-L^1 estimate with non-doubling measures

In this section, we consider a Radon measure μ on $\mathbb{R}^d$ which satisfies the polynomial growth condition: there exists a constant $C_\mu > 0$ and $0 < n \le d$ such that

$$\mu(B(c, r)) \le C_\mu r^n \tag{1.7}$$

for any balls $B(c, r)$. In this case, unlike in the case of the Lebesgue measure, the Hardy space $H^1(\mu)$ (see Mateu *et al.* [8]) is not suitable for the Calderón–Zygmund theory since Verdera [15] pointed out that the Cauchy integral does not satisfy the $H^1(\mu)$-$L^1(\mu)$ estimate in general. After that, Tolsa [12] developed the atomic block Hardy space $H^1_{\mathrm{atb}}(\mu)$ and established the $H^1_{\mathrm{atb}}(\mu)$-$L^1(\mu)$ estimate of singular integral operators (Theorem C). We will show that our BMO Hörmander condition still works in this setting with a little modification.

Recall that $H^1_{\mathrm{atb}}(\mu)$ and $\mathrm{RBMO}(\mu)$ are defined as follows.

The coefficient $\delta(B_0, B)$: Let (B_0, B) be a pair of balls such that $B_0 \subset B$. The coefficient $\delta(B_0, B)$ is defined by

$$\delta(B_0, B) := \int_{y \in 2B \setminus B_0} \frac{1}{|y - c(B_0)|^n} \, d\mu(y). \tag{4.1}$$

The atomic block: A function $b \in L^1_{\mathrm{loc}}(\mu)$ is called an atomic block if there exist a ball B, a pair of balls $\{B_j\}_{j=1}^2$, functions $\{a_j\}_{j=1}^2$ and numbers $\{\lambda_j\}_{j=1}^2$ such that

$$\begin{gathered} \operatorname{supp} b \subset B, \quad \operatorname{supp} a_j \subset B_j, \quad B_j \subset B, \\ \int_B b \, d\mu = 0, \\ \|a_j\|_{L^\infty(\mu)} \le ((1 + \delta(B_j, B))\mu(2B_j))^{-1}, \\ b = \lambda_1 a_1 + \lambda_2 a_2 \end{gathered} \tag{4.2}$$

and write

$$|b|_{H^1_{\mathrm{atb}}(\mu)} := |\lambda_1| + |\lambda_2|. \tag{4.3}$$

The atomic block Hardy space $H^1_{\mathrm{atb}}(\mu)$: The atomic block Hardy space $H^1_{\mathrm{atb}}(\mu)$ is defined by

$$H^1_{\mathrm{atb}}(\mu) := \left\{ \sum_{j=1}^\infty b_j : b_j \text{ are atomic blocks such that } \sum_{j=1}^\infty |b_j|_{H^1_{\mathrm{atb}}(\mu)} < \infty \right\} \tag{4.4}$$

and its norm is

$$\|f\|_{H^1_{\mathrm{atb}}(\mu)} := \inf \left\{ \sum_{j=1}^\infty |b_j|_{H^1_{\mathrm{atb}}(\mu)} : b_j \text{ are atomic blocks such that } f = \sum_{j=1}^\infty b_j \right\}. \tag{4.5}$$

The regularized bounded mean oscillation space $\mathrm{RBMO}(\mu)$: A function $f \in L^1_{\mathrm{loc}}(\mu)$ is in $\mathrm{RBMO}(\mu)$ if there exists a collection of numbers $\{m_B\}_B$ such that

$$\sup_{B \subset \mathbb{R}^d} \frac{1}{\mu(2B)} \int_{y \in B} |f(y) - m_B| \, d\mu(y) + \sup_{B_0 \subset B \subset \mathbb{R}^d} \frac{1}{1 + \delta(B_0, B)} |m_{B_0} - m_B| < \infty, \tag{4.6}$$

and its norm is defined by $\|f\|_{\mathrm{RBMO}(\mu)} := \inf_{\{m_B\}_B}$ (LHS of (4.6)), where supremum $\sup_{B \subset \mathbb{R}^d}$, $\sup_{B_0 \subset B \subset \mathbb{R}^d}$ and infimum $\inf_{\{m_B\}_B}$ are taken over all balls B with $\mu(B) > 0$, all pairs of balls (B_0, B) such that $B_0 \subset B$, and all collections $\{m_B\}_B$, respectively.

It is known that these spaces $H^1_{\mathrm{atb}}(\mu)$ and $\mathrm{RBMO}(\mu)$ satisfy properties analogous to those of usual $H^1(\mathbb{R}^d)$ and $\mathrm{BMO}(\mathbb{R}^d)$ with the Lebesgue measure, such as the John–Nirenberg inequality, the $H^1_{\mathrm{atb}}(\mu)$-$\mathrm{RBMO}(\mu)$ duality, interpolation inequalities, $T1$ and Tb theorems (see [10], [12], [14], [16]).

Now we introduce an RBMO Hörmander condition: there exists a collection of functions $\{m_B\}_B$ such that

$$\begin{aligned}(4.7)\qquad &\sup_{B\subset\mathbb{R}^d}\frac{1}{\mu(2B)}\int_{y\in B}\int_{x\in\mathbb{R}^d\setminus 2B}|K(x,y)-m_B(x)|\,d\mu(x)\,d\mu(y)\\ &+\sup_{B_0\subset B\subset\mathbb{R}^d}\frac{1}{1+\delta(B_0,B)}\int_{x\in\mathbb{R}^d\setminus 2B}|m_{B_0}(x)-m_B(x)|\,d\mu(x)<\infty,\end{aligned}$$

and write $[K]_{H_{**}} := \inf_{\{m_B\}_B}$ (LHS of (4.7)). Note that we can easily see $[K]_{H_{**}} \le 2[K]_{H_\infty}$. We are going to prove the non-doubling version of Proposition 3.

Theorem 3. *Let T be a singular integral operator associated with a kernel K. Suppose that T is bounded on $L^{p_0}(\mu)$ for some $1 < p_0 < \infty$ and its kernel K satisfies the* RBMO *Hörmander condition $[K]_{H_{**}} < \infty$ and $|K(x,y)| \le A|x-y|^{-n}$ for some constant $A > 0$. Then T is bounded from $H^1_{\rm atb}(\mu)$ to $L^1(\mu)$ with a constant proportional to $A + \|T\|_{L^{p_0}(\mu)\to L^{p_0}(\mu)} + [K]_{H_{**}}$.*

Proof of Theorem 3. Let $b = \sum_{j=1}^{2}\lambda_j a_j \in H^1_{\rm atb}(\mu)$ be an atomic block. Since T is bounded on $L^{p_0}(\mu)$, it is enough to show that

$$\|Tb\|_{L^1(\mu)} \lesssim (A + \|T\|_{L^{p_0}(\mu)\to L^{p_0}(\mu)} + [K]_{H_{**}})|b|_{H^1_{\rm atb}(\mu)}.$$

We decompose $\|Tb\|_{L^1(\mu)}$ as

$$\|Tb\|_{L^1(\mu)} \le \sum_{j=1}^{2}\lambda_j(\|Ta_j\|_{L^1(2B_j,\mu)} + \|Ta_j\|_{L^1(2B\setminus 2B_j,\mu)}) + \|Tb\|_{L^1(\mathbb{R}^d\setminus 2B,\mu)}$$

and prove

$$\|Ta_j\|_{L^1(2B_j,\mu)} \lesssim \|T\|_{L^{p_0}(\mu)\to L^{p_0}(\mu)}, \tag{4.8}$$

$$\|Ta_j\|_{L^1(2B\setminus 2B_j,\mu)} \lesssim A, \tag{4.9}$$

$$\|Tb\|_{L^1(\mathbb{R}^d\setminus 2B,\mu)} \lesssim [K]_{H_{**}}|b|_{H^1_{\rm atb}(\mu)}. \tag{4.10}$$

(4.8). By the Hölder inequality and the $L^{p_0}(\mu)$ boundedness of T, we have

$$\begin{aligned}\|Ta_j\|_{L^1(2B_j,\mu)} &\le \mu(2B)^{1/p_0'}\|Ta_j\|_{L^{p_0}(\mu)}\\ &\le \|T\|_{L^{p_0}(\mu)\to L^{p_0}(\mu)}\mu(2B_j)^{1/p_0'}\|a_j\|_{L^{p_0}(\mu)}\\ &\le \|T\|_{L^{p_0}(\mu)\to L^{p_0}(\mu)}\mu(2B_j)^{1/p_0'}\mu(B_j)^{1/p_0}\|a_j\|_{L^\infty(\mu)}\\ &\le \|T\|_{L^{p_0}(\mu)\to L^{p_0}(\mu)}.\end{aligned}$$

(4.9). For each $x \in 2B \setminus 2B_j$, we have a pointwise estimate

$$\begin{aligned}
|Ta_j(x)| &= \left| \int_{y\in B_j} K(x,y)a_j(y)\,d\mu(y) \right| \\
&\le \int_{y\in B_j} |K(x,y)a_j(y)|\,d\mu(y) \\
&\le \|a_j\|_{L^\infty(\mu)} \int_{y\in B_j} \frac{A}{|x-y|^n}\,d\mu(y) \\
&\le \frac{1}{(1+\delta(B_j,B))\mu(2B_j)} \int_{y\in B_j} \frac{2^nA}{|x-c_j|^n}\,d\mu(y) \\
&\le \frac{2^nA}{1+\delta(B_j,B)} \frac{1}{|x-c_j|^n}.
\end{aligned}$$

Therefore, we obtain

$$\|Ta_j\|_{L^1(2B\setminus 2B_j,\mu)} \le \frac{2^nA}{1+\delta(B_j,B)}\delta(B_j,B) \le 2^nA.$$

(4.10). Since

$$\begin{aligned}
&\|Tb\|_{L^1(\mathbb{R}^d\setminus 2B,\mu)} \\
&= \int_{x\in\mathbb{R}^d\setminus 2B} \left| \int_{y\in B} K(x,y)b(y)\,d\mu(y) \right| d\mu(x) \\
&= \int_{x\in\mathbb{R}^d\setminus 2B} \left| \int_{y\in B} K(x,y)b(y)\,d\mu(y) - m_B(x)\int_{y\in B} b(y)\,d\mu(y) \right| d\mu(x) \\
&\le \int_{y\in B} \int_{x\in\mathbb{R}^d\setminus 2B} |\,K(x,y)-m_B(x)\,|\,d\mu(x)\,|b(y)|\,d\mu(y) \\
&\le \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^\infty(\mu)} \int_{y\in B_j} \int_{x\in\mathbb{R}^d\setminus 2B} |\,K(x,y)-m_B(x)\,|\,d\mu(x)\,d\mu(y) \\
&\le \sum_{j=1}^{2} |\lambda_j| \left(\frac{1}{(1+\delta(B_j,B))\mu(2B_j)} \int_{y\in B_j} \int_{x\in\mathbb{R}^d\setminus 2B_j} |\,K(x,y)-m_{B_j}(x)\,|\,d\mu(x)\,d\mu(y) \right. \\
&\qquad \left. + \frac{1}{(1+\delta(B_j,B))\mu(2B_j)} \int_{y\in B_j} \int_{x\in\mathbb{R}^d\setminus 2B} |\,m_{B_j}(x)-m_B(x)\,|\,d\mu(x)\,d\mu(y) \right) \\
&\le \sum_{j=1}^{2} |\lambda_j| \left(\frac{1}{\mu(2B_j)} \int_{y\in B_j} \int_{x\in\mathbb{R}^d\setminus 2B_j} |\,K(x,y)-m_{B_j}(x)\,|\,d\mu(x)\,d\mu(y) \right. \\
&\qquad \left. + \frac{1}{1+\delta(B_j,B)} \int_{x\in\mathbb{R}^d\setminus 2B} |\,m_{B_j}(x)-m_B(x)\,|\,d\mu(x) \right)
\end{aligned}$$

for any collections $\{m_B\}_B$, we have $\|Tb\|_{L^1(\mathbb{R}^d\setminus 2B)} \le [K]_{H_{**}} |b|_{H^1_{\mathrm{atb}}(\mu)}$. □

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Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, Aichi, 464-8602, Japan

Email address: `m18020a@math.nagoya-u.ac.jp, soichiro.suzuki.m18020a@gmail.com`