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AMPLIFIED GRAPH C*-ALGEBRAS II: RECONSTRUCTION

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ABSTRACT. Let E be a countable directed graph that is amplified in the sense that whenever there is an edge from v to w, there are infinitely many edges from v to w. We show that E can be recovered from $C^*(E)$ together with its canonical gauge-action, and also from $L_{\mathbb{K}}(E)$ together with its canonical grading.

1. INTRODUCTION

The purpose of this paper is to investigate the gauge-equivariant isomorphism question for C^*-algebras of countable amplified graphs, and the graded isomorphism question for Leavitt path algebras of countable amplified graphs. A directed graph E is called an *amplified* graph if for any two vertices v, w, the set of edges from v to w is either empty or infinite.

The geometric classification (that is, classification by the underlying graph modulo the equivalence relation generated by a list of allowable graph moves) of the C^*-algebras of finite-vertex amplified graph C^*-algebras was completed in [12], and was an important precursor to the eventual geometric classification of all finite graph C^*-algebras [13]. But there has been increasing recent interest in understanding isomorphisms of graph C^*-algebras that preserve additional structure: for example the canonical gauge action of the circle; or the canonical diagonal subalgebra isomorphic to the algebra of continuous functions vanishing at infinity on the infinite path space of the graph; or the smaller coefficient algebra generated by the vertex projections; or some combination of these (see, for example, [5–10, 19]).

A program of geometric classification for these various notions of isomorphism was initiated by the first two authors in [11]. They discuss xyz-isomorphism of graph C^*-algebras, where x is 1 if we require exact isomorphism, and 0 if we require only stable isomorphism; y is 1 if the isomorphism is required to be gauge-equivariant, and 0 otherwise; and z is 1 if the isomorphism is required to preserve the diagonal subalgebra and 0 otherwise. They also identified a set of moves on graphs that preserve various kinds of xyz-isomorphism, and conjectured that for all xyz other than $\mathsf{x10}$, the equivalence relation on graphs with finitely many vertices induced by xyz-isomorphism of C^*-algebras is generated by precisely those of their moves that induce xyz-isomorphisms.

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This was an important motivation for the present paper. None of the moves in [11] takes an amplified graph to an amplified graph. And although we know of one important instance where one amplified graph can be transformed into another via a sequence of 101-preserving moves passing through nonamplified graphs (see Diagram (3.1) in Remark 3.5), we had given up on envisioning such a sequence consisting only of x1z-preserving moves. Based on the main conjecture of [11], this led us to expect that an amplified graph C^*-algebra together with its gauge action should remember the graph itself.

Our main theorem shows that, indeed, any countable amplified graph E can be reconstructed from either the circle-equivariant K_0-group of its C^*-algebra or the graded K_0-group of its Leavitt path algebra over any field. That is:

Theorem A. *Let E and F be countable amplified graphs and let $\mathbb{K}$ be a field. Then the following are equivalent:*

(1) $E \cong F$;
(2) *there is a $\mathbb{Z}[x, x^{-1}]$-module order-isomorphism $K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E)) \cong K_0^{\mathrm{gr}}(L_{\mathbb{K}}(F))$; and*
(3) *there is a $\mathbb{Z}[x, x^{-1}]$-module order-isomorphism of $\mathbb{T}$-equivariant K_0-groups $K_0^{\mathbb{T}}(C^*(E), \gamma) \cong K_0^{\mathbb{T}}(C^*(F), \gamma)$.*

We spell out a number of consequences of this theorem in Remark 3.9, Theorem 3.4, and Theorem 3.8. The headline is that for amplified graphs, and for any x, z, the graph C^*-algebras $C^*(E)$ and $C^*(F)$ are x1z-isomorphic if and only if E and F are isomorphic. Combined with results of [4, 13], this confirms the main conjecture of [11] for amplified graphs (see Remark 3.5).

Another immediate consequence is that, since ordered graded K_0 is an isomorphism invariant of graded rings, and ordered $\mathbb{T}$-equivariant K_0 is an isomorphism invariant of C^*-algebras carrying circle actions, our theorem confirms a special case of Hazrat's conjecture: ordered graded K_0 is a complete graded-isomorphism invariant for amplified Leavitt path algebras; and we also obtain that ordered $\mathbb{T}$-equivariant K_0 is a complete gauge-isomorphism invariant of amplified graph C^*-algebras.

A third consequence is related to different graded stabilisations of Leavitt path algebras (and different equivariant stabilisations of graph C^*-algebras). Each Leavitt path algebra has a canonical grading, and, as alluded to above, significant work led by Hazrat has been done on determining when graded K-theory completely classifies graded Leavitt path algebras. Historically, in the classification program for C^*-algebras, significant progress has been made by first considering classification up to stable isomorphism; so it is natural to consider the same approach to Hazrat's graded classification question. But almost immediately, there is a difficulty: which grading on $L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})$ should we consider? It seems natural enough to use the grading arising from the graded tensor product of the graded algebras $L_{\mathbb{K}}(E)$ and $M_\infty(\mathbb{K})$. But there are many natural gradings on $M_\infty(\mathbb{K})$: given any $\overline{\delta} \in \prod_i \mathbb{Z}$, we obtain a grading of $M_\infty(\mathbb{K})$ in which the m, n matrix unit is homogeneous of degree $\overline{\delta}_m - \overline{\delta}_n$. Different nonzero choices for $\overline{\delta}$ correspond to different ways of stabilising $L_{\mathbb{K}}(E)$ by modifying the graph E (for example by adding heads [23]), while taking $\overline{\delta} = (0, 0, 0, \dots)$ corresponds to stabilising the associated groupoid by taking its cartesian product with the (trivially graded) full equivalence relation $\mathbb{N} \times \mathbb{N}$.

In Section 3.2, we show that for amplified graphs it doesn't matter what value of $\overline{\delta}$ we pick. Specifically, using results of Hazrat, we prove that $K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E) \otimes$

$M_\infty(\mathbb{K})(\overline{\delta})) \cong K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ regardless of $\overline{\delta}$. Consequently, for any choice of $\overline{\delta}$ we have $L_{\mathbb{K}}(E)\otimes M_\infty(\mathbb{K})(\overline{\delta}) \cong L_{\mathbb{K}}(F)\otimes M_\infty(\mathbb{K})(\overline{\delta})$ if and only if there exists a $\mathbb{Z}[x,x^{-1}]$-module order-isomorphism $K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E)) \cong K_0^{\mathrm{gr}}(L_{\mathbb{K}}(F))$. A similar result holds for C^*-algebras with the gradings on Leavitt path algebras replaced by gauge actions on graph C^*-algebras, and the gradings of $M_\infty(\mathbb{K})$ corresponding to different elements $\overline{\delta}$ replaced by the circle actions on $\mathcal{K}(\ell^2)$ implemented by different strongly continuous unitary representations of the circle on ℓ^2.

We prove our main theorem in Section 2. We use general results to see that the graded K_0-group of $L_{\mathbb{K}}(E)$ and the equivariant K_0-group of $C^*(E)$ are isomorphic as ordered $\mathbb{Z}[x,x^{-1}]$-modules to the K_0-groups of the Leavitt path algebra and the graph C^*-algebra (respectively) of the skew-product graph $E\times_1\mathbb{Z}$. These are known to coincide, and their lattice of order ideals (with canonical $\mathbb{Z}$-action) is isomorphic to the lattice of hereditary subsets of $(E\times_1\mathbb{Z})^0$ with the $\mathbb{Z}$-action of translation in the second variable. So the bulk of the work in Section 2 goes into showing how to recover E from this lattice. We then go on in Section 3.2 to establish the consequences of our main theorem for stabilisations. Here the hard work goes into showing that $K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E)\otimes M_\infty(\mathbb{K})(\overline{\delta})) \cong K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ for any $\overline{\delta}\in\prod_i\mathbb{Z}$ and that $K_0^{\mathbb{T}}(C^*(E)\otimes\mathcal{K}, \gamma^E\otimes\mathrm{Ad}_u) \cong K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ for any strongly continuous unitary representation u of $\mathbb{T}$.

2. Gauge-invariant classification of amplified graph C^*-algebras

In this paper, a countable directed graph E is a quadruple $E=(E^0,E^1,r,s)$ where E^0 is a countable set whose elements are called *vertices*, E^1 is a countable set whose elements are called *edges*, and $r,s\colon E^1\to E^0$ are functions. We think of the elements of E^0 as points or dots, and each element e of E^1 as an arrow pointing from the vertex $s(e)$ to the vertex $r(e)$. We follow the conventions of, for example [14], where a path is a sequence $e_1\dots e_n$ of edges in which $s(e_{n+1})=r(e_n)$. This is not the convention used in Raeburn's monograph [21], but is the convention consistent with all of the Leavitt path algebra literature as well as much of the graph C^*-algebra literature. In keeping with this, for $v,w\in E^0$ and $n\geq 0$, we define

$$vE^1=s^{-1}(v),\quad E^1w=r^{-1}(w),\quad \text{and}\quad vE^1w=s^{-1}(v)\cap r^{-1}(w).$$

We will also write vE^n for the sets of paths of length n that are emitted by v, E^nw for the set of paths of length n received by w, and vE^nw for the set of paths of length n pointing from v to w.

A vertex v is *singular* if vE^1 is either empty or infinite, so v is either a sink or an infinite emitter; it is *regular* if it is not singular. For any edge e, we have $s_e^*s_e=p_{r(e)}$ and $p_{s(e)}\geq s_es_e^*$ in the graph C^*-algebra $C^*(E)$. We will also consider the Leavitt path algebras, $L_{\mathbb{K}}(E)$ for any field $\mathbb{K}$, the so-called algebraic cousin of graph C^*-algebras. Leavitt path algebras are defined via generators and relations similar to those for graph C^*-algebras (see [1]).

Countable directed graphs E and F are *isomorphic*, denoted $E\cong F$, if there is a bijection $\phi: E^0\sqcup E^1\to F^0\sqcup F^1$ that restricts to bijections $\phi^0\colon E^0\to F^0$ and $\phi^1\colon E^1\to F^1$ such that

$$\phi^0(r(e))=r(\phi^1(e))\quad\text{and}\quad\phi^0(s(e))=s(\phi^1(e)).$$

In this paper, we consider amplified graphs. The classification of amplified graph C^*-algebras was the starting point in the classification of unital graph C^*-algebras via moves (see [12] and [13]).

Definition 2.1 (Amplified graph and amplified graph algebra)**.** A directed graph E is an *amplified graph* if for all $v, w \in E^0$, the set $vE^1w = s^{-1}(v) \cap r^{-1}(w)$ is either empty or infinite. An *amplified graph C^*-algebra* is a graph C^*-algebra of an amplified graph and an *amplified Leavitt path algebra* is a Leavitt path algebra of an amplified graph.

Observe that in an amplified graph, every vertex is singular.

Recall that a set $H \subseteq E^0$ is *hereditary* if $s(e) \in H$ implies $r(e) \in H$ for every $e \in E^1$, and is *saturated* if whenever v is a regular vertex such that $r(vE^1) \subseteq H$, we have $v \in H$. Again since every vertex in an amplified graph is singular, every set of vertices is saturated.

Recall from [18] that if E is a directed graph, then the skew-product graph $E \times_1 \mathbb{Z}$ is the graph with vertices $E^0 \times \mathbb{Z}$ and edges $E^1 \times \mathbb{Z}$ with $s(e, n) = (s(e), n)$ and $r(e, n) = (r(e), n + 1)$. If E is an amplified graph, then so is $E \times_1 \mathbb{Z}$.

For a countable amplified graph, E, we write $\mathcal{H}(E \times_1 \mathbb{Z})$ for the lattice (under set inclusion) of hereditary subsets of the vertex-set of the skew-product graph $E \times_1 \mathbb{Z}$. The action of $\mathbb{Z}$ on $E \times_1 \mathbb{Z}$ given by $n \cdot (e, m) = (e, n + m)$ induces an action lt of $\mathbb{Z}$ on $\mathcal{H}(E \times_1 \mathbb{Z})$.

Throughout this section, given $v \in E^0$ and $n \in \mathbb{Z}$, we write $H(v, n)$ for the smallest hereditary subset of $(E \times_1 \mathbb{Z})^0$ containing (v, n). So $H(v, n) = \{(r(\mu), n + |\mu|) : \mu \in vE^*\}$ is the set of vertices that can be reached from (v, n) in $E \times_1 \mathbb{Z}$.

If $(\mathcal{L}, \preceq)$ is a lattice, we say that $L \in \mathcal{L}$ has a unique predecessor if there exists $K \in \mathcal{L}$ such that $K \prec L$, and every K' with $K' \prec L$ satisfies $K' \preceq K$.

Let E be a countable amplified graph. Define $\mathcal{H}_{\text{vert}} \subseteq \mathcal{H}(E \times_1 \mathbb{Z})$ to be the subset

$$\mathcal{H}_{\text{vert}} = \{H \in \mathcal{H}(E \times_1 \mathbb{Z}) : H \text{ has a unique predecessor}\}.$$

The argument of [12, Lemma 5.2] shows that

$$\mathcal{H}_{\text{vert}} = \{H(v, n) : v \in E^0, n \in \mathbb{Z}\}.$$

Note that $H \in \mathcal{H}(E \times_1 \mathbb{Z})$ has a unique predecessor if and only if it is of the form $H(v, n)$ for some $v \in E^0$ and $n \in \mathbb{Z}$.

Proposition 2.2 is the engine-room of our main result.

Proposition 2.2. *Let E be a countable amplified graph. Let*

$$\overline{E}^0 := \{H(v, 0) : v \in E^0\}.$$

Define $\overline{E}^1 := \{(H, n, K) : H, K \in \overline{E}^0, \mathrm{lt}_1(K) \subseteq H,$ and $n \in \mathbb{N}\}$. Define $\bar{s}, \bar{r} : \overline{E}^1 \to \overline{E}^0$ by $\bar{s}(H, n, K) = H$ and $\bar{r}(H, n, K) = K$. Then $\overline{E} := (\overline{E}^0, \overline{E}^1, \bar{r}, \bar{s})$ is a countable amplified directed graph, and there is an isomorphism $E \cong \overline{E}$ that carries each $v \in E^0$ to $H(v, 0)$.

Proof. Since $E \times_1 \mathbb{Z}$ is acyclic, the $H(v, 0)$ are distinct, and we deduce that $\theta^0 : v \mapsto H(v, 0)$ is a bijection from E^0 to $\overline{E}^0$.

Fix $v, w \in E^0$. We have $\mathrm{lt}_1(H(w, 0)) = H(w, 1)$, and since $(w, 1) \in H(v, 0)$ if and only if $vE^1w \neq \emptyset$, we have $H(w, 1) \subseteq H(v, 0)$ if and only if $vE^1w \neq \emptyset$, in which case vE^1w is infinite because E is amplified. It follows that $|H(v, 0)\overline{E}^1 H(w, 0)| =$

$|vE^1w|$ for all v, w, so we can choose a bijection $\theta^1 : E^1 \to \overline{E}^1$ that restricts to bijections $vE^1w \to \theta^0(v)\overline{E}^1\theta^0(w)$ for all $v, w \in E^0$. The pair (θ^0, θ^1) is then the desired isomorphism $E \cong \overline{E}$. □

In order to use Proposition 2.2 to prove Theorem A, we need to know that if $(\mathcal{H}(E\times_1\mathbb{Z}), \mathrm{lt}^E)$ is order isomorphic to $(\mathcal{H}(F\times_1\mathbb{Z}), \mathrm{lt}^F)$ then there is an isomorphism from $(\mathcal{H}(E \times_1 \mathbb{Z}), \mathrm{lt}^E)$ to $(\mathcal{H}(F \times_1 \mathbb{Z}), \mathrm{lt}^F)$ that carries $\{\mathcal{H}(v, 0) : v \in E^0\}$ to $\{\mathcal{H}(w, 0) : w \in F^0\}$. We do this by showing that if E is connected, then, up to translation, we can recognise the set $\{\mathcal{H}(v, 0) : v \in E^0\}$ amongst subsets of $\mathcal{H}_{\mathrm{vert}}$ using just the order-structure and the action lt.

Recalling that vE^nw denotes the set of paths of length n from v to w, we have

$$H(w, n) \subseteq H(v, m) \quad \text{if and only if} \quad vE^{n-m}w \neq \emptyset. \tag{2.1}$$

Recall that a graph E is said to be *weakly connected* if the smallest equivalence relation on E^0 containing $\{(s(e), r(e)) : e \in E^1\}$ is all of $E^0 \times E^0$.

Let E be a weakly connected, countable amplified graph. The set $V_0^E := \{H(v, 0) : v \in E^0\}$ satisfies the following

- for each $H \in \mathcal{H}_{\mathrm{vert}}$ there is a unique $n \in \mathbb{Z}$ such that $\mathrm{lt}_n(H) \in V_0^E$;
- the smallest equivalence relation on V_0^E containing $\{(H, K) : \mathrm{lt}_1(K) \subseteq H\}$ is all of $V_0^E \times V_0^E$; and
- if H, K are distinct elements of V_0^E, and if $n \geq 0$, then $H \not\subseteq \mathrm{lt}_n(K)$.

Lemma 2.3 shows that for weakly connected graphs, these properties characterise V_0^E up to translation.

Lemma 2.3. *Suppose that E is a weakly connected, countable amplified graph. Suppose that $V \subseteq \mathcal{H}_{\mathrm{vert}}$ satisfies*

(1) *for each $K \in \mathcal{H}_{\mathrm{vert}}$ there is a unique $n \in \mathbb{Z}$ such that $\mathrm{lt}_n(K) \in V$;*
(2) *the smallest equivalence relation on V containing $\{(H, K) : \mathrm{lt}_1(K) \subseteq H\}$ is all of $V \times V$; and*
(3) *if K, K' are distinct elements of V, and if $n \geq 0$, then $K \not\subseteq \mathrm{lt}_n(K')$.*

Then there exists $n \in \mathbb{Z}$ such that $V = \mathrm{lt}_n(V_0) = \{\mathcal{H}(v, n) : v \in E^0\}$.

Proof. For each $v \in E^0$, item (1) applied to $K = H(v, 0)$ shows that there exists a unique $n_v \in \mathbb{Z}$ such that $H(v, n_v) = \mathrm{lt}_{n_v}(K) \in V$. So $V = \{H(v, n_v) : v \in E^0\}$. We must show that $n_v = n_w$ for all $v, w \in E^0$. To do this, it suffices to show that for any $u \in E^0$, we have $n_w \geq n_u$ for all $w \in E^0$.

So fix $u \in E^0$. Define

$$L_u := \{v \in E^0 : n_v < n_u\} \quad \text{and} \quad G_u := \{w \in E^0 : n_w \geq n_u\}.$$

We prove that if $v \in L_u$ and $w \in G_u$, then

$$\mathrm{lt}_1(H(v, n_v)) \not\subseteq H(w, n_w) \quad \text{and} \quad \mathrm{lt}_1(H(w, n_w)) \not\subseteq H(v, n_v). \tag{2.2}$$

For this, fix $v \in L_u$ and $w \in G_u$; note that in particular $v \neq w$.

To see that $\mathrm{lt}_1(H(v, n_v)) \not\subseteq H(w, n_w)$, suppose otherwise for contradiction. Then $H(v, n_v + 1) \subseteq H(w, n_w)$. Hence (2.1) shows that $wE^{n_v+1-n_w}v \neq \emptyset$, which forces $n_v \geq n_w - 1$. Since $v \in L_u$ and $w \in G_u$, we also have $n_v \leq n_w - 1$, and we conclude that $n_v + 1 - n_w = 0$. This forces $wE^0v \neq \emptyset$, contradicting that $v \neq w$.

To see that $\mathrm{lt}_1(H(w, n_w)) \not\subseteq H(v, n_v)$, we first claim that there is no $e \in E^1$ satisfying $s(e) \in L_u$ and $r(e) \in G_u$. To see this, fix $x \in L_u$ and $y \in G_u$. Then $n_y > n_x$, and in particular $n_y - 1 - n_x \geq 0$. Hence Item (3) shows that $H(y, n_y) \not\subseteq$

$\mathrm{lt}_{n_y-1-n_x}(H(x,n_x))$. Applying lt_{1-n_y} on both sides shows that $\mathrm{lt}_1(H(y,0)) \not\subseteq H(x,0)$, and so $xE^1y = \emptyset$. This proves the claim.

Since $v \in L_u$, applying the claim $n_w + 1 - n_v$ times shows that for any path $\mu \in vE^{n_w+1-n_v}$, we have $r(\mu) \in L_u$. In particular, $vE^{n_w+1-n_v}w = \emptyset$. Thus (2.1) implies that $\mathrm{lt}_1(H(w,n_w)) \not\subseteq H(v,n_v)$.

We have now established (2.2). Set

$$\overline{L}_u = \{H(v,n_v) : v \in L_u\} \quad \text{and} \quad \overline{G}_u = \{H(w,n_w) : w \in G_u\}.$$

Then (2.2) shows that $(\overline{L}_u \times \overline{L}_u) \sqcup (\overline{G}_u \times \overline{G}_u)$ is an equivalence relation on V containing $\{(H,K) : \mathrm{lt}_1(K) \subseteq H\}$. Thus item (2) implies that either $\overline{L}_u$ or $\overline{G}_u$ is empty. Since $H(u,n_u) \in \overline{G}_u$, we deduce that $\overline{L}_u = \emptyset$ which implies that $L_u = \emptyset$. Hence $G_u = E^0$, and so $n_w \geq n_u$ for all $w \in E^0$ as required. □

Corollary 2.4. *Suppose that E and F are amplified graphs. If there exists an isomorphism $\rho : (\mathcal{H}(E \times_1 \mathbb{Z}), \subseteq, \mathrm{lt}^E) \cong (\mathcal{H}(F \times_1 \mathbb{Z}), \subseteq, \mathrm{lt}^F)$, then there exists an isomorphism $\overline{\rho} : (\mathcal{H}(E \times_1 \mathbb{Z}), \subseteq, \mathrm{lt}^E) \to (\mathcal{H}(F \times_1 \mathbb{Z}), \subseteq, \mathrm{lt}^F)$ such that $\overline{\rho}(V_0^E) = V_0^F$.*

Proof. First suppose that E and F are weakly connected as in Lemma 2.3. Since $H \in \mathcal{H}^E_{\mathrm{vert}}$ if and only if H has a unique predecessor in $\mathcal{H}(E \times_1 \mathbb{Z})$ and similarly for F, the map ρ restricts to an inclusion-preserving bijection $\rho : \mathcal{H}^E_{\mathrm{vert}} \to \mathcal{H}^F_{\mathrm{vert}}$. Since V_0^E satisfies (1)–(3) of Lemma 2.3, so does $\{\rho(H) : H \in V_0^E\}$. So Lemma 2.3 shows that ρ restricts from a bijection from V_0^E to $\mathrm{lt}_n(F_0^F)$ for some $n \in \mathbb{Z}$, and therefore $\overline{\rho} := \mathrm{lt}_{-n} \circ \rho$ is the desired isomorphism.

Now suppose that E and F are not weakly connected. Let $\mathcal{WC}(E)$ denote the set of equivalence classes for the equivalence relation on E^0 generated by $\{(s(e), r(e)) : e \in E^1\}$; so the elements of $\mathcal{WC}(E)$ are the weakly connected components of E. Similarly, let $\mathcal{WC}(F)$ be the set of weakly connected components of F.

Using that vE^*w is nonempty if and only if $\mathrm{lt}_n(H(w,0)) \subseteq H(v,0)$ for some $n \in \mathbb{Z}$, we see that $vE^*w \neq \emptyset$ if and only if $\bigcup_n \mathrm{lt}_n(H(w,i)) \subseteq \bigcup_n \mathrm{lt}_n(H(v,j))$ for some (equivalently for all) $i,j \in \mathbb{Z}$. Since the same is true in F, we see that for $v,w \in E^0$, writing $x,y \in F^0$ for the elements such that $\rho(H(v,0)) \in \mathrm{lt}_{\mathbb{Z}}(H(x,0))$ and $\rho(H(w,0)) \in \mathrm{lt}_{\mathbb{Z}}(H(y,0))$, we have $vE^*w \neq \emptyset$ if and only if $xF^*y \neq \emptyset$. Now an induction shows that there is a bijection $\tilde{\rho} : \mathcal{WC}(E) \to \mathcal{WC}(F)$ such that for each $C \in \mathcal{WC}(E)$, we have $\rho(\{H(v,n) : v \in C, n \in \mathbb{Z}\}) = \{H(w,m) : w \in \tilde{\rho}(C), m \in \mathbb{Z}\}$. For each $C \in \mathcal{WC}(E)$, write E_C for the subgraph (C, CE^1C, r, s) of E and similarly for F. Then the inclusions $E_C \hookrightarrow E$ induce inclusions $(H(E_C \times_1 \mathbb{Z}), \mathrm{lt}) \hookrightarrow (H(E \times_1 \mathbb{Z}), \mathrm{lt})$ whose ranges are lt-invariant and mutually incomparable with respect to $\subseteq$. Hence ρ induces isomorphisms $\rho_C : (\mathcal{H}(E_C \times_1 \mathbb{Z}), \mathrm{lt}) \cong (\mathcal{H}(F_{\tilde{\rho}(C)} \times_1 \mathbb{Z}), \mathrm{lt})$. The first paragraph then shows that for each $C \in \mathcal{WC}(E)$ there is an isomorphism $\overline{\rho}_C : (\mathcal{H}(E_C \times_1 \mathbb{Z}), \mathrm{lt}) \to (\mathcal{H}(F_{\tilde{\rho}(C)} \times_1 \mathbb{Z}), \mathrm{lt})$ that carries $V_0^{E_C}$ to $V_0^{F_{\tilde{\rho}(C)}}$, and these then assemble into an isomorphism $\overline{\rho} : (\mathcal{H}(E \times_1 \mathbb{Z}), \subseteq, \mathrm{lt}^E) \to (\mathcal{H}(F \times_1 \mathbb{Z}), \subseteq, \mathrm{lt}^F)$ such that $\overline{\rho}(V_0^E) = V_0^F$. □

We are now ready to prove Theorem A.

Proof of Theorem A. That (1) implies (2) and that (1) implies (3) are clear.

By [3, Proposition 5.7] the graded $\mathcal{V}$-monoid $\mathcal{V}^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ is isomorphic to the $\mathcal{V}$-monoid $\mathcal{V}(L_{\mathbb{K}}(E \times_1 \mathbb{Z}))$, and that this isomorphism is equivariant for the canonical $\mathbb{Z}[x, x^{-1}]$ actions arising from the grading on $\mathcal{V}^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ and from the action on $\mathcal{V}(L_{\mathbb{K}}(E \times_1 \mathbb{Z}))$ induced by translation in the $\mathbb{Z}$-coordinate in $E \times_1 \mathbb{Z}$. Hence

$K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ is order isomorphic to $K_0(L_{\mathbb{K}}(E\times_1\mathbb{Z}))$ as $\mathbb{Z}[x,x^{-1}]$-modules. Hence condition (2) holds if and only if $K_0(L_{\mathbb{K}}(E\times_1\mathbb{Z}))\cong K_0(L_{\mathbb{K}}(F\times_1\mathbb{Z}))$ as ordered $\mathbb{Z}[x,x^{-1}]$-modules.

Likewise [20, Theorem 2.7.9] shows that the equivariant K-group $K_0^{\mathbb{T}}(C^*(E))$ is order isomorphic, as a $\mathbb{Z}[x,x^{-1}]$-module, to the K_0-group $K_0(C^*(E)\times_\gamma\mathbb{T})$. The canonical isomorphism $C^*(E)\times_\gamma\mathbb{T}\cong C^*(E\times_1\mathbb{Z})$ is equivariant for the dual action $\hat{\gamma}$ of $\mathbb{Z}$ on the former and the action of $\mathbb{Z}$ on the latter induced by translation in $E\times_1\mathbb{Z}$. It therefore induces an isomorphism $K_0(C^*(E)\times_\gamma\mathbb{T})\cong K_0(C^*(E\times_1\mathbb{Z}))$ of ordered $\mathbb{Z}[x,x^{-1}]$-modules. So condition (3) holds if and only if $K_0(C^*(E\times_1\mathbb{Z}))\cong K_0(C^*(F\times_1\mathbb{Z}))$ as ordered $\mathbb{Z}[x,x^{-1}]$-modules.

By [16, Theorem 3.4 and Corollary 3.5] (see also [2]), for any directed graph E there is an isomorphism $K_0(L_{\mathbb{K}}(E))\cong K_0(C^*(E))$ that carries the class of the module $L_{\mathbb{K}}(E)v$ to the class of the projection p_v in $C^*(E)$ for each $v\in E^0$. It follows that $K_0(L_{\mathbb{K}}(E\times_1\mathbb{Z}))\cong K_0(C^*(E\times_1\mathbb{Z}))$ as ordered $\mathbb{Z}[x,x^{-1}]$-modules. This shows that conditions (2) and (3) are equivalent. So it now suffices to show that (2) implies (1).

So suppose that (2) holds. Since E, and therefore $E\times_1\mathbb{Z}$, is an amplified graph, it admits no breaking vertices with respect to any saturated hereditary set, and every hereditary subset of $E\times_1\mathbb{Z}$ is a saturated hereditary subset. So the lattice $\mathcal{H}(E\times_1\mathbb{Z})$ of hereditary sets is identical to the lattice of admissible pairs in the sense of [22] via the map $H\mapsto(H,\emptyset)$. By [3, Theorem 5.11], there is a lattice isomorphism from $\mathcal{H}(E\times_1\mathbb{Z})$ to the lattice of order ideals of $K_0(L_{\mathbb{K}}(E\times_1\mathbb{Z}))$ that carries a hereditary set H to the class of the module $L_{\mathbb{K}}(E\times_1\mathbb{Z})H$. This isomorphism clearly intertwines the action of $\mathbb{Z}$ induced by the module structure on $K_0(L_{\mathbb{K}}(E\times_1\mathbb{Z}))$ and the action lt^E of $\mathbb{Z}$ on $\mathcal{H}(E\times_1\mathbb{Z})$ induced by translation. By the same argument applied to F, we see that $\big(\mathcal{H}(E\times_1\mathbb{Z}),\subseteq,\mathrm{lt}^E\big)\cong\big(\mathcal{H}(F\times_1\mathbb{Z}),\subseteq,\mathrm{lt}^F\big)$.

Now Corollary 2.4 implies that $\big(\mathcal{H}(E\times_1\mathbb{Z}),\mathrm{lt}^E,H_0^E\big)\cong\big(\mathcal{H}(F\times_1\mathbb{Z}),\mathrm{lt}^F,H_0^F\big)$. This isomorphism induces an isomorphism $\overline{E}\cong\overline{F}$ of the graphs constructed from these data in Proposition 2.2. Thus two applications of Proposition 2.2 give $E\cong\overline{E}\cong\overline{F}\cong F$, which is (1). ☐

3. Equivariant K-theory and graded K-theory are stable invariants

In this section, we prove that equivariant K-theory and graded K-theory are stable invariants. We suspect that these are well-known results but we have been unable to find a reference in the literature. For the convenience of the reader, we include their proofs here. We use these results to deduce the consequences of Theorem A for graded stable isomorphisms of amplified Leavitt path algebras, and gauge-equivariant stable isomorphisms of amplified graph C^*-algebras.

3.1. Stability of equivariant K-theory.

Theorem 3.1. *Let G be a compact group and let α be an action of G on a C^*-algebra A. Suppose that A has an increasing approximate identity consisting of G-invariant projections. Then the natural $R(G)$-module isomorphism from $K_0^G(A,\alpha)$ to $K_0(C^*(G,A,\alpha))$ is an order isomorphism.*

Proof. First suppose A has a unit. Then the theorem follows from the proof of Julg's Theorem [17] (see also [20, Theorem 2.7.9]). The isomorphism is given by the composition of two isomorphisms:

$$K_0^G(A,\alpha) \to K_0(L^1(G,A,\alpha)) \quad \text{and}$$
$$K_0(L^1(G,A,\alpha)) \to K_0(C^*(G,A,\alpha)).$$

The proof that these maps are isomorphisms shows that the maps are order isomorphisms (see the proof of [20, Lemma 2.4.2 and Theorem 2.6.1]).

Now suppose that A has an increasing approximate identity S consisting of G-invariant projections. Fix $p \in S$. Let

$$\lambda_A \colon K_0^G(A,\alpha) \to K_0(C^*(G,A,\alpha)), \text{ and}$$
$$\lambda_p \colon K_0^G(pAp,\alpha) \to K_0(C^*(G,pAp,\alpha)), \quad p \in S$$

be the natural $R(G)$-isomorphisms given in Julg's Theorem. Note that α does indeed induce an action on pAp since p is G-invariant. Let ι_p be the G-equivariant inclusion of pAp into A and let $\widetilde{\iota_p}$ be the induced $*$-homomorphism from $C^*(G,pAp,\alpha)$ to $C^*(G,A,\alpha)$.

Let $x \in K_0^G(A,\alpha)_+$. By [20, Corollary 2.5.5], there exist $p \in S$ and $x' \in K_0^G(pAp,\alpha)_+$ such that $(\iota_p)_*(x') = x$. Naturality of the maps λ_A and λ_p gives $\lambda_A(x) = (\widetilde{\iota_p})_* \circ \lambda_p(x')$. Consequently, $\lambda_A(x) \in K_0(C^*(G,A,\alpha))_+$ since $(\widetilde{\iota_p})_* \circ \lambda_p(x') \in K_0(C^*(G,A,\alpha))_+$. Fix $y \in K_0(C^*(G,A,\alpha))_+$. For $f \in L^1(G)$ and $a \in A$ we write $f \otimes a : G \to A$ for the function $(f \otimes a)(g) = f(g)a$. Since S is an approximate identity of A and since

$$\{f \otimes a : f \in L^1(G), a \in A\}$$

is dense in $C^*(G,A,\alpha)$, the set $\bigcup_{p\in S} \widetilde{\iota_p}(C^*(G,pAp,\alpha))$ is dense in $C^*(G,A,\alpha)$. Thus, there exists a projection $p \in S$ and there exists $y' \in K_0(C^*(G,pAp,\alpha))_+$ such that $(\widetilde{\iota_p})_*(y') = x$. Since λ_p is an order isomorphism, $\lambda_p^{-1}(y') \in K_0^G(pAp,\alpha)_+$. Then $(\iota_p)_* \circ \lambda_p^{-1}(y') \in K_0^G(A,\alpha)_+$. Naturality of the maps λ_A and λ_p implies that $\lambda_A \circ (\iota_p)_* \circ \lambda_p^{-1}(y') = y$. We have shown that $\lambda_A(K_0^G(A,\alpha)_+) = K_0(C^*(G,A,\alpha))_+$ which implies that λ_A is an order isomorphism. □

Lemma 3.2. *Let G be a compact group and let A be a separable C^*-algebra and let α be an action of G on A. If B is a hereditary subalgebra of A such that*

1. *B has an increasing approximate identity of G-invariant projections,*
2. *A has an increasing approximate identity of G-invariant projections,*
3. *$\overline{ABA} = A$, and*
4. *$\alpha_g(B) \subseteq B$ for all $g \in G$,*

then the inclusion $\iota \colon B \to A$ induces an isomorphism $K_0^G(B) \cong K_0^G(A)$ of ordered $R(G)$-modules.

Proof. Since B is G-invariant, α restricts to an action on B and ι is G-equivariant. Let $\lambda_B \colon K_0^G(B,\alpha) \to K_0(C^*(G,B,\alpha))$ and $\lambda_A \colon K_0^G(A) \to K_0(C^*(G,A,\alpha))$ be the natural $R(G)$-module order isomorphisms given in Theorem 3.1. Naturality of λ_B and λ_A implies that the diagram

$$\begin{array}{ccc} K_0^G(B) & \xrightarrow{\iota_*} & K_0^G(A) \\ \downarrow{\scriptstyle \lambda_B} & & \downarrow{\scriptstyle \lambda_A} \\ K_0(C^*(G,B,\alpha)) & \xrightarrow[\widetilde{\iota}_*]{} & K_0(C^*(G,A,\alpha)) \end{array}$$

is commutative. As in the proof of [20, Proposition 2.9.1], $C^*(G, B, \alpha)$ is a hereditary subalgebra of $C^*(G, A, \alpha)$ such that the closed two-sided ideal of $C^*(G, A, \alpha)$ generated by $C^*(G, B, \alpha)$ is $C^*(G, A, \alpha)$. This $\widetilde{\iota}_*$ is an order isomorphism, and so ι_* is also an order isomorphism. □

Corollary 3.3 implies that the equivariant K_0-group is a stable invariant.

Corollary 3.3. *Let G be a compact group, let α be an action of G on a separable C^*-algebra A, and let β be an action of G on $\mathcal{K}(\ell^2)$. If both A and $\mathcal{K}(\ell^2)$ admit increasing approximate identities consisting of G-invariant projections, then there is an $R(G)$-module order isomorphism from $K_0^G(A, \alpha)$ to $K_0^G(A \otimes \mathcal{K}(\ell^2), \alpha \otimes \beta)$.*

In particular, if $u : G \to \mathcal{U}(\ell^2)$ is a continuous (in the strong operator topology) unitary representation of G and $\beta_g = \mathrm{Ad}(u_g)$, then there is an $R(G)$-module order isomorphism from $K_0^G(A, \alpha)$ and $K_0^G(A \otimes \mathcal{K}(\ell^2), \alpha \otimes \beta)$.

Proof. Let $\{p_n\}_{n\in\mathbb{N}}$ be an increasing approximate identity consisting of G-invariant projections in $\mathcal{K}(\ell^2)$. We may assume $p_1 \neq 0$. Then $A \otimes p_1$ is a G-invariant hereditary subalgebra of $A \otimes \mathcal{K}(\ell^2)$ such that $\overline{(A \otimes \mathcal{K}(\ell^2))(A \otimes p_1)(A \otimes \mathcal{K}(\ell^2))} = A \otimes \mathcal{K}(\ell^2)$. From the assumption on A and $\mathcal{K}(\ell^2)$, both $A \otimes p_1$ and $A \otimes \mathcal{K}(\ell^2)$ have increasing approximate identities consisting of G-invariant projections. Lemma 3.2 implies that there is an $R(G)$-module order isomorphism from $K_0^G(A \otimes p_1, \alpha \otimes \beta)$ to $K_0^G(A \otimes \mathcal{K}(\ell^2), \alpha \otimes \beta)$. The result now follows since the map $a \mapsto a \otimes p_1$ is a G-equivariant $*$-isomorphism from A to $A \otimes p_1$.

For the last part of the corollary, since G is compact, u is a direct sum of finite dimensional representations. Thus, $\mathcal{K}(\ell^2)$ has an increasing approximate identity consisting of G-invariant projections. □

To finish this subsection, we describe the consequences of Theorem A for equivariant stable isomorphism of amplified graph C^*-algebras. For Theorem 3.4, given a strong-operator continuous unitary representation $u : \mathbb{T} \to \mathcal{U}(\ell^2)$ of $\mathbb{T}$ on a Hilbert space H, we will write β^u for the action of $\mathbb{T}$ on $\mathcal{B}(\ell^2)$ given by $\beta^u_z = \mathrm{Ad}(u_z)$.

Theorem 3.4. *Let E and F be countable amplified graphs. Then the following are equivalent:*

1. *$E \cong F$;*
2. *$(C^*(E), \gamma^E) \cong (C^*(F), \gamma^F)$;*
3. *$(C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \beta^u) \cong (C^*(F) \otimes \mathcal{K}, \gamma^F \otimes \beta^u)$, for every strongly continuous representation $u : \mathbb{T} \to \mathcal{U}(\ell^2)$;*
4. *there exists a strongly continuous unitary representation $u : \mathbb{T} \to \mathcal{U}(\ell^2)$ such that $(C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \beta^u) \cong (C^*(F) \otimes \mathcal{K}, \gamma^F \otimes \beta^u)$; and*
5. *there exist strongly continuous unitary representations $u, v : \mathbb{T} \to \mathcal{U}(\ell^2)$ such that $(C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \beta^u) \cong (C^*(F) \otimes \mathcal{K}, \gamma^F \otimes \beta^v)$.*

Proof. If $\phi : E \to F$ is an isomorphism, it induces an isomorphism $C^*(E) \cong C^*(F)$, which is gauge invariant because it carries generators to generators. This gives (1) $\implies$ (2).

If (2) holds, say $\phi : C^*(E) \to C^*(F)$ is a gauge-equivariant isomorphism, then for any u the map $\phi \otimes \mathrm{id}_{\mathcal{K}}$ is an equivariant isomorphism from $(C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \beta^u)$ to $(C^*(F) \otimes \mathcal{K}, \gamma^F \otimes \beta^u)$, giving (3). Clearly (3) implies (4). And if (4) holds for a given $u : \mathbb{T} \to \mathcal{B}(\ell^2)$, then (5) holds with $u = v$. Finally, if (5) holds, then two applications of Corollary 3.3 show that

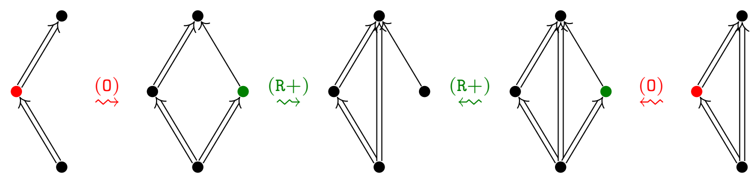
$$\begin{aligned} K_0^{\mathbb{T}}(C^*(E),\gamma^E) &\cong K_0^{\mathbb{T}}(C^*(E)\otimes\mathcal{K}(\ell^2),\gamma^E\otimes\beta^u) \\ &\cong K_0^{\mathbb{T}}(C^*(F)\otimes\mathcal{K}(\ell^2),\gamma^F\otimes\beta^v) \cong K_0^{\mathbb{T}}(C^*(F),\gamma^F) \end{aligned}$$

as ordered $\mathbb{Z}[x,x^{-1}]$-modules, and so Theorem A gives (1). □

Remark 3.5. In this remark, we outline the conclusions that can be drawn from the results presented here for the project in [11] by two of the authors. The goal of this work is to characterise eight different notions of isomorphism between two unital graph C^*-algebras—denoted $\overline{\mathtt{xyz}}$ with $\mathtt{x},\mathtt{y},\mathtt{z}\in\{\mathtt{0},\mathtt{1}\}$—by showing that they are each the smallest equivalence relation containing an appropriate collection of so-called moves on the graphs. As will be detailed in [11], Theorem 3.4 completes this program for amplified graphs in all eight cases.

Indeed, Theorem 3.4 shows that for amplified graphs all the four notions of equivalence $\overline{\mathtt{x1z}}$ (requiring the isomorphisms to commute with the canonical gauge action) coincide, and degenerate to isomorphism of the underlying graphs. Hence it becomes vacuously true that these equivalence relations are generated by graph moves.

A similar result holds for the relations $\overline{\mathtt{x0z}}$ (not requiring the isomorphisms to commute with the canonical gauge action). Recall from [12] that if E is an amplified graph then its amplified transitive closure tE is the amplified graph with $tE^0 = E^0$ and $v(tE^1)w \neq \emptyset$ if and only if $vE^*w\setminus\{v\}\neq\emptyset$. Theorem 1.1 of [12] shows that for amplified graphs, if $(E,F)\in\overline{\mathtt{000}}$, then $tE\cong tF$. By the construction

(3.1) 

one sees that the operation which, given vertices u,v,w such that uE^1v and vE^1w are infinite, adds infinitely many new edges to uE^1w can be obtained using the two graph moves (O) (out-splitting) and (R+) (reduction). By [11], moves (O) and (R+) preserve $\overline{\mathtt{101}}$, so this leads to the conclusion that the four equivalence relations $\overline{\mathtt{x0z}}$ are identical, coincide with isomorphism of amplified transitive closures of the underlying graphs, and are generated by (O) and (R+), as required.

3.2. **Stability of graded algebraic K_0.** Next we establish the stable invariance of graded K-theory. Let Γ be an additive abelian group and let A be a Γ-graded ring. For $\overline{\delta}\in\Gamma^n$, we write $M_n(A)(\overline{\delta})$ for the Γ-graded ring $M_n(A)$ with grading given by $(a_{i,j})\in M_n(A)_\lambda$ if and only if $a_{i,j}\in A_{\lambda+\delta_j-\delta_i}$. Similarly, for $\overline{\delta}\in\prod_n\Gamma$, we write $M_\infty(A)(\overline{\delta})$ for the Γ-graded ring $M_\infty(A)$ with grading given by $(a_{i,j})\in M_\infty(A)(\overline{\delta})_\lambda$ if and only if $a_{i,j}\in A_{\lambda+\delta_j-\delta_i}$.

Since the tensor product of two graded modules will be key in the proof, we recall the construct given in [15, Section 1.2.6]. Let Γ be an additive abelian group, let A be a Γ-graded ring, let M be a graded right A-module, and let N be a graded left A-module. Then $M\otimes_A N$ is defined to be $M\otimes_{A_0} N$ modulo the subgroup generated by

$$\{ma\otimes n - m\otimes an : m\in M, n\in N, \text{ and } a\in A \text{ are homogeneous}\}$$

with grading induced by the grading on $M \otimes_{A_0} N$ given by

$$(M \otimes_{A_0} N)_\lambda = \left\{ \sum_i m_i \otimes n_i : m_i \in M_{\alpha_i}, n_i \in N_{\beta_i} \text{ with } \alpha_i + \beta_i = \lambda \right\}.$$

Theorem 3.6. *Let Γ be an additive abelian group, let A be a unital Γ-graded ring, and let $\overline{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in \Gamma^n$. Then the inclusion $\iota\colon A \to M_n(A)(\overline{\delta})$ into the $e_{1,1}$ corner induces a $\mathbb{Z}[\Gamma]$-module order isomorphism $K_0^{\mathrm{gr}}(\iota)\colon K_0^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(M_n(A)(\overline{\delta}))$ given by $K_0^{\mathrm{gr}}(\iota)([P]) = [P \otimes_A M_n(A)(\overline{\delta})]$ (the left A-module structure on $M_n(A)(\overline{\delta})$ is given by the inclusion ι).*

Proof. Let $\overline{\alpha} = (0, \delta_2 - \delta_1, \ldots, \delta_n - \delta_1)$. By [15, Corollary 2.1.2], there is an equivalence of categories $\phi\colon \text{Pgr-}A \to \text{Pgr-}M_n(A)(\overline{\alpha})$ given by $\phi(P) = P \otimes_A A^n(\overline{\alpha})$. Moreover, ϕ commutes with the suspension map. Since

$$M_n(A)(\overline{\alpha})_\lambda = \begin{pmatrix} A_\lambda & A_{\lambda+\alpha_2-\alpha_1} & \cdots & A_{\lambda+\alpha_n-\alpha_1} \\ A_{\lambda+\alpha_1-\alpha_2} & A_\lambda & \cdots & A_{\lambda+\alpha_n-\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\lambda+\alpha_1-\alpha_n} & A_{\lambda+\alpha_2-\alpha_n} & \cdots & A_\lambda \end{pmatrix}$$
$$= \begin{pmatrix} A_\lambda & A_{\lambda+\delta_2-\delta_1} & \cdots & A_{\lambda+\delta_n-\delta_1} \\ A_{\lambda+\delta_1-\delta_2} & A_\lambda & \cdots & A_{\lambda+\delta_n-\delta_2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\lambda+\delta_1-\delta_n} & A_{\lambda+\delta_2-\delta_n} & \cdots & A_\lambda \end{pmatrix} = M_n(A)(\overline{\delta})_\lambda,$$

we have $M_n(A)(\overline{\alpha}) = M_n(A)(\overline{\delta})$. Therefore, $\phi(P) = P \otimes_A A^n(\overline{\alpha})$ is an equivalence of categories from Pgr-A to Pgr-$M_n(A)(\overline{\delta})$ and ϕ commutes with the suspension map. Hence, ϕ induces a $\mathbb{Z}[\Gamma]$-module order isomorphism from $K_0^{\mathrm{gr}}(A)$ to $K_0^{\mathrm{gr}}(M_n(A)(\overline{\delta}))$.

We claim that $\phi = K_0^{\mathrm{gr}}(\iota)$. Let M be a graded right A-module. We will show that $M \otimes_A A^n(\overline{\alpha})$ and $M \otimes_A M_n(A)(\overline{\delta})$ are isomorphic as graded modules. Since $1_A \in A_0$ and $M1_A = M$,

$$M \otimes_A M_n(A)(\overline{\delta}) \cong_{\mathrm{gr}} M \otimes_A \iota(1_A) M_n(A)(\overline{\delta}) = M \otimes_A e_{1,1} M_n(A)(\overline{\delta}).$$

By the definitions of the gradings on $e_{1,1}M_n(A)(\overline{\delta})$ and $A^n(\alpha)$, the right $M_n(A)$-module isomorphism

$$e_{1,1}X \mapsto (x_{1,1}, x_{1,2}, \ldots, x_{1,n})$$

is a graded isomorphism. Hence,

$$M \otimes_A M_n(A)(\overline{\delta}) \cong_{\mathrm{gr}} M \otimes_A e_{1,1} M_n(A)(\overline{\delta}) \cong_{\mathrm{gr}} M \otimes_A A^n(\alpha).$$

Thus, $\phi = K_0^{\mathrm{gr}}(\iota)$. Consequently, $K_0^{\mathrm{gr}}(\iota)$ is a $\mathbb{Z}[\Gamma]$-module order isomorphism. □

Corollary 3.7. *Let Γ be an additive abelian group and let A be a Γ-graded ring with a sequence of idempotents $\{e_n\}_{n=1}^\infty \subseteq A_0$ such that $e_n e_{n+1} = e_n$ for all n, and $\bigcup_n e_n A e_n = A$. For $\overline{\delta} \in \prod_i \Gamma$, the embedding $\iota\colon A \to M_\infty(A)(\overline{\delta})$ into the $e_{1,1}$ corner of $M_\infty(A)(\overline{\delta})$ induces a $\mathbb{Z}[\Gamma]$-module order isomorphism $K_0^{\mathrm{gr}}(\iota)\colon K^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(M_\infty(A)(\overline{\delta}))$.*

In particular, if E is a countable directed graph and $\overline{\delta} \in \prod_i \mathbb{Z}$, then the inclusion $\iota\colon L_{\mathbb{K}}(E) \to M_\infty(L_{\mathbb{K}}(E))(\overline{\delta})$ of $L_{\mathbb{K}}(E)$ in the $e_{1,1}$ corner of $M_\infty(L_{\mathbb{K}}(E))(\overline{\delta})$ induces a $\mathbb{Z}[x, x^{-1}]$-module order isomorphism from $K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ to $K_0^{\mathrm{gr}}(M_\infty(L_{\mathbb{K}}(E))(\overline{\delta}))$ for any field $\mathbb{K}$.

Proof. Let $\iota_n \colon e_n A e_n \to M_\infty(e_n A e_n)(\overline{\delta})$ be the inclusion of $e_n A e_n$ into the $e_{1,1}$ corner of $M_\infty(e_n A e_n)(\overline{\delta})$. Note that $A = \varinjlim e_n A e_n$, that $M_\infty(A) = \varinjlim M_\infty(e_n A e_n)$, and that the diagram

$$\begin{array}{ccc} e_n A e_n & \xrightarrow{\subseteq} & A \\ {\scriptstyle \iota_n}\downarrow & & \downarrow{\scriptstyle \iota} \\ M_\infty(e_n A e_n)(\overline{\delta}) & \xrightarrow{\subseteq} & M_\infty(A)(\overline{\delta}) \end{array}$$

commutes. Therefore, if each $K_0^{\mathrm{gr}}(\iota_n)$ is a $\mathbb{Z}[\Gamma]$-module order isomorphism, then $K_0^{\mathrm{gr}}(\iota)$ is a $\mathbb{Z}[\Gamma]$-module order isomorphism since the graded K_0-group respects direct limits [15, Theorem 3.2.4]. Hence, without loss of generality, we may assume that A is a unital Γ-graded ring.

Let $\overline{\delta}_n = (\delta_1, \delta_2, \ldots, \delta_n)$. Let $j_n \colon A \to M_n(A)(\overline{\delta}_n)$ be the inclusion of A into the $e_{1,1}$ corner of $M_n(A)(\overline{\delta}_n)$, and let $\iota_n : M_n(A)(\overline{\delta}_n) \to M_\infty(A)(\overline{\delta})$ be the inclusion map. Then $\varinjlim M_n(A)(\overline{\delta}_n) = M_\infty(A)(\overline{\delta})$ and the diagram

$$\begin{array}{ccc} A & & \\ {\scriptstyle j_n}\downarrow & \searrow^{\iota} & \\ M_n(A)(\overline{\delta}_n) & \xrightarrow{\iota_n} & M_\infty(A)(\overline{\delta}) \end{array}$$

commutes. By Theorem 3.6, $K_0^{\mathrm{gr}}(j_n)$ is a $\mathbb{Z}[\Gamma]$-module order isomorphism. Since the graded-K_0 functor respects direct limits, $K_0^{\mathrm{gr}}(\iota)$ is $\mathbb{Z}[\Gamma]$-module order isomorphism.

For the last part of the corollary, let $\{X_n\}$ be a sequence of finite subsets of E^0 such that $X_n \subseteq X_{n+1}$ and $\bigcup_n X_n = E^0$. Then $e_n := \sum_{v \in X_n} v$ defines idempotents of degree zero such that $\bigcup_n e_n L_{\mathbb{K}}(E) e_n = L_{\mathbb{K}}(E)$. □

As in the preceding subsection, we finish by describing the consequences of Theorem A for graded stable isomorphism of amplified Leavitt path algebras.

Theorem 3.8. *Let E and F be countable amplified graphs and let $\mathbb{K}$ be a field. Then the following are equivalent:*

(1) $E \cong F$;
(2) $L_{\mathbb{K}}(E) \cong^{\mathrm{gr}} L_{\mathbb{K}}(F)$;
(3) $L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})(\overline{\delta}) \cong^{\mathrm{gr}} L_{\mathbb{K}}(F) \otimes M_\infty(\mathbb{K})(\overline{\delta})$ *for every* $\overline{\delta} \in \prod_i \mathbb{Z}$;
(4) $L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})(\overline{\delta}) \cong^{\mathrm{gr}} L_{\mathbb{K}}(F) \otimes M_\infty(\mathbb{K})(\overline{\delta})$ *for some* $\overline{\delta} \in \prod_i \mathbb{Z}$; *and*
(5) $L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})(\overline{\delta}) \cong^{\mathrm{gr}} L_{\mathbb{K}}(F) \otimes M_\infty(\mathbb{K})(\overline{\varepsilon})$ *for some* $\overline{\delta}, \overline{\varepsilon} \in \prod_i \mathbb{Z}$.

Proof. The argument is similar to that of Theorem 3.4, so we summarise. Any isomorphism of graphs induces a graded isomorphism of their Leavitt path algebras, and any graded isomorphism $\phi : L_{\mathbb{K}}(E) \cong L_{\mathbb{K}}(F)$ amplifies to a graded isomorphism $\phi \otimes \mathrm{id} : L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})(\overline{\delta}) \cong L_{\mathbb{K}}(F) \otimes M_\infty(\mathbb{K})(\overline{\delta})$, giving (1) $\implies$ (2) $\implies$ (3). The implications (3) $\implies$ (4) $\implies$ (5) are trivial. The second statement of Corollary 3.7 shows that if (5) holds then $K^{\mathrm{gr}}(L_{\mathbb{K}}(E)) \cong K^{\mathrm{gr}}(L_{\mathbb{K}}(F))$ as ordered $\mathbb{Z}[x, x^{-1}]$-modules, and then Theorem A gives (1). □

Remark 3.9. Since statement (1) of Theorem 3.8 does not depend on the field $\mathbb{K}$, we deduce that each of the other four statements holds for some field $\mathbb{K}$ if and only if it holds for every field $\mathbb{K}$. In particular the graded-isomorphism problem for amplified Leavitt path algebras is field independent, so it suffices, for example, to consider the field $\mathbb{F}_2$.

Remark 3.10. Let E and F be amplified graphs. Theorem 3.4 shows that the existence of an isomorphism $(C^*(E)\otimes\mathcal{K}, \gamma^E\otimes\beta^u) \cong (C^*(F)\otimes\mathcal{K}, \gamma^F\otimes\beta^u)$ for every u is equivalent to the existence of such an isomorphism for some u, and indeed to the existence of an isomorphism $(C^*(E)\otimes\mathcal{K}, \gamma^E\otimes\beta^u) \cong (C^*(F)\otimes\mathcal{K}, \gamma^F\otimes\beta^v)$ for some u, v. All of these conditions are formally weaker than the existence of isomorphisms $(C^*(E)\otimes\mathcal{K}, \gamma^E\otimes\beta^u) \cong (C^*(F)\otimes\mathcal{K}, \gamma^F\otimes\beta^v)$ for every pair of strongly continuous representations $u, v : \mathbb{T} \to \mathcal{U}(\ell^2)$, and this in turn is clearly equivalent to the existence of an isomorphism $(C^*(E)\otimes\mathcal{K}, \gamma^E\otimes\beta^u) \cong (C^*(F)\otimes\mathcal{K}, \gamma^F\otimes\mathrm{id})$ for every u. So it is natural to ask for which amplified graphs E, F and which strongly continuous representations $u : \mathbb{T} \to \mathcal{U}(\ell^2)$ we have $(C^*(E)\otimes\mathcal{K}, \gamma^E\otimes\beta^u) \cong (C^*(F)\otimes\mathcal{K}, \gamma^F\otimes\mathrm{id})$.

This is an intriguing question to which we do not know a complete answer, but we can certainly show that the condition that $(C^*(E)\otimes\mathcal{K}, \gamma^E\otimes\beta^u) \cong (C^*(F)\otimes\mathcal{K}, \gamma^F\otimes\mathrm{id})$ for every u is in general strictly stronger than the equivalent conditions of Theorem 3.4. Specifically, let $E = F$ be the directed graph with $E^0 = \{v, w\}$ and $E^1 = \{e_n : n \in \mathbb{N}\}$ with $s(e_n) = v$ and $r(e_n) = w$ for all $\mathbb{N}$. Then the only nonzero spectral subspaces for the gauge action on $C^*(E)$ are those corresponding to $-1, 0, -1$, and so the same is true for the spectral subspaces of $C^*(E)\otimes\mathcal{K}$ with respect to $\gamma^E\otimes\mathrm{id}$. On the other hand, if $u : \mathbb{T} \to B(\ell^2(\mathbb{Z}))$ is given by $u_z e_n = z^n e_n$, then each spectral subspace of $C^*(E)\otimes\mathcal{K}$ for $\gamma^E\otimes\beta^u$ is nonempty, so $(C^*(E)\otimes\mathcal{K}, \gamma^E\otimes\beta^u) \not\cong (C^*(E)\otimes\mathcal{K}, \gamma^E\otimes\mathrm{id})$. We do not, however, know of an example in which $C^*(E)$ is simple.

A similar question can be posed for amplified Leavitt path algebras: for which amplified graphs E, F and elements $\overline{\delta} \in \prod_i \mathbb{Z}$ do we have $L_{\mathbb{K}}(E)\otimes M_\infty(\mathbb{K})(\overline{\delta}) \cong^{\mathrm{gr}} L_{\mathbb{K}}(F)\otimes M_\infty(\mathbb{K})(\overline{0})$? The same example shows that the existence of such an isomorphism for every $\overline{\delta}$ is in general strictly stronger than the equivalent conditions of Theorem 3.8.

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