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ON THE TYPE OF THE VON NEUMANN ALGEBRA OF AN OPEN SUBGROUP OF THE NERETIN GROUP

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ABSTRACT. The Neretin group $\mathcal{N}_{d,k}$ is the totally disconnected locally compact group consisting of almost automorphisms of the tree $\mathcal{T}_{d,k}$. This group has a distinguished open subgroup $\mathcal{O}_{d,k}$. We prove that this open subgroup is not of type I. This gives an alternative proof of the recent result of P.-E. Caprace, A. Le Boudec and N. Matte Bon which states that the Neretin group is not of type I, and answers their question whether $\mathcal{O}_{d,k}$ is of type I or not.

1. Introduction

The Neretin group $\mathcal{N}_{d,k}$ was introduced by Yu. A. Neretin in [11] as an analogue of the diffeomorphism group of the circle. This group $\mathcal{N}_{d,k}$ consists of almost automorphisms of the tree $\mathcal{T}_{d,k}$ and is a totally disconnected locally compact Hausdorff group. It has a distinguished open subgroup $\mathcal{O}_{d,k}$; for an accurate definition, see Section 3. Recently, P.-E. Caprace, A. Le Boudec and N. Matte Bon proved that the Neretin group $\mathcal{N}_{d,k}$ is not of type I by constructing two weakly equivalent but inequivalent irreducible representations of $\mathcal{N}_{d,k}$ [4]. In their paper, they conjectured that the subgroup $\mathcal{O}_{d,k}$ of the Neretin group $\mathcal{N}_{d,k}$ is not type I either [4, Remark 4.8]. Our main theorem answers their question.

Theorem 1.1. *The group von Neumann algebra of $L(\mathcal{O}_{d,k})$ of the open subgroup $\mathcal{O}_{d,k}$ of the Neretin group $\mathcal{N}_{d,k}$ is of type II. In particular, the open subgroup $\mathcal{O}_{d,k}$ of the Neretin group $\mathcal{N}_{d,k}$ is not of type I.*

This theorem gives an alternative proof of the fact that the Neretin group $\mathcal{N}_{d,k}$ is not of type I, since the type I property is inherited to open subgroups. In the proof of our main theorem, we construct a nontrivial central sequence in the corner of the group von Neumann algebra $L(\mathcal{O}_{d,k})$.

2. Preliminaries

2.1. **von Neumann algebras.** We refer the reader to [6] for basics about von Neumann algebras. We review several topologies we use. Let H be a separable Hilbert space. For $\xi \in H$, seminorms p_ξ, p_ξ^* on $B(H)$ are defined by $p_\xi(x) = \|x\xi\|$ and $p_\xi^*(x) = \|x^*\xi\|$. The topology defined by these seminorms $\{p_\xi \mid \xi \in H\} \cup \{p_\xi^* \mid \xi \in H\}$ on $B(H)$ is called **strong-$*$ operator topology**. For $\{\xi_n\} \in$

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$\ell^2 \otimes H = \{\{\xi_n\} \mid \xi_n \in H, \sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty\}$, seminorms $q_{\{\xi_n\}}, q^*_{\{\xi_n\}}$ are defined by $q_{\{\xi_n\}}(x) = (\sum_{n=1}^{\infty} \|x\xi_n\|^2)^{\frac{1}{2}}$ and $q^*_{\{\xi_n\}}(x) = (\sum_{n=1}^{\infty} \|x^*\xi_n\|^2)^{\frac{1}{2}}$. The topology defined by these seminorms $\{q_{\{\xi_n\}} \mid \{\xi_n\} \in \ell^2 \otimes H\} \cup \{q^*_{\{\xi_n\}} \mid \{\xi_n\} \in \ell^2 \otimes H\}$ on $B(H)$ is called **ultrastrong-$*$ topology**. Note that these two topologies coincide on bounded subsets of $B(H)$.

We also review definitions of types of von Neumann algebras (see [3, Section 1.3]). A von Neumann algebra M is of **type I** if it is isomorphic to $\prod_{j\in J} \mathcal{A}_j \bar{\otimes} B(H_j)$ for some set J of cardinal numbers, where $\mathcal{A}_j$ is an abelian von Neumann algebra and H_j is a Hilbert space of dimension j. A von Neumann algebra M is of **type $\mathbf{II_1}$** if it has no nonzero summand of type I and there exists a separating family of normal tracial states. A von Neumann algebra M is of **type $\mathbf{II}_\infty$** if it has no nonzero summand of type I or II_1 but there exists an increasing net of projections $\{p_i\}_{i\in I} \subset M$ converging strongly to 1_M such that $p_i M p_i$ is of type II_1 for every $i \in I$. A von Neumann algebra M is of **type II** if it is a direct sum of a type II_1 and a type II_∞ von Neumann algebra. A von Neumann algebra M is of **type III** if it has no nonzero summand of type I, II_1 or II_∞. Every von Neumann algebra M has a unique decomposition $M \cong M_{\mathrm{I}} \oplus M_{\mathrm{II}} \oplus M_{\mathrm{III}}$ where $M_{\mathrm{I}}, M_{\mathrm{II}}, M_{\mathrm{III}}$ are of type I, type II, type III respectively.

We review types of von Neumann algebras from the perspective of central sequences and obtain a criterion of having no nonzero type I summand.

Definition 2.1. Let M be a separable von Neumann algebra. A **central sequence** of M is a sequence $\{u_n\}$ of unitary elements in M such that $[x, u_n]$ converges to 0 in the ultrastrong-$*$ topology for all $x \in M$. A central sequence $\{u_n\}$ of M is **trivial** if there exists a sequence $\{z_n\}$ of unitary elements of the center of M such that $u_n - z_n$ converges to 0 in the ultrastorong-$*$ topology.

Remark 2.2. A sequence $\{u_n\}$ of unitary elements in M is a central sequence if and only if there exists $M_0 \subset M$ such that $M_0'' = M$ and for all $x \in M_0$, $[x, u_n] \to 0$ in the ultrastrong-$*$ topology.

A. Connes showed that any type I factor has no nontrivial central sequence [5, Corollary 3.10] and this fact can be easily extended to type I von Neumann algebras.

Lemma 2.3. *Let M be a separable von Neumann algebra. If M is of type I, then every central sequence of M is trivial.*

Proof. We may assume that M is isomorphic to $\mathcal{A} \bar{\otimes} B(H)$ for some separable abelian von Neumann algebra $\mathcal{A}$ and some separable Hilbert space H. Let $\{u_n\}$ be a central sequence in M. Take some unit vector $\eta_0 \in H$ and let $p \in B(H)$ be the projection onto $\mathbb{C}\eta_0$. Then there exist $a_n \in \mathcal{A}$ such that $(1\otimes p)u_n(1\otimes p) = a_n \otimes p \in \mathcal{A} \bar{\otimes} pB(H)p \cong \mathcal{A} \bar{\otimes} \mathbb{C}p$. Since $\mathcal{A}$ is abelian, there exists a unitary element $v_n \in \mathcal{A}$ such that $a_n = v_n|a_n|$. We will show $u_n - v_n \otimes 1 \to 0$ in the strong-$*$ topology. First, we will show $u_n - a_n \otimes 1 \to 0$ in the strong-$*$ topology. Fix a faithful representation $\mathcal{A} \subset B(K)$ and take $\xi \in K, \eta \in H$ arbitrarily. Then, for sufficiently large n,

$$\begin{aligned} u_n(\xi \otimes \eta) &\approx (1 \otimes (\eta \otimes \eta_0^*))u_n(\xi \otimes \eta_0) \\ &= (1 \otimes (\eta \otimes \eta_0^*))(a_n \otimes p)(\xi \otimes \eta_0) \\ &= (a_n \otimes 1)(\xi \otimes \eta), \end{aligned}$$

where $\eta\otimes\eta_0^*$ is a Schatten form; $\eta\otimes\eta_0^*(\zeta) = \langle\zeta,\eta_0\rangle\eta$. Similarly, one has $u_n^*(\xi\otimes\eta) \approx (a_n^*\otimes 1)(\xi\otimes\eta)$ for sufficiently large n. Finally, we should prove $|a_n| \to 1$ in $\mathcal{A}$ in the ultrastrong-$*$ topology; if this holds, then $a_n\otimes 1 - v_n\otimes 1 = v_n((|a_n|-1)\otimes 1) \to 0$ in the ultrastrong-$*$ topology. Since $t \mapsto \sqrt{t\vee 0}$ is a linear growth function, it suffices to prove $a_n^* a_n \to 1$ in the strong-$*$ topology. For arbitrary $\xi \in K$,

$$\begin{aligned}\|a_n^*a_n\xi - \xi\| &= \|(a_n^*a_n\otimes p)\xi\otimes\eta_0 - \xi\otimes\eta_0\| \\ &= \|(1\otimes p)u_n^*(1\otimes p)u_n(1\otimes p)\xi\otimes\eta_0 - \xi\otimes\eta_0\| \\ &\to 0.\end{aligned}$$

Therefore, a central sequence $\{u_n\}$ in M is trivial. □

Lemma 2.4. *Let M be a separable von Neumann algebra. Suppose there exist a faithful normal state φ and two central sequences $\{u_n\}, \{v_n\}$ such that $\varphi((u_nv_nu_n^*v_n^*)^k)$ converges to 0 for every $k \in \mathbb{Z}\setminus\{0\}$. Then M has no nonzero type I summand.*

Proof. For simplicity, we write $u_nv_nu_n^*v_n^*$ as w_n. Note that for every $f \in C(\mathbb{T})$, $\varphi(f(w_n)) \to \int_{\mathbb{T}} f(z)\,dz$ where $\mathbb{T} = \{z\in\mathbb{C} \mid |z| = 1\}$, since trigonometric polynomials are dense in $C(\mathbb{T})$. Let $p \in M$ be a central projection such that pM is of type I. Since every central sequence in a type I von Neumann algebra is trivial and $\{pu_n\}$ and $\{pv_n\}$ are central sequences in pM, pw_n converges to p in the ultrastrong-$*$ topology. Then for every $f\in C(\mathbb{T})$, $\varphi(pf(w_n)) \to \varphi(p)f(1)$. Take $\varepsilon > 0$ arbitrarily and $f \in C(\mathbb{T})$ such that $f \geq 0$, $f(1) = 1$ and $\int_{\mathbb{T}} f(z)\,dz < \varepsilon$. Then $\varphi(f(w_n)) \geq \varphi(pf(w_n))$, so $\varphi(p) \leq \int_{\mathbb{T}} f(z)\,dz < \varepsilon$. Since ε is arbitrary, $\varphi(p) = 0$, i.e., $p = 0$. Therefore M has no nonzero type I summand. □

2.2. **Hecke algebras.** We refer the reader to [9] and [10] for definitions and basic properties of Hecke algebras.

Suppose (G,H) is a Hecke pair and $H\backslash G$ is a discrete space. Then the Hecke algebra $\mathcal{H}(G,H)$ acts on $\ell^2(H\backslash G)$ from left; define $\lambda\colon \mathcal{H}(G,H) \to B(\ell^2(H\backslash G))$ by

$$[\lambda(f)\xi]\,(Hx) = \sum_{Hy\in H\backslash G} f(Hxy^{-1})\xi(Hy)$$

for $f \in \mathcal{H}(G,H)$ and $\xi \in \ell^2(H\backslash G)$. We may omit λ and write $\mathcal{H}(G,H) \subset B(\ell^2(H\backslash G))$.

Let $\rho\colon G \to B(\ell^2(H\backslash G))$ be the right quasi-regular representation defined by $[\rho_s\xi](x) = \xi(xs)$. One can easily check that $\mathcal{H}(G,H) \subset \rho(G)'$. Moreover, one has $\mathcal{H}(G,H)'' = \rho(G)'$ (see [1, Theorem 1.4]). The unit vector $\delta_H \in \ell^2(H\backslash G)$ is a separating vector for $\mathcal{H}(G,H)$, since δ_H is a $\rho(G)$-cyclic vector. Moreover, if $R(x) = R(x^{-1})$ for every $x\in G$, then it is not hard to see that δ_H is a tracial vector, i.e., the vector state associated with δ_H is a trace on $\lambda(\mathcal{H}(G,H))$. In particular, the vector state $x \mapsto \langle x\delta_H, \delta_H\rangle$ is a faithful tracial state of $\mathcal{H}(G,H)$ for a unimodular locally compact group G and its compact open subgroup H.

For a finite group G and its subgroup $H \leq G$, note that the Hecke algebra $\mathcal{H}(G,H)$ is identical to $p_H\mathbb{C}[G]p_H$ where $p_H = \frac{1}{|H|}\sum_{h\in H} h \in \mathbb{C}[G]$ is a projection (see [9, Corollary 4.4]).

Proposition 2.5 is a special case of [10, Proposition 1.3].

Proposition 2.5. *Let G be a finite group acting on a finite group V, and let Γ be a subgroup of G leaving a subgroup V_0 of V invariant. Then we have a canonical* *embedding $\mathcal{H}(V, V_0)^\Gamma \hookrightarrow \mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma)$. Moreover, the canonical traces are consistent with this embedding.*

Proof. We will prove that there exists a canonical, trace preserving embedding $(p_{V_0}\mathbb{C}[V]p_{V_0})^\Gamma \hookrightarrow p_{V_0\rtimes\Gamma}\mathbb{C}[V \rtimes G]p_{V_0\rtimes\Gamma}$ where $p_H = \frac{1}{|H|}\sum_{h\in H} h$ for a subgroup H. Since Γ leaves V_0 invariant, p_{V_0} commutes with every element of Γ in $\mathbb{C}[V_0 \rtimes \Gamma]$. In particular, p_{V_0} commutes with p_Γ and $p_{V_0\rtimes\Gamma} = p_{V_0}p_\Gamma = p_\Gamma p_{V_0}$. Note that p_Γ commutes with every element in $\mathbb{C}[V]^\Gamma$. Therefore, multiplication with p_Γ is a $*$-homomorphism from $(p_{V_0}\mathbb{C}[V]p_{V_0})^\Gamma \cong p_{V_0}\mathbb{C}[V]^\Gamma p_{V_0}$ to $p_{V_0\rtimes\Gamma}\mathbb{C}[V]^\Gamma p_{V_0\rtimes\Gamma} \subset p_{V_0\rtimes\Gamma}\mathbb{C}[V \rtimes G]p_{V_0\rtimes\Gamma}$. This map preserves the canonical trace, since it is spatially implemented by the canonical isometry $W \colon \ell^2(V_0\backslash V) \to \ell^2((V_0 \rtimes \Gamma)\backslash(V \rtimes G))$, and $W^*\delta_{V_0\rtimes\Gamma} = \delta_{V_0}$. Since the canonical traces are faithful, this $*$-homomorphism is an embedding. ☐

Corollary 2.6. *In addition to the assumptions of Proposition* 2.5*, suppose G leaves V_0 invariant. Then there is a canonical trace preserving embedding $\mathcal{H}(G, \Gamma) \hookrightarrow \mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma)$ and $\mathcal{H}(V, V_0)^G \subset \mathcal{H}(G, \Gamma)'$ in $\mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma)$.*

Proof. The same argument as above shows that the first assertion holds. To show the second assertion, we identify $\mathcal{H}(V, V_0)^G$ and $\mathcal{H}(G, \Gamma)$ with $p_{V_0}\mathbb{C}[V]^G p_{V_0}$ and $p_\Gamma\mathbb{C}[G]p_\Gamma$, respectively. The assertion follows from the fact that $p_{V_0}p_\Gamma = p_\Gamma p_{V_0}$ and $\mathbb{C}[V]^G \subset \mathbb{C}[G]'$. ☐

2.3. **Locally compact groups.** In this paper, topological groups are assumed to be Hausdorff. Let G be a locally compact second countable group and μ be its left Haar measure. The **left regular representation** of G is a unitary representation $\lambda \colon G \to \mathcal{U}(L^2(G))$ defined by $(\lambda_g f)(h) = f(g^{-1}h)$ for $f \in L^2(G)$ where $L^2(G)$ is a Haar square integrable function on G. The von Neumann algebra $\{\lambda_g \mid g \in G\}'' \subset B(L^2(G))$ is called the group von Neumann algebra. The representation λ extends to a representation of $L^1(G)$: $\lambda(f)g = f * g$ for $f \in L^1(G)$ and $g \in L^2(G)$.

A unitary representation (π, H) of G is called of being **type I** if the associated von Neumann algebra $\pi(G)'' \subset B(H)$ is of type I. A locally compact group G is called of being **type I** if all its unitary representations are of type I. See [2, Chapter 6, 7] for more details and properties of type I groups.

3. Neretin groups

Let $d, k \geq 2$ be integers and $\mathcal{T}_{d,k}$ be a rooted tree such that the root has k adjacent vertices and the others have $d + 1$ adjacent vertices. An **almost automorphism** of $\mathcal{T}_{d,k}$ is a triple (A, B, φ) where $A, B \subset \mathcal{T}_{d,k}$ are finite subtrees containing the root with $|\partial A| = |\partial B|$ and $\varphi \colon \mathcal{T}_{d,k} \setminus A \to \mathcal{T}_{d,k} \setminus B$ is an isomorphism. The **Neretin group** $\mathcal{N}_{d,k}$ is the quotient of the set of all almost automorphisms by the relation which identifies two almost automorphisms (A_1, B_1, φ_1), (A_2, B_2, φ_2) if there exists a finite subtree $\tilde{A} \subset \mathcal{T}_{d,k}$ containing the root such that $A_1, A_2 \subset \tilde{A}$ and $\varphi_1|_{\mathcal{T}_{d,k}\setminus\tilde{A}} = \varphi_2|_{\mathcal{T}_{d,k}\setminus\tilde{A}}$. One can easily check that $\mathcal{N}_{d,k}$ is a group.

Let d be the graph metric on $\mathcal{T}_{d,k}$, v_0 be the root of $\mathcal{T}_{d,k}$ and $B_n := \{v \in \mathcal{T}_{d,k} \mid d(v_0, v) \leq n\}$ for $n \geq 0$. Every automorphism of $\mathcal{T}_{d,k}$ leaves B_n invariant. For each $n \geq 0$, $\mathcal{O}_{d,k}^{(n)}$ denotes the subgroup consisting of automorphisms on $\mathcal{T}_{d,k} \setminus B_n$ and we write $\mathcal{O}_{d,k} := \bigcup_{n=0}^\infty \mathcal{O}_{d,k}^{(n)}$. Each $\mathcal{O}_{d,k}^{(n)}$ is a subgroup of $\mathcal{N}_{d,k}$ containing

$K := \mathrm{Aut}\,(\mathcal{T}_{d,k})$. Let $V_n := \partial B_n = \{v \in \mathcal{T}_{d,k} \mid d(v, v_0) = n\}$. Note that $\mathcal{O}_{d,k}^{(n)} \cong \mathrm{Aut}\,(\mathcal{T}_{d,d}) \wr \mathfrak{S}_{|V_n|} = \mathrm{Aut}\,(\mathcal{T}_{d,d})^{|V_n|} \rtimes \mathfrak{S}_{|V_n|}$ and $\mathcal{O}_{d,d}^{(l)} \wr \mathfrak{S}_{|V_n|} < \mathcal{O}_{d,k}^{(n+l)}$.

The Neretin group $\mathcal{N}_{d,k}$ admits a totally disconnected locally compact group topology such that the inclusion map $K \hookrightarrow \mathcal{N}_{d,k}$ is continuous and open [7, Theorem 4.4]. The Neretin group $\mathcal{N}_{d,k}$ is compactly generated and simple; see [7].

The group $\mathcal{O}_{d,k}$ is an open subgroup of $\mathcal{N}_{d,k}$. It is unimodular and amenable since $\mathcal{O}_{d,k}$ is an increasing union $\bigcup_{n=1}^{\infty} \mathcal{O}_{d,k}^{(n)}$ of its compact subgroups.

4. Proof of theorem

We normalize the Haar measure μ on $\mathcal{O}_{d,k}$ so that $\mu(K) = 1$. Let $p = \lambda(\chi_K)$ be the projection onto the subspace of left K-invariant functions. This subspace can be identified with $\ell^2(K\backslash\mathcal{O}_{d,k})$. The Hecke algebra $\mathcal{H}(\mathcal{O}_{d,k}, K) \subset B(\ell^2(K\backslash\mathcal{O}_{d,k}))$ is a dense subalgebra of the corner $pL(\mathcal{O}_{d,k})p \subset B(\ell^2(K\backslash\mathcal{O}_{d,k}))$ with respect to the weak operator topology. We will show that $pL(\mathcal{O}_{d,k})p$ is of type II.

Since K acts on V_n, there exists a canonical group homomorphism $K \to \mathrm{Aut}\,(V_n) \cong \mathfrak{S}_{|V_n|}$. The range of this homomorphism is denoted by $P_n = \mathrm{Aut}\,(B_n) < \mathfrak{S}_{|V_n|}$. Similarly, let Q_n be the range of the canonical group homomorphism $\mathrm{Aut}\,(\mathcal{T}_{d,d}) \to \mathrm{Aut}\,(W_n)$, where W_n is the subset $\{v \in \mathcal{T}_{d,d} \mid d(v, v_0) = n\}$ of $\mathcal{T}_{d,d}$. One has $\mathcal{H}(\mathcal{O}_{d,k}, K) \cong \cup_{n=1}^{\infty} \mathcal{H}(\mathcal{O}_{d,k}^{(n)}, K)$ and $\mathcal{H}(\mathcal{O}_{d,k}^{(n)}, K) \cong \mathcal{H}(\mathfrak{S}_{|V_n|}, P_n)$. We use this identification freely. For finite groups G_1, G_2 and their subgroups $H_i < G_i$, $\mathcal{H}(G_1, H_1) \otimes \mathcal{H}(G_2, H_2) \cong \mathcal{H}(G_1 \times G_2, H_1 \times H_2)$. Proposition 2.5 for $G = \mathfrak{S}_{|V_n|}, \Gamma = P_n, V = \mathfrak{S}_{d^l}^{|V_n|}, V_0 = Q_l^{|V_n|}$ implies

$$\begin{aligned}((\mathcal{H}(\mathcal{O}_{d,d}^{(l)}, \mathrm{Aut}\,(\mathcal{T}_{d,d})))^{\otimes|V_n|})^{P_n} &\cong (\mathcal{H}(\mathfrak{S}_{d^l}^{|V_n|}, Q_l^{|V_n|}))^{P_n} \\ &\hookrightarrow \mathcal{H}(\mathfrak{S}_{d^l}^{|V_n|} \rtimes \mathfrak{S}_{|V_n|}, Q_l^{|V_n|} \rtimes P_n) \\ &= \mathcal{H}(\mathfrak{S}_{d^l}^{|V_n|} \rtimes \mathfrak{S}_{|V_n|}, P_{n+l}) \\ &\subset \mathcal{H}(\mathfrak{S}_{|V_{n+l}|}, P_{n+l})\end{aligned}$$

for $l \in \mathbb{N}$. Moreover, Corollary 2.6 implies $(\mathcal{H}(\mathfrak{S}_{d^l}^{|V_n|}, Q_l^{|V_n|}))^{\mathfrak{S}_{|V_n|}} \subset \mathcal{H}(\mathfrak{S}_{|V_n|}, P_n)'$. Since $\mathcal{H}(\mathcal{O}_{d,d}^{(l)}, \mathrm{Aut}\,(\mathcal{T}_{d,d})) \cong \mathcal{H}(\mathfrak{S}_{d^l}, Q_l)$ and $(\mathfrak{S}_{d^l}, Q_l)$ is not a Gelfand pair for $l \geq 3$ (see [8, Theorem 1.2]), $\mathcal{H}(\mathcal{O}_{d,d}^{(3)}, \mathrm{Aut}\,(\mathcal{T}_{d,d}))$ is noncommutative.

Let τ be the vector state associated with $\delta_K \in \ell^2(K\backslash\mathcal{O}_{d,k})$. This is a trace, since $\mathcal{O}_{d,k}$ is a unimodular locally compact group and K is its compact open subgroup. Note that $\tau(x^{\otimes|V_n|}) = (\tau(x))^{|V_n|}$ for $x \in \mathcal{H}(\mathcal{O}_{d,d}^{(3)}, \mathrm{Aut}\,(\mathcal{T}_{d,d}))$ where τ also denotes the canonical trace on $\mathcal{H}(\mathcal{O}_{d,d}^{(3)}, \mathrm{Aut}\,(\mathcal{T}_{d,d}))$. Since $\mathcal{H}(\mathcal{O}_{d,d}^{(3)}, \mathrm{Aut}\,(\mathcal{T}_{d,d}))$ is a noncommutative finite dimensional algebra, there exist two unitaries $u, v \in \mathcal{H}(\mathcal{O}_{d,d}^{(3)}, \mathrm{Aut}\,(\mathcal{T}_{d,d}))$ such that $|\tau((u^*v^*uv)^k)| < 1$ and $|\tau((v^*u^*vu)^k)| < 1$ for all $k \in \mathbb{Z} \setminus \{0\}$. Set $u_n := u^{\otimes|V_n|} \in \mathcal{H}(\mathcal{O}_{d,k}^{(n)}, K)'$ and $v_n := v^{\otimes|V_n|} \in \mathcal{H}(\mathcal{O}_{d,k}^{(n)}, K)'$. Then for every $x \in \mathcal{H}(\mathcal{O}_{d,k}, K)'' = pL(\mathcal{O}_{d,k})p$, $[x, u_n] \to 0$ and $[x, v_n] \to 0$ in the ultrastrong-$*$ topology. Thus $\{u_n\}$ and $\{v_n\}$ are central sequences. In addition, $\tau((u_n v_n u_n^* v_n^*)^k) = \tau((uvu^*v^*)^k)^n \to 0$ as $n \to \infty$ for every $k \in \mathbb{Z} \setminus \{0\}$. So by Lemma 2.4, $pL(\mathcal{O}_{d,k})p$ has no nonzero type I summand and it is of type II.

Let $K_n := \{\varphi \in K \mid \varphi|_{B_n} = \mathrm{id}_{B_n}\}$ and $p_n := \frac{1}{\mu(K_n)}\lambda(\chi_{K_n}) \in L(\mathcal{O}_{d,k})$. Then $\{p_n\}$ converges $1_{L(\mathcal{O}_{d,k})}$ in the strong operator topology. Applying the same argument as above to $p_n L(\mathcal{O}_{d,k})p_n$, one finds that $p_n L(\mathcal{O}_{d,k})p_n$ is of type II. Therefore $L(\mathcal{O}_{d,k})$ is of type II.

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