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MEASURE INDUCED HANKEL AND TOEPLITZ TYPE OPERATORS ON WEIGHTED DIRICHLET SPACES

NINA ZORBOSKA

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ABSTRACT. For a complex Borel measure μ on the open unit disk, and for a weighted Dirichlet space $\mathcal{H}_s$ with $0 < s < 1$, we characterize the boundedness of the measure induced Hankel type operator $H_{\mu,s} : \mathcal{H}_s \to \overline{\mathcal{H}_s}$, extending the results of Xiao [Bull. Austral. Math. Soc. 62 (2000), pp. 135–140] for the classical Hardy space $H^2 = \mathcal{H}_1$, and of Arcozzi, Rochberg, Sawyer, and Wick [J. Lond. Math. Soc. (2) 83 (2011), pp. 1–18] for the classical Dirichlet space $\mathcal{D} = \mathcal{H}_0$. Our approach relies on some recent results about weak products of complete Nevanlinna-Pick reproducing kernel Hilbert spaces. We also include some related results on Hankel measures, Carleson measures, and Toeplitz type operators on weighted Dirichlet spaces $\mathcal{H}_s$, $0 < s < 1$.

1. Introduction

Toeplitz and Hankel operators are two classes of widely researched operators, acting on a variety of spaces of holomorphic functions. The exploration of their operator theoretic properties and the connections to the behaviour of their inducing symbols have provided a deeper understanding of operator theory, complex analysis, measure theory, and other related areas of mathematics.

The goal of this paper is to determine the boundedness of measure induced Hankel type operators acting on a specific range of weighted Dirichlet spaces, and to provide some further related results on Hankel measures, Carleson measures, and measure induced Toeplitz type operators. Along the way, we use some more recent results on complete Nevanlinna-Pick reproducing kernel Hilbert spaces and their weak products, thus contributing to the already existing connections between these topics.

We start with some basic notation, a few definitions, and some known facts which we need in the presentation of our results.

In what follows, the open unit disk in the complex plane is denoted by $\mathbb{D}$ and $Hol(\mathbb{D})$ stands for the space of holomorphic, complex valued functions on $\mathbb{D}$.

For $s \in \mathbb{R}$, and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $Hol(\mathbb{D})$, we say that f belongs to the *weighted Hardy space* $\mathcal{H}_s$ if

$$\Sigma_{n=0}^{\infty}(n+1)^{1-s}|a_n|^2 < \infty.$$

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The class of spaces $\mathcal{H}_s$ contains the classical Hardy space H^2 when $s = 0$, $\mathcal{H}_2$ is the Bergman space A^2, and $\mathcal{H}_0$ is the Dirichlet space $\mathcal{D}$. Each space $\mathcal{H}_s$ is a separable (function) Hilbert space, thus also a reproducing kernel Hilbert space (RKHS), with the set $Hol(\overline{\mathbb{D}})$ being dense in $\mathcal{H}_s$. For $s < 0$, $\mathcal{H}_s$ is also a Banach algebra contained in the disk algebra $\mathcal{A}(\mathbb{D})$.

We denote the normalized Lebesgue area measure on the unit disk by dA, and for $s > -1$, we write $dA_s(z) = (1 - |z|^2)^s dA(z)$.

When $s > 1$, the standard norm of the space $\mathcal{H}_s$ is given by:

$$\|f\|_s^2 = \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{s-2} dA(z) = \|f\|^2_{L^2(dA_{s-2})}.$$

We will also refer to this range of spaces as *weighted Bergman spaces.*

Recall that for $s = 1$, the norm of f in the Hardy space $\mathcal{H}_1 = H^2$ is

$$\|f\|_1^2 = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta),$$

where dm is the normalized Lebesgue length measure on the unit circle $\mathbb{T}$.

When $-1 < s < 1$, we have that $\mathcal{H}_s \subset \mathcal{H}_1 = H^2$, and we use the norm:

$$\|f\|_s^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^s dA(z) = |f(0)|^2 + \|f'\|^2_{L^2(dA_s)}.$$

When $s = 1$ we can also use an equivalent norm for the Hardy space H^2, namely

$$\|f\|_1^2 \approx |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2) dA(z).$$

Furthermore, since for any $s > -1$, and any f such that $f(0) = 0$, we have that $\|f\|_{L^2(dA_s)} \approx \|f'\|_{L^2(dA_{s+2})}$, $f \in H_s$ if and only if $f' \in \mathcal{H}_{s+2}$. Thus, for any $s > -1$, and any f with $f(0) = 0$, we get an equivalent $\mathcal{H}_s$ norm, namely

$$\|f\|_s^2 \approx \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^s dA(z) = \|f'\|^2_{L^2(dA_s)}.$$

In this paper we are mostly concerned with the range $0 < s < 1$, and we refer to these $\mathcal{H}_s$ spaces as *weighted Dirichlet spaces.* Of course, as mentioned before, when $s = 0$ we have that $\mathcal{H}_0$ is the classical Dirichlet space $\mathcal{D}$.

A special feature of the weighted Hardy spaces $\mathcal{H}_s$ with $0 < s \leq 1$ is that these spaces are RKHS with a complete Nevanlinna-Pick kernel (see [1]), and this will be an essential property used in the proof of our results below. In such a case we will also say that the space is a complete Nevanlinna-Pick (CNP) space. Note that when $s = 0$, $\mathcal{H}_0 = \mathcal{D}$ is also a CNP space, but only when we use a suitable, equivalent norm. The kernel formula for $s = 0$ below is with respect to that norm. More specifically, the positive definite kernels $K_s : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ for the range of $\mathcal{H}_s$ spaces with $s \geq 0$ are given by:

$$K_s(w, z) = \begin{cases} \frac{1}{(1-\bar{z}w)^s}, & s > 0 \\ \\ \frac{1}{\bar{z}w} \log \frac{1}{1-\bar{z}w}, & s = 0. \end{cases}$$

A complex Borel measure μ on $\mathbb{D}$ is a *Carleson measure for* a Hilbert space $\mathcal{H}$ of functions on $\mathbb{D}$ if there exists $C > 0$ such that for all f in $\mathcal{H}$

$$\int_{\mathbb{D}} |f(z)|^2 d|\mu|(z) \leq C \|f\|^2_{\mathcal{H}},$$

where $|\mu|$ is the total variation of μ. The smallest such constant C will be denoted by $\|\mu\|_{CM(\mathcal{H})}$.

We will say that a complex Borel measure μ on $\mathbb{D}$ is a *Hankel measure for* $\mathcal{H}$ if there exists $C > 0$ such that for all f in $\mathcal{H}$

$$\left| \int_{\mathbb{D}} f^2(z) d\mu(z) \right| \leq C \|f\|^2_{\mathcal{H}},$$

and we denote the smallest such constant C with $\|\mu\|_{HM(\mathcal{H})}$.

If a complex Borel measure μ on $\mathbb{D}$ is Carleson measure for $\mathcal{H}$, then it is also a Hankel measure for $\mathcal{H}$. That is easy to see, since for $f \in \mathcal{H}$,

$$\left| \int_{\mathbb{D}} f^2(z) d\mu(z) \right| \leq \int_{\mathbb{D}} |f(z)|^2 d|\mu|(z) \leq \|\mu\|_{CM(\mathcal{H})} \|f\|^2_{\mathcal{H}}.$$

For a complex Borel measure μ on $\mathbb{D}$, $s > 0$, and $f \in Hol(\overline{\mathbb{D}})$, a measure induced Toeplitz type operator $T_{\mu,s}$ is defined by

$$T_{\mu,s} f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \overline{w} z)^s} d\mu(w).$$

If μ is such that $T_{\mu,s} f \in \mathcal{H}_s$ we have that for any $g \in \mathcal{H}_s$,

$$\langle T_{\mu,s} f, g \rangle_s = \int_{\mathbb{D}} f(z) \overline{g(z)} d\mu(z).$$

The measure induced Toeplitz type operators on the $\mathcal{H}_s$ spaces were introduced by Luecking in [11]. The paper contains more details on the claims we make here, and much more. For example, when μ is a finite positive Borel measure on $\mathbb{D}$, the Toeplitz type operator $T_{\mu,s}$ is a positive operator, and so $T_{\mu,s}$ is bounded on $\mathcal{H}_s$ if and only if μ is Carleson measure for $\mathcal{H}_s$. When the measure μ is complex, the Carleson measure condition is in general only a sufficient condition for the boundedness of the corresponding Toeplitz type operator.

For $s > 1$, i.e. when $\mathcal{H}_s$ is a weighted Bergman space, the measure induced Toeplitz type operators are generalizations of the classical Toeplitz operators. Namely, if the complex measure μ is absolutely continuos with respect to the weighted area Lebesgue measure, i.e. if $d\mu(z) = \phi(z) dA_s(z)$, for some $\phi \in L^1(dA_s(z))$, the Toeplitz type operator $T_{\mu,s}$ is just the classical Topelitz operator $T_{\phi,s}$ on $\mathcal{H}_s$. This is not anymore the case when $s \leq 1$.

Similarly, for a complex Borel measure μ on $\mathbb{D}$, $s > 0$ and $f \in Hol(\overline{\mathbb{D}})$, we define the measure induced Hankel type operator $H_{\mu,s}$ by

$$H_{\mu,s} f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \overline{z} w)^s} d\mu(w).$$

Note that $H_{\mu,s} f$ is a conjugate analytic function, i.e. the range of the Hankel type operator acting on $\mathcal{H}_s$ will eventually be in $\overline{\mathcal{H}_s}$, where $\overline{\mathcal{H}_s} = \{\overline{f} : f \in \mathcal{H}_s\}$ is a Hilbert space with the inner product $\langle \overline{f}, \overline{g} \rangle_{\overline{\mathcal{H}_s}} = \langle g, f \rangle_s$. It is not too hard to see that if μ and f are such that $H_{\mu,s} f \in \overline{\mathcal{H}_s}$, and $g \in \mathcal{H}_s$, then

$$\langle H_{\mu,s} f, \overline{g} \rangle_{\overline{\mathcal{H}_s}} = \int_{\mathbb{D}} f(z) g(z) d\mu(z).$$

The measure induced Hankel type operators on the $\mathcal{H}_s$ spaces were also introduced by Luecking in [11], but with a slightly different definition, so that the operator's range is instead contained in $Hol(\mathbb{D})$. They are generalizations of the classical (small) Hankel operators, and are also related to the so called Hankel forms. For more details on this see [11], [14], [12], [2].

The connection between Hankel type operators and Hankel measures was established for the Hardy space $H^2 = \mathcal{H}_1$, and for the weighted Bergman spaces $\mathcal{H}_s, s > 1$, by Xiao in [16] and [17], correspondingly. Hankel measures for $\mathcal{H}_1$ were also explored further in the more recent work [6]. The Hankel measures for the Dirichlet space $\mathcal{D} = \mathcal{H}_0$ were classified in [4], by using explicitlly the connections to the weak product $\mathcal{D} \odot \mathcal{D}$, and to their further results on its dual.

2. Main results

Before presenting our main results, we need to define two more notions: weak products of RKHS, and the special Banach spaces that represent their duals. We use [12] and [2] as sources for these topics. The interested reader can find more details and a further list of references there.

For $\mathcal{H}$ a RKHS of holomorphic functions on $\mathbb{D}$, define the weak product of $\mathcal{H}$ as

$$\mathcal{H} \odot \mathcal{H} = \left\{ h = \sum_{i=1}^{\infty} f_i g_i : f_i, g_i \in \mathcal{H}, \sum_{i=1}^{\infty} \|f_i\| \|g_i\| < \infty \right\},$$

and

$$\|h\|_{\mathcal{H}\odot\mathcal{H}} = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\| \|g_i\| : h = \sum_{i=1}^{\infty} f_i g_i \right\}.$$

The space $\mathcal{H} \odot \mathcal{H}$ is a Banach space of holomorphic functions on $\mathbb{D}$ such that the point evaluations are continuous. For example, it is well known that $H^2 \odot H^2 = H^1$, and that $A^2 \odot A^2 = A^1$.

Weak products, and in particular their duals, are closely related to bilinear forms and Hankel operators. Namely, when $Hol(\overline{\mathbb{D}})$ is dense in $\mathcal{H}$, one can define a set $\mathcal{X}(\mathcal{H}) \subset \mathcal{H}$, of symbols of Hankel operators, characterized via densely defined bilinear forms, which can be extended boundedly to $\mathcal{H} \oplus \mathcal{H}$. In many cases, including the ones that we are interested in, one can identify $\mathcal{X}(\mathcal{H})$ with the dual of $\mathcal{H} \odot \mathcal{H}$.

Motivated by these examples, it is natural to look more closely at the weak products of $\mathcal{H}_s$ spaces and their duals, and to explore their connections not only to classical Hankel operators, but also to measure induced Hankel type operators on the $\mathcal{H}_s$ spaces. For this particular class of Hilbert spaces, the space $\mathcal{X}(\mathcal{H}_s)$ can be described via special type of Carleson measures.

For $s \geq 0$ and f in $\mathcal{H}_s$, define the positive finite Borel measure $\mu_{f,s}$ by

$$d\mu_{f,s}(z) = |f'(z)|^2 dA_s(z),$$

and let $\mathcal{X}_s$ be a subset of $\mathcal{H}_s$ of functions that induce Carleson measures for $\mathcal{H}_s$, namely

$$\mathcal{X}_s = \{ f \in \mathcal{H}_s : \|\mu_{f,s}\|_{CM(\mathcal{H}_s)} < \infty \}.$$

It was shown in [14] and [12] that the Hankel operator $H_{b,s}$ with a holomorphic symbol b, densely defined on $\mathcal{H}_s$ for $f \in Hol(\overline{\mathbb{D}})$ and $g \in \mathcal{H}_s$ by

$$\langle H_{b,s} f, \overline{g} \rangle_{\overline{\mathcal{H}_s}} = \langle g, \overline{f} b \rangle_s,$$

can be boundedly extended to $\mathcal{H}_s$ if and only if $b \in \mathcal{X}_s$. Namely, $\mathcal{X}_s$ is exactly the space $\mathcal{X}(\mathcal{H}_s)$ described above, and furthermore, $(\mathcal{H}_s \odot \mathcal{H}_s)^* = \mathcal{X}_s$.

For example, it is well known that the dual of H^1 is BMOA, the space of holomorphic functions of bounded mean oscillation, and the dual of A^1 is the Bloch space $\mathcal{B}$, i.e. $\mathcal{X}_1 = BMOA$, and $\mathcal{X}_2 = \mathbb{B}$. Correspondingly, a Hankel operator with holomorphic symbol b is bounded on H^2 if and only if $b \in BMOA$, and it is bounded on A^2 if and only if $b \in \mathcal{B}$. Actually, it turns out that $\mathcal{X}_s = \mathbb{B}$, for all $s > 1$. For a direct proof of the last claim, and for other equivalent definitions of the spaces BMOA and $\mathbb{B}$, see also [18].

In the result that follows we determine the boundedness of measure induced Hankel type operators on the weighted Dirichlet spaces, and give equivalent characterizations of Hankel measures, extending the results from [16] for the Hardy space $H^2 = \mathcal{H}_1$, and from [4] for the Dirichlet space $\mathcal{D} = \mathcal{H}_0$. The core of the proof relies on the properties of weak products of normalized CNP spaces and their duals, developed in [4], [12], [10], [9], [2].

Theorem 1. *Let μ be a complex Borel measure on $\mathbb{D}$, and let $0 < s < 1$. Then the following are equivalent:*

(i) *The Hankel type operator $H_{\mu,s}$ is bounded on $\mathcal{H}_s$.*
(ii) *$\exists C > 0$ such that $\left|\int_{\mathbb{D}} f(z)g(z)d\mu(z)\right| \leq C\|f\|_s\|g\|_s$, $\forall f, g \in \mathcal{H}_s$.*
(iii) *μ is a Hankel measure for $\mathcal{H}_s$.*
(iv) *$\exists C > 0$ such that $\left|\int_{\mathbb{D}} h(z)d\mu(z)\right| \leq C\|h\|_{\mathcal{H}_s \odot \mathcal{H}_s}$, $\forall h \in \mathcal{H}_s \odot \mathcal{H}_s$.*
(v) *The function $b_{\mu,s} = \overline{H_{\mu,s}\mathbb{1}}$, where $\mathbb{1}$ denotes the constant function 1, is in $\mathcal{X}_s$.*

Proof. The equivalence of (i) and (ii) is obvious, and follows by the definition of boundedness of an operator on a general Hilbert space.

Clearly, (ii) implies (iii) by taking $g = f$. To prove that (iii) implies (ii), we use an idea from [4] to represent fg for $f, g \in \mathcal{H}_s$ as a difference of squares,

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right].$$

If one of f or g is zero, the inequality in (ii) turns into $0 = 0$. If both f and g are nonzero, we start with the case when $\|f\|_s = \|g\|_s = 1$. Then

$$\begin{aligned}\left|\int_{\mathbb{D}} f(z)g(z)d\mu(z)\right| &= \frac{1}{4}\left|\int_{\mathbb{D}}\left[(f+g)^2 - (f-g)^2\right]d\mu(z)\right| \\ &\leq \frac{1}{4}\left[\left|\int_{\mathbb{D}}(f+g)^2 d\mu(z)\right| + \left|\int_{\mathbb{D}}(f-g)^2\ d\mu(z)\right|\right] \\ &\leq \frac{1}{4}\|\mu\|_{HM(\mathcal{H}_s)}\left(\|f+g\|_s^2 + \|f-g\|_s^2\right) \\ &= \frac{1}{2}\|\mu\|_{HM(\mathcal{H}_s)}\left(\|f\|_s^2 + \|g\|_s^2\right) = \|\mu\|_{HM(\mathcal{H}_s)},\end{aligned}$$

where the second inequality follows by using that μ is a Hankel measure, and the first equal sign in the last line follows by the parallelogram law.

For general nonzero $f, g \in \mathcal{H}_s$, replacing f by $\frac{f}{\|f\|_s}$, and g by $\frac{g}{\|g\|_s}$ in the equations above, we get that

$$\left|\int_{\mathbb{D}} f(z)g(z)d\mu(z)\right| \leq \|\mu\|_{HM(\mathcal{H}_s)}\|f\|_s\|g\|_s,$$

which shows that (iii) implies (ii).

That (iv) implies (ii) follows easily, since for f, g in $\mathcal{H}_s$, $fg \in \mathcal{H}_s \odot \mathcal{H}_s$, and since by the definition of the $\mathcal{H}_s \odot \mathcal{H}_s$ norm,

$$\|fg\|_{\mathcal{H}_s \odot \mathcal{H}_s} \leq \|f\|_{\mathcal{H}_s}\|g\|_{\mathcal{H}_s}.$$

To show that (ii) implies (iv) we use recent results on weak products of particular CNP spaces of holomorphic function, sometimes referred to as "first order weighted Besov spaces". Namely, since for $0 < s < 1$ each RKHS $\mathcal{H}_s$ is a normalized CNP space [1]. Hence, by result in [9], each $\mathcal{H}_s$ has the column-raw property with constant 1. Thus, by [10, Theorem 1.3], for each $h \in \mathcal{H}_s \odot \mathcal{H}_s$, we can find f and g in $\mathcal{H}_s$, and $C > 0$ independent of f and g, such that $h = fg$ and $\|f\|_{\mathcal{H}_s}\|g\|_{\mathcal{H}_s} \leq C\|h\|_{\mathcal{H}_s\odot\mathcal{H}_s}$, and so (ii) implies (iv).

Before we show the equivalence of (iv) and (v), recall that a complex Borel measure μ on $\mathbb{D}$ is by definition a finite measure, i.e. $|\mu(\mathbb{D})| < \infty$. Applying the Hankel operator $H_{\mu,s}$ to the constant function $\mathbb{1}$ gives

$$H_{\mu,s}\mathbb{1}(w) = \langle H_{\mu,s}\mathbb{1}, \overline{K_{w,s}}\rangle_{\overline{\mathcal{H}_s}} = \int_{\mathbb{D}} K_s(z,w)d\mu(z).$$

Since for $0 < s < 1$, $K_s(z,0) = \mathbb{1}$, we get that $H_{\mu,s}\mathbb{1}(0) = \int_{\mathbb{D}} d\mu(z) = \mu(\mathbb{D})$. We will use this fact in one of the equations below.

The main idea in the proof of the equivalence of (iv) and (v) relies on the characterization of the dual of the weak product space $\mathcal{H}_s \odot \mathcal{H}_s$. By the results in [12] on the so called weighted Besov spaces with admmisable radial weights, or by using the more recent general results in [2] on first order weighted Besov spaces, we know that

$$(\mathcal{H}_s \odot \mathcal{H}_s)^* = \mathcal{X}_s.$$

More specifically, this was achieved by showing that for $f \in \mathcal{X}_s$ and $h \in \mathcal{H}_s$, the liner map $L_f(h) = \langle h, f\rangle_s$ extends boundedly to $\mathcal{H}_s \odot \mathcal{H}_s$.

On the other hand, if $0 < s < 1$, μ is a complex Borel measure on $\mathbb{D}$, and $h \in \mathcal{H}_s$, we have that

$$\begin{aligned}
\int_{\mathbb{D}} h(z)d\mu(z) &= \int_{\mathbb{D}} \langle h, K_{z,s}\rangle_s d\mu(z) \\
&= \int_{\mathbb{D}} \left(h(0)\overline{K_{z,s}(0)} + \int_{\mathbb{D}} h'(w)\overline{\frac{\partial}{\partial w}K_{z,s}(w)}dA_s(w)\right) d\mu(z) \\
&= h(0)\int_{\mathbb{D}} d\mu(z) + \int_{\mathbb{D}} h'(w)\left(\overline{\int_{\mathbb{D}} \frac{\partial}{\partial w}K_{z,s}(w)d\overline{\mu}(z)}\right) dA_s(w) \\
&= h(0)\mu(\mathbb{D}) + \int_{\mathbb{D}} h'(w)\left(\overline{\frac{\partial}{\partial w}\int_{\mathbb{D}} \overline{K_{w,s}(z)}d\overline{\mu}(z)}\right) dA_s(w) \\
&= h(0)H_{\mu,s}\mathbb{1}(0) + \int_{\mathbb{D}} h'(w)\overline{\frac{\partial}{\partial w}\left(\overline{H_{\mu,s}\mathbb{1}(w)}\right)}dA_s(w) \\
&= h(0)\overline{b_{\mu,s}(0)} + \int_{\mathbb{D}} h'(w)\overline{b'_{\mu,s}(w)}dA_s(w) \\
&= \langle h, b_{\mu,s}\rangle_s.
\end{aligned}$$

Thus, (iv) is true if and only if the linear map $L_{\mu,s}$ defined on $\mathcal{H}_s$ by

$$L_{\mu,s}(h) = \int_{\mathbb{D}} h(z)d\mu(z) = \langle h, b_{\mu,s}\rangle_s$$

can be extended boundedly to $\mathcal{H}_s \odot \mathcal{H}_s$. But by [12], this is equivalent to $b_{\mu,s} \in \mathcal{X}_s$, and so we have that (iv) is equivalent to (v). $\square$

Another important subset of a function Hilbert spaces $\mathcal{H}$ is its set of multipliers, i.e. the multiplier algebra $Mult(\mathcal{H}) = \{\phi \in \mathcal{H} : \phi f \in \mathcal{H}, \forall f \in \mathcal{H}\}$. For the $\mathcal{H}_s$ spaces with $s \geq 0$, Stegenga's results in [15] determine that

$$Mult(\mathcal{H}_s) = H^\infty \cap \mathcal{X}_s.$$

Thus, any $b \in Mult(\mathcal{H}_s)$ generates a bounded Hankel operator $H_{b,s}$ on $\mathcal{H}_s$, and a bounded Toeplitz operator $T_{b,s}$, which in this case is often also referred to as a multiplication operator. It is well known that for $s \geq 1$, $Mult(\mathcal{H}_s) = H^\infty$. This can also be seen by noting that $H^\infty \subset \mathcal{X}_1 = BMOA \subset \mathcal{X}_2 = \mathcal{B}$. We also have that

$$Hol(\overline{\mathbb{D}}) \subset Mult(\mathcal{H}_s) \subset \mathcal{X}_s \subset \mathcal{H}_s,$$

and so $Mult(\mathcal{H}_s)$ is dense in $\mathcal{H}_s$, which plays an important role in many results on RKHS, and in particular on CNP spaces.

The next theorem shows that for measure induced Hankel and Toeplitz operators, the boundedness of the special "determining" function $b_{\mu,s}$ implies the boundedness of the corresponding operators, and furthermore implies that $b_{\mu,s} \in Mult(\mathcal{H}_s)$. It is known that a similar situation occurs also for special measures supported on the unit circle $\mathbb{T}$, and we will discuss this further in Section 3.

Theorem 2. *Let $0 < s < 1$, and let μ be a finite positive Borel measure on $\mathbb{D}$. If the holomorphic function $b_{\mu,s} = \overline{H_{\mu,s}\mathbb{1}} = T_{\mu,s}\mathbb{1}$ is bounded on $\mathbb{D}$, then the Toeplitz type operator $T_{\mu,s}$ and the Hankel type operator $H_{\mu,s}$ are bounded on $\mathcal{H}_s$. Furthermore, $b_{\mu,s}$ is in the multiplier algebra for $\mathcal{H}_s$.*

Proof. The kernel function K_s is conjugate symmetric, and when $0 < s < 1$, ReK_s is comparable to $|K_s|$. Thus, by a result from [3, Lemma 24], a finite positive Borel measure μ is Carleson measure for $\mathcal{H}_s$ if and only if

$$\sup_{\|g\|_{L^2(d\mu)} \leq 1} \int_{\mathbb{D}} \int_{\mathbb{D}} |K_s(z,w) g(z) g(w)| d\mu(z) d\mu(w) < \infty.$$

Using that $|K_s(z,w)| = |K_s(w,z)|$ and the Cauchy–Schwatrz inequality, it is easy to see that the condition

$$\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} |K_s(z,w)| d\mu(w) < \infty$$

implies the finiteness of the supremum above. Moreover, since

$$b_{\mu,s}(z) = \int_{\mathbb{D}} K_s(z,w) d\mu(w),$$

and since K_s is conjugate symmetric and ReK_s is comparable to $|K_s|$, the boundedness of $b_{\mu,s}(z)$ on $\mathbb{D}$ is equivalent to the boundedness of $\int_{\mathbb{D}} |K_s(z,w)| d\mu(w)$ on $\mathbb{D}$ as a function in z. Thus, the assumption that $b_{\mu,s}$ is bounded implies that μ is a Carleson measure for $\mathcal{H}_s$, and so the Toeplitz type operator $T_{\mu,s}$ is bounded on $\mathcal{H}_s$.

As noted before, if the measure μ is Carleson measure for $\mathcal{H}_s$, it is also a Hankel mesure for $\mathcal{H}_s$. Thus, using Theorem 1, we have that the Hankel type operator $H_{\mu,s}$ is bounded from $\mathcal{H}_s$ into $\overline{\mathcal{H}_s}$.

Furthermore, Theorem 1, part (iv), says that then $b_{\mu,s} \in \mathcal{X}_s$. By the result from [15] for the multiplier algebra of $\mathcal{H}_s$, $0 < s < 1$, we know that $Mult(\mathcal{H}_s) = H^\infty \cap \mathcal{X}_s$. Hence, using the assumption that $b_{\mu,s} \in H^\infty$, we get that $b_{\mu,s} \in Mult(\mathcal{H}_s)$. $\square$

In the last part of this section we consider a special class of positive finite Borel measures with support in the interval $[0,1)$. They are sometimes also referred to as radial measures. Their behaviour is much more regular than for general positive measures, in particular when we are interested in Hankel and Carleson measures for the $\mathcal{H}_s$ spaces. For example, as we will see in Lemma 1, a measure μ on $[0,1)$ is a Carleson measure for $\mathcal{H}_s$ if and only if it is an s-Carleson measure. We define s-Carleson measures next.

Let $I \subset \mathbb{T}$ be an arc with a normalized arc length $|I|$, and let

$$S(I) = \{z \in \mathbb{D} : 1 - |I| < |z| < 1, z/|z| \in I\}$$

be the Carleson box determined by I. For $s > 0$ and for a finite positive Borel measure μ on $\mathbb{D}$, we say that μ is an *s-Carleson measure* if there exists a positive constant C, such that for all $I \subset \mathbb{T}$,

$$\mu(S(I)) \leq C|I|^s.$$

It is well known that when $s \geq 1$, μ is an s-Carleson measure if and only if μ is a Carleson measure for $\mathcal{H}_s$. When $0 < s < 1$, then μ is Carleson measure for $\mathcal{H}_s$ implies that μ is an s-Carleson measure, but the converse is not true in general. The geometric condition that characterizes Carleson measures for the $\mathcal{H}_s$ spaces when $0 < s < 1$ involves Bessel capacities, and is much more complicated than the condition characterizing the s-Carleson measures. For more details on this see [15].

When μ is a positive finite Borel measure on $[0,1)$, the results in [13, Lemma 3] provide a simpler characterization of such Carleson measures for $\mathcal{H}_s$, even when $0 < s < 1$. This characterization can be used to show the equivalence between such Carleson measures and s-Carleson measures. We state the details in Lemma 1.

Lemma 1 ([15], [13]). *Let $s > 0$ and let μ be a positive finite Borel measure on $[0,1)$. Then the following are equivalent:*

(i) *μ is an s-Carleson measure.*
(ii) *$\mu([1-t,1)) = O(t^s)$ for all $0 < t < 1$.*
(iii) *μ is a Carleson measure for $\mathcal{H}_s$.*

Proof. That (ii) is equivalent to (iii) is Lemma 3 in [13], proven by using the characterization of Carleson measures for $\mathcal{H}_s$ in [15].

The equivalence of (i) and (ii) follows from the observation that for any arc $I \subset \mathbb{T}$, either $1 \in I$ and then $S(I) \cap [0,1) = [1-t,1)$, where $|I| = t$ and $0 < t < 1$, or $1 \notin I$ and then $S(I) \cap [0,1) = \emptyset$. □

For the proof of the next result we will need to mention yet another type of spaces related to Carleson measures, namely the so-called Q_s spaces. These spaces are related to s-Carleson measures, the same way the $\mathcal{X}_s$ spaces are related to Carleson measures for $\mathcal{H}_s$ spaces. For $s > 0$, $f \in Hol(\mathbb{D})$, and the corresponding positive Borel measure $d\mu_{f,s}$ as defined previously,

$$Q_s = \{f \in Hol(\mathbb{D}) : \mu_{f,s} \text{ is an s-Carleson measure}\}.$$

As mentioned before for general finite positive Borel measures, every Carleson measures for $\mathcal{H}_s$ is an s-Carleson measure. Thus, $\mathcal{X}_s \subset Q_s$ for all $s > 0$. For $s \geq 1$, Carleson measures for $\mathcal{H}_s$ are also s-Carleson measure, and so $\mathcal{X}_s = Q_s$, namely $\mathcal{X}_1 = BMOA = Q_1$, and for $s > 1$, $\mathcal{X}_s = \mathcal{B} = Q_s$. For more details about the Q_s spaces, see [18].

We also recall that for a finite positive Borel measure μ with support in $[0,1)$, and each $n \in \mathbb{Z}_+$, the *n-th moment* of μ is defined by

$$\mu[n] = \int_0^1 t^n d\mu(t).$$

Naturally, the properties of the moment sequence are closely related to the properties of the measure, and our next result shows how that relation plays out in our context. The equivalence of (ii) and (iii) below is a combination of Lemma 1 and a result from [5, Theorem 2.1].

Theorem 3. *Let $0 < s < 1$ and let μ be a positive finite Borel measure on $[0,1)$. Then the following are equivalent:*

(i) *μ is a Hankel measure for $\mathcal{H}_s$.*
(ii) *μ is a Carleson measure for $\mathcal{H}_s$.*
(iii) *$\mu[n] = O(\frac{1}{n^s})$.*

Proof. It was already mentioned in Section 1 that (ii) implies (i) in general, i.e. for any complex Borel measure μ on $\mathbb{D}$.

Next we show that if μ is positive finite Borel measure supported on $[0,1)$, then (i) implies (iii). We know from Theorem 1 that μ is a Hankel measure for $\mathcal{H}_s$ if and only if the function $b_{\mu,s} = \overline{H_{\mu,s}\mathbb{1}}$ is in $\mathcal{X}_s$. Using the binomial expansion formula

$$\frac{1}{(1-x)^s} = \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{\Gamma(s)n!} x^n,$$

we have that

$$\begin{aligned} b_{\mu,s}(z) &= \int_0^1 \frac{1}{(1-tz)^s} d\mu(t) = \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{\Gamma(s)n!} \left(\int_0^1 t^n d\mu(t) \right) z^n \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{\Gamma(s)n!} \mu[n] z^n = \sum_{n=0}^{\infty} a_n z^n. \end{aligned}$$

The Taylor coefficients a_n of the holomorphic function $b_{\mu,s}$ are non-negative, and it is easy to see that the sequence (a_n) is a decreasing. In that case, since $b_{\mu,s} \in \mathcal{X}_s \subset Q_s$, we can use Corolarry 3.3.1 in [18] which says that $b_{\mu,s} \in Q_s$ if and only if $a_n = O(\frac{1}{n})$. By Stirling's formula for the gamma function, $a_n \approx \frac{1}{n^{1-s}}\mu[n]$, and so $\mu[n] = O(\frac{1}{n^s})$, which proves that (ii) implies (iii).

To prove (iii) implies (ii), we first use a result from [5, Theorem 2.1] saying that if μ is a positive finite Borel measure on $[0,1)$ and $s > 0$, then μ is an s-Carleson measure if and only if $\mu[n] = O(\frac{1}{n^s})$. But then by Lemma 1, μ is also a Carleson measure for $\mathcal{H}_s$.

Thus, we have shown that (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii), and we are done with the proof. □

As a corollary to Theorem 3, we also get an interesting $T1$ type boundedness characterization of the corresponding special class of Toeplitz operators.

Corollary 1. *Let $0 < s < 1$ and let μ be a positive finite Borel measure on $[0,1)$. Then the Toeplitz type operator $T_{\mu,s}$ is bounded on $\mathcal{H}_s$ if and only if $T_{\mu,s}\mathbb{1}$ is in $\mathcal{X}_s$.*

Proof. Since μ is a finite positive Borel measure, $T_{\mu,s}$ is bounded on $\mathcal{H}_s$ if and only if μ is a Carleson measure for $\mathcal{H}_s$. But since μ is also a measure on $[0,1)$, by

Theorem 2, μ is a Carleson measure for $\mathcal{H}_s$ if and only if μ is a Hankel measure for $\mathcal{H}_s$. By Theorem 1, this is equivalent to $b_{\mu,s} = \overline{H_{\mu,s}\mathbb{1}}$ is in $\mathcal{X}_s$. It is easy to see that when μ is a measure on $[0,1)$, we also have that $\overline{H_{\mu,s}\mathbb{1}} = T_{\mu,s}\mathbb{1}$, and so the claim follows. □

2.1. **Further remarks.**

Remark 1. Our first remark is about the boundedness of the determining function $b_{\mu,s}$. Recall that, for example, on the Hardy space $H^2 = \mathcal{H}_1$, the classical Toeplitz operator T_ϕ is bounded if and only if the symbol ϕ is bounded. As we can see from Theorem 2, for $0 < s < 1$, and a positive Borel measure μ on $\mathbb{D}$, if $b_{\mu,s} = \overline{H_{\mu,s}\mathbb{1}} = T_{\mu,s}\mathbb{1}$ is bounded on $\mathbb{D}$, then the Toeplitz type operator $T_{\mu,s}$ and the Hankel type operator $H_{\mu,s}$ are bounded on $\mathcal{H}_s$.

Even though Theorem 1 and Corollary 1 say that the operator boundedness is equivalent to $b_{\mu,s} \in \mathcal{X}_s$, one may wonder if the boundedness of $b_{\mu,s}$ is also a necessary condition in case of some special measures, such as the measures on $[0,1)$. The following example provides a negative answer to this question. The idea for the example comes from a similar example in [8], given while considering the one-box condition for Carleson measures on the classical Dirichlet space.

Example 1. Let $0 < s < 1$ and let μ be the finite positive Borel measure on $[0,1)$ defined by $\mu([1-t,1)) = t^s$, for $0 < t < 1$. By Lemma 1, μ is a Carleson measure for $\mathcal{H}_s$, and so the operators $T_{\mu,s}$ and $H_{\mu,s}$ are bounded on $\mathcal{H}_s$.

We will show that the determining function $b_{\mu,s} = \overline{H_{\mu,s}\mathbb{1}} = T_{\mu,s}\mathbb{1}$ is not bounded, by showing that for $w \in (0,1)$, $b_{\mu,s}(w) \to \infty$, as $w \to 1^-$. Using the Fubini's theorem and integration by parts, we have that

$$\begin{aligned} b_{\mu,s}(w) &= \int_0^1 \frac{1}{(1-wt)^s} d\mu(t) \\ &\geq \int_{1-w}^1 \mu(\{t \in [0,1) : 1 - wt \leq x\}) \frac{s}{x^{1+s}} dx \\ &= \frac{s}{w^s} \int_{1-w}^1 \frac{(x-1+w)^s}{x^{1+s}} dx, \end{aligned}$$

and it is easy to see that then $b_{\mu,s}(w)$ goes to infinity, as w approaches 1^-. ◊

Remark 2. When $s \leq 1$, each space $\mathcal{H}_s$ is contained in H^2, and so the functions in $\mathcal{H}_s$ have radial limits almost everywhere on $\mathbb{T}$. Hence, when looking at the measure induced operators $T_{\mu,s}$ and $H_{\mu,s}$ on $\mathcal{H}_s$, we can also consider measures with support on $\overline{\mathbb{D}}$, and in particular, measures with support in $\mathbb{T}$.

We recall some potential theory notions for a finite positive Borel measure μ with support in $\mathbb{T}$. Namely, for $0 < s < 1$ the function

$$b_{\mu,s}(z) = \int_{\mathbb{T}} \frac{1}{(1-\overline{\zeta}z)^s} d\mu(\zeta)$$

is called the holomorphic $\mathcal{H}_s$-potential of the measure μ, and the $\mathcal{H}_s$-energy of μ is defined as

$$\varepsilon(\mu, \mathcal{H}_s) = \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{1}{|1-\overline{\zeta}\xi|^s} d\mu(\zeta) d\mu(\xi).$$

Note that since $b_{\mu,s} = \overline{H_{\mu,s}\mathbb{1}} = T_{\mu,s}\mathbb{1}$, a necessary condition for the boundedness of the operators $T_{\mu,s}$ and $H_{\mu,s}$ is that $b_{\mu,s} \in \mathcal{H}_s$.

We want to point out some interesting related ideas and results from a recent paper [7]. For example, by Proposition 3.2 in [7], for a finite positive Borel measure μ on $\mathbb{T}$, $b_{\mu,s} \in \mathcal{H}_s$ if and only if the measure's $\mathcal{H}_s$ energy $\varepsilon(\mu, \mathcal{H}_s) < \infty$, in which case $\|b_{\mu,s}\|_s \approx \varepsilon(\mu, \mathcal{H}_s)$. Note that for $0 < s < 1$, the s-Bessel capacities of (compact) subsets of $\mathbb{T}$, used in the Stegenga's geometric description of Carleson measures for $\mathcal{H}_s$ in [15], are equivalent to the so called $\mathcal{H}_s$ capacities defined via (finite) $\mathcal{H}_s$-energies of positive Borel measures on $\mathbb{T}$ (see [7] for the definitions, and for more details). These connections point to interesting possible further exploration of Carleson measures with support in $\mathbb{T}$, and Topelitz type operators on the weighted Dirichlet spaces.

Lastly, we want to mention [7, Proposition 3.6], stating that for a finite positive Borel measure μ on $\mathbb{T}$, if the holomorphic potential $b_{\mu,s}$ is bounded, then $b_{\mu,s} \in Mult(\mathcal{H}_s)$. Recall that in Theorem 2 we have shown that this also holds true for $b_{\mu,s}$ with μ a finite positive Borel measure on $\mathbb{D}$. $\diamondsuit$

Remark 3. The last comment we want to make in the case of measures μ with support in $\mathbb{T}$ is about the Fourier coefficients of such measures, and their relations to the measure induced Toeplitz and Hankel type operators. Note that the Fourier coefficients play here a similar role as the sequence of moments, in the case when μ is supported on the interval $[0, 1)$.

For a complex Borel measure μ with support in $\mathbb{T}$, and for $n \in \mathbb{Z}$, the n-th Fourier coefficient $\hat{\mu}(n)$ of μ is defined by

$$\hat{\mu}(n) = \int_{\mathbb{T}} \overline{\zeta^n} d\mu(\zeta).$$

Hence, for $s > 0$ we get the corresponding series expansion of the function $b_{\mu,s}$:

$$\begin{aligned} b_{\mu,s}(z) &= \int_{\mathbb{T}} \frac{1}{(1 - \overline{\zeta} z)^s} d\mu(\zeta) = \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{\Gamma(s) n!} \left(\int_{\mathbb{T}} \overline{\zeta^n} d\mu(\zeta) \right) z^n \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{\Gamma(s) n!} \hat{\mu}(n) z^n \approx \sum_{n=0}^{\infty} \frac{1}{n^{1-s}} \hat{\mu}(n) z^n. \end{aligned}$$

Since $b_{\mu,s} \in \mathcal{H}_s$ is a necessary condition for the boundedness of the operators $T_{\mu,s}$ and $H_{\mu,s}$ on $\mathcal{H}_s$, we must have that $\sum_{n=0}^{\infty} \frac{|\hat{\mu}(n)|^2}{n^{1-s}} < \infty$.

Note that when $s = 1$, i.e. for the Hardy space $H^2 = \mathcal{H}_1$, the condition becomes $\sum_{n=0}^{\infty} |\hat{\mu}(n)|^2 < \infty$. It is well known that, by the F. and M. Riesz theorem, this implies that μ is absolutely continuous with respect to the Lebesgue measure dm, i.e $d\mu = \phi dm$, with $\phi \in L^1(\mathbb{T})$. Thus, the Toeplitz type operator $T_{\mu,1}$ is the classical Toeplitz operator T_ϕ, which is bounded if and only if $\phi \in L^\infty(\mathbb{T})$. Note that in that case, $b_{\mu,1}$ must also be bounded.

We end the remark with the following question: For $0 < s < 1$, and for a complex Borel measure μ with support in $\mathbb{T}$, is the boundedness of $b_{\mu,s}$ a necessary condition for the boundedness of the Toeplitz type operator $T_{\mu,s}$ on $\mathcal{H}_s$? $\diamondsuit$

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Department of Mathematics, University of Manitoba, Winnipeg, R3T 2N2, Manitoba, Canada

Email address: Nina.Zorboska@umanitoba.ca