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PATHWISE INTEGRATION AND CHANGE OF VARIABLE FORMULAS FOR CONTINUOUS PATHS WITH ARBITRARY REGULARITY

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Abstract. We construct a pathwise integration theory, associated with a change of variable formula, for smooth functionals of continuous paths with arbitrary regularity defined in terms of the notion of pth variation along a sequence of time partitions. For paths with finite pth variation along a sequence of time partitions, we derive a change of variable formula for p times continuously differentiable functions and show pointwise convergence of appropriately defined compensated Riemann sums.

Results for functions are extended to regular path-dependent functionals using the concept of vertical derivative of a functional. We show that the pathwise integral satisfies an "isometry" formula in terms of pth order variation and obtain a "signal plus noise" decomposition for regular functionals of paths with strictly increasing pth variation. For less regular (C^{p-1}) functions we obtain a Tanaka-type change of variable formula using an appropriately defined notion of local time.

These results extend to multidimensional paths and yield a natural higher-order extension of the concept of "reduced rough path". We show that, while our integral coincides with a rough path integral for a certain rough path, its construction is canonical and does not involve the specification of any rough-path superstructure.

Introduction

In his seminal paper *Calcul d'Itô sans probabilités* [14], Hans Föllmer provided a pathwise proof of the Itô formula, using the concept of *quadratic variation along a sequence of partitions*, defined as follows. A path $S \in C([0,T],\mathbb{R})$ is said to have finite quadratic variation along the sequence of partitions $\pi_n = (0 = t_0^n < t_1^n < \cdots < t_{N(\pi_n)}^n = T)$ if for any $t \in [0,T]$, the sequence of measures

$$\mu^n := \sum_{[t_j^n, t_{j+1}^n] \in \pi_n} \delta(\cdot - t_j)|S(t_{j+1}^n) - S(t_j^n)|^2$$

converges weakly to a measure μ without atoms. The continuous increasing function $[S]\colon [0,T] \to \mathbb{R}_+$ defined by $[S](t) = \mu([0,t])$ is then called the quadratic variation of S along π. Extending this definition to vector-valued paths Föllmer [14] showed that, for integrands of the form $\nabla f(S(t))$ with $f \in C^2(\mathbb{R}^d)$, one may define a

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pathwise integral $\int \nabla f(S(t))dS$ as a pointwise limit of Riemann sums along the sequence of partitions (π_n) and he obtained an Itô (change of variable) formula for $f(S(t))$ in terms of this pathwise integral: for $f \in C^2(\mathbb{R}^d), t \in [0,T]$,

$$f(S(t)) = \int_0^t \langle \nabla f(S(s)), \mathrm{d}S(s)\rangle + \frac{1}{2}\int_0^t \langle \nabla^2 f(S(s)), \mathrm{d}[S](s)\rangle,$$

where

$$\int_0^t \langle \nabla f(S(s)), \mathrm{d}S(s)\rangle := \lim_{n\to\infty} \sum_{[t_j^n, t_{j+1}^n]\in\pi_n} \langle \nabla f(S(t)), (S(t_{j+1}^n \wedge t) - S(t_j^n \wedge t))\rangle.$$

This result has many interesting ramifications and applications in the pathwise approach to stochastic analysis, and has been extended in different ways, to less regular functions using the notion of pathwise local time [2, 10, 24], as well as to path-dependent functionals and integrands [1, 7, 8, 25].

The central role played by the concept of quadratic variation has led to the presumption that they do not extend to less regular paths with infinite quadratic variation. Integration theory and change of variables formulas for processes with infinite quadratic variation, such as fractional Brownian motion and other fractional processes, have relied on probabilistic, rather than pathwise constructions [5, 9, 18]. Furthermore, the change of variable formulae obtained using these methods are valid for a restricted range of Hurst exponents (see [23] for an overview).

In this work, we show that Föllmer's pathwise Itô calculus may be extended to paths with arbitrary regularity, in a strictly pathwise setting, using the concept of pth variation along a sequence of time partitions. For paths with finite pth variation along a sequence of time partitions, we derive a change of variable formula for p times continuously differentiable functions and show pointwise convergence of appropriately defined compensated Riemann sums. This result may be seen as the natural extension of the results of Föllmer [14] to paths of lower regularity. Our results apply in particular to paths of fractional Brownian motions with arbitrary Hurst exponent, and yield pathwise proofs for results previously derived using probabilistic methods, without any restrictions on the Hurst exponent.

Using the concept of the vertical derivative of a functional [8], we extend these results to regular path-dependent functionals of such paths. We obtain an "isometry" formula in terms of pth order variations for the pathwise integral and a "signal plus noise" decomposition for regular functionals of paths with strictly increasing pth variation, extending the results of [1] obtained for the case $p = 2$ to arbitrary even integers $p \geq 2$.

The extension to less regular (i.e., not p times differentiable) functions is more delicate and requires defining an appropriate higher-order analogue of semimartingale local time, which we introduce through an appropriate spatial localization of the pth order variation. Using this higher-order concept of local time, we obtain a Tanaka-type change of variable formula for less regular (i.e., $p-1$ times differentiable) functions. We conjecture that these results apply in particular to paths of fractional Brownian motion and other fractional processes.

Finally, we consider extensions of these results to multidimensional paths and link them with rough path theory; the corresponding concepts yield a natural higher-order extension to the concept of "reduced rough path" introduced by Friz and Hairer [17, Chapter 5].

Outline. Section 1 introduces the notion of pth variation along a sequence of partitions and derives a change of variable formula for p times continuously differentiable functions of paths with finite pth variation (Theorem 1.5). An extension of these results to path-dependent functionals is discussed in Section 1.3: Theorem 1.10 gives a functional change of variable formula for regular functionals of paths with finite pth variation.

Section 2 studies the corresponding pathwise integral in more detail. We first show (Theorem 2.1) that the integral exhibits an "isometry" property in terms of the pth order variation and use this property to obtain a unique "signal plus noise" decomposition where the components are discriminated in terms of their pth order variation (Theorem 2.3).

The extension of these concepts to multidimensional paths and the relation to the concept of "reduced rough paths" are discussed in Section 4.

1. Pathwise calculus for paths with finite pth variation

1.1. **pth variation along a sequence of partitions.** We introduce, in the spirit of Föllmer [14], the concept of pth variation along a sequence of partitions $\pi_n = \{t_0^n, \ldots, t_{N(\pi_n)}^n\}$ with $t_0^n = 0 < \ldots < t_k^n < \ldots < t_{N(\pi_n)}^n = T$. Define the *oscillation* of $S \in C([0,T],\mathbb{R})$ along π_n as

$$\operatorname{osc}(S,\pi_n) := \max_{[t_j,t_{j+1}]\in\pi_n} \max_{r,s\in[t_j,t_{j+1}]} |S(s)-S(r)|.$$

Here and in the following we write $[t_j,t_{j+1}] \in \pi_n$ to indicate that t_j and t_{j+1} are both in π_n and are immediate successors (i.e., $t_j < t_{j+1}$ and $\pi_n \cap (t_j,t_{j+1}) = \emptyset$).

Definition 1.1 (pth variation along a sequence of partitions)**.** Let $p > 0$. A continuous path $S \in C([0,T],\mathbb{R})$ is said to have a pth variation along a sequence of partitions $\pi = (\pi_n)_{n\geq 1}$ if $\operatorname{osc}(S,\pi_n) \to 0$ and the sequence of measures

$$\mu^n := \sum_{[t_j,t_{j+1}]\in\pi_n} \delta(\cdot - t_j)|S(t_{j+1}) - S(t_j)|^p$$

converges weakly to a measure μ without atoms. In that case we write $S \in V_p(\pi)$ and $[S]^p(t) := \mu([0,t])$ for $t \in [0,T]$, and we call $[S]^p$ the *pth variation* of S.

Remark 1.2. (1) Functions in $V_p(\pi)$ do not necessarily have finite p-variation in the usual sense. Recall that the p-variation of a function $f \in C([0,T],\mathbb{R})$ is defined as [11]

$$\|f\|_{\text{p-var}} := \Big(\sup_{\pi\in\Pi([0,T])} \sum_{[t_j,t_{j+1}]\in\pi} |f(t_{j+1}) - f(t_j)|^p\Big)^{1/p},$$

where the supremum is taken over the set $\Pi([0,T])$ of all partitions π of $[0,T]$. A typical example is the Brownian motion B, which has quadratic variation $[B]^2(t) = t$ along any refining sequence of partitions almost surely while at the same time having infinite 2-variation almost surely [11, 29]:

$$\mathbb{P}\left(\|B\|_{\text{2-var}} = \infty\right) = 1.$$

(2) If $S \in V_p(\pi)$ and $q > p$, then $S \in V_q(\pi)$ with $[S]^q \equiv 0$.

The following lemma gives a simple characterization of this property.

Lemma 1.3. *Let $S \in C([0,T],\mathbb{R})$. $S \in V_p(\pi)$ if and only if there exists a continuous function $[S]^p$ such that*

$$\forall t \in [0,T], \qquad \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_j\le t}} |S(t_{j+1})-S(t_j)|^p \overset{n\to\infty}{\longrightarrow} [S]^p(t). \tag{1}$$

If this property holds, then the convergence in (1) *is uniform.*

Indeed, the weak convergence of measures on $[0,T]$ is equivalent to the pointwise convergence of their cumulative distribution functions at all continuity points of the limiting cumulative distribution function, and if the limiting cumulative distribution function is continuous, the convergence is uniform.

Example 1.4. If B is a fractional Brownian motion with Hurst index $H \in (0,1)$ and $\pi_n = \{kT/n : k \in \mathbb{N}_0\} \cap [0,T]$, then $B \in V_{1/H}(\pi)$ and $[B]^{1/H}(t) = t\mathbb{E}[|B_1|^{1/H}]$; see [26, 27].

1.2. **Pathwise integral and change of variable formula.** A key observation of Föllmer [14] was that, for $p = 2$, Definition 1.1 is sufficient to obtain a pathwise Itô formula for (C^2) functions of $S \in V_2(\pi)$. We will show that in fact Föllmer's argument may be applied for any even integer p.

Theorem 1.5 (Change of variable formula for paths with finite pth variation)**.** *Let $p \in \mathbb{N}$ be even, let (π_n) be a given sequence of partitions, and let $S \in V_p(\pi)$. Then for every $f \in C^p(\mathbb{R},\mathbb{R})$ the pathwise change of variable formula*

$$f(S(t)) - f(S(0)) = \int_0^t f'(S(s))\mathrm{d}S(s) + \frac{1}{p!}\int_0^t f^{(p)}(S(s))\mathrm{d}[S]^p(s)$$

holds, where the integral

$$\int_0^t f'(S(s))\mathrm{d}S(s) := \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!}(S(t_{j+1}\wedge t) - S(t_j\wedge t))^k$$

is defined as a (pointwise) limit of compensated Riemann sums.

Proof. Applying a Taylor expansion at order p to the increments of $f(S)$ along the partition, we obtain

(2)

$$\begin{aligned}
&f(S(t)) - f(S(0))\\
&\quad= \sum_{[t_j,t_{j+1}]\in\pi_n} (f(S(t_{j+1}\wedge t)) - f(S(t_j\wedge t)))\\
&\quad= \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p} \frac{f^{(k)}(S(t_j))}{k!}(S(t_{j+1}\wedge t) - S(t_j\wedge t))^k\\
&\qquad+ \sum_{[t_j,t_{j+1}]\in\pi_n} \int_0^1 \mathrm{d}\lambda \frac{(1-\lambda)^{p-1}}{(p-1)!}(S(t_{j+1}\wedge t) - S(t_j\wedge t))^p\\
&\qquad\qquad\times \big(f^{(p)}(S(t_j) + \lambda(S(t_{j+1}\wedge t) - S(t_j\wedge t))) - f^{(p)}(S(t_j))\big).
\end{aligned}$$

Since the image of $(S(t))_{t\in[0,T]}$ is compact, we may assume without loss of generality that f is compactly supported; then the remainder on the right hand side is bounded

by

$$\Big| \sum_{[t_j,t_{j+1}]\in\pi_n} \int_0^1 \mathrm{d}\lambda \frac{(1-\lambda)^{p-1}}{(p-1)!}(S(t_{j+1}\wedge t)-S(t_j\wedge t))^p \\ \times \big(f^{(p)}(S(t_j)+\lambda(S(t_{j+1}\wedge t)-S(t_j\wedge t)))-f^{(p)}(S(t_j))\big)\Big| \\ \leq C(f,S,\pi_n,p)\mu_n([0,t])$$

with a constant $C(f,S,\pi_n,p)>0$ that converges to zero for $n\to\infty$, and therefore the remainder vanishes for $n\to\infty$. Since $S\in V_p(\pi)$ we know that

$$\lim_{n\to\infty}\sum_{[t_j,t_{j+1}]\in\pi_n}\frac{f^{(p)}(S(t_j))}{p!}(S(t_{j+1}\wedge t)-S(t_j\wedge t))^p=\frac{1}{p!}\int_0^t f^{(p)}(S(s))\mathrm{d}[S]^p(s),$$

and therefore we obtain from (2)

$$\lim_{n\to\infty}\sum_{[t_j,t_{j+1}]\in\pi_n}\sum_{k=1}^{p-1}\frac{f^{(k)}(S(t_j))}{k!}(S(t_{j+1}\wedge t)-S(t_j\wedge t))^k \\ = f(S(t))-f(S(0))-\frac{1}{p!}\int_0^t f^{(p)}(S(s))\mathrm{d}[S]^p(s),$$

and we simply define $\int_0^t f'(S(s))\mathrm{d}S(s)$ as the limit on the left hand side. □

Remark 1.6 (Relation with Young integration and rough path integration). The expression

$$\sum_{[t_j,t_{j+1}]\in\pi_n}\sum_{k=1}^{p-1}\frac{f^{(k)}(S(t_j))}{k!}(S(t_j\wedge t)-S(t_j\wedge t))^k$$

is a "compensated Riemann sum". Note however that, given the assumptions on S, the pathwise integral appearing in the formula cannot be defined as a Young integral, *even* after substracting the compensating terms. This relates to the observation in Remark 1.2 that p-variation can be infinite for $S\in V_p(\pi)$.

When $p=2$ it reduces to an ordinary (left) Riemann sum. For $p>2$ such compensated Riemann sums appear in the construction of "rough path integrals" [17, 19]. Let $X\in C^\alpha([0,T],\mathbb{R})$ be α-Hölder continuous for some $\alpha\in(0,1)$, and write $q=\lfloor\alpha^{-1}\rfloor$. We can enhance X uniquely into a (weakly) geometric rough path $(\mathbb{X}^1_{s,t},\mathbb{X}^2_{s,t},\dots,\mathbb{X}^q_{s,t})_{0\leq s\leq t\leq T}$, where $\mathbb{X}^k_{s,t}:=(X(t)-X(s))^k/k!$. Moreover, for $g\in C^{q+1}(\mathbb{R},\mathbb{R})$ the function $g'(X)$ is controlled by X with Gubinelli derivatives

$$g'(X(t))-g'(X(s))=\sum_{k=1}^{q-1}\frac{g^{(k+1)}(X(s))}{k!}(X(t)-X(s))^k+O(|t-s|^{q\alpha}) \\ =\sum_{k=1}^{q-1}g^{(k+1)}(X(s))\mathbb{X}^k_{s,t}+O(|t-s|^{q\alpha}),$$

and therefore the controlled rough-path integral $\int_0^t g'(X(s))\mathrm{d}X(s)$ is given by

$$\lim_{|\pi|\to 0}\sum_{[t_j,t_{j+1}]\in\pi}\sum_{k=1}^{q}g^{(k)}(X(s))\mathbb{X}^k_{s,t}=\lim_{|\pi|\to 0}\sum_{[t_j,t_{j+1}]\in\pi}\sum_{k=1}^{q}g^{(k)}(X(s))\frac{(X(t)-X(s))^k}{k!},$$

where $|\pi|$ denotes the mesh size of the partition π, and which is exactly the type of compensated Riemann sum that we used to define our integral. The link between our approach and rough-path integration is explained in more detail in Section 4.2 below.

Remark 1.7. In principle we could apply similar arguments for odd integers p if instead of $S \in V_p(\pi)$ we assumed that $\sum_{[t_j,t_{j+1}]\in\pi_n} \delta(\cdot - t_j)(S(t_{j+1}) - S(t_j))^p$ converges to a signed measure. However, for odd p we typically expect the limit to be zero; see the appendix for a prototypical example. So to slightly simplify the presentation, we restrict our attention to even p.

Remark 1.8. A notion similar to our definition of pth variation was introduced by Errami and Russo [13], in the (probabilistic and not pathwise) context of stochastic calculus via regularization [28]. For $p = 3$, Errami and Russo prove an Itô-type formula that is similar to the one in Theorem 1.5. However, since they use a definition of the integral $\int_0^t f'(S(s))\mathrm{d}S(s)$ that does not take the higher-order compensation terms into account, their approach is limited to $p = 3$. Gradinaru, Russo, and Vallois [18] extended this approach to $p = 4$ for functions of a fractional Brownian motion with Hurst index $H \geq 1/4$, a result which relies heavily on the Gaussian properties of fractional Brownian motion.

The key ingredient of our approach is to define the integral using compensated Riemann sums which, compared with previous work, drastically simplifies the derivation of the change of variable formula for arbitrary (even) p in a strictly pathwise setting without any use of probabilistic notions of convergence.

1.3. **Extension to path-dependent functionals.** An important generalization of Föllmer's pathwise Itô formula is to the case of path-dependent functionals [8] of paths $S \in V_2(\pi)$ using Dupire's functional derivative [12]; see [7] for an overview. We extend here the functional change of variable formula of Cont and Fournié [8] to functionals of paths $S \in V_p(\pi)$, where p is any even integer.

Let $D([0,T],\mathbb{R})$ be the space of càdlàg paths from $[0,T]$ to $\mathbb{R}$ and write

$$\omega_t(s) = \omega(s \wedge t)$$

for the path ω stopped at time t. Let

$$\Lambda_T := \{(t,\omega_t) : (t,\omega) \in [0,T] \times D([0,T],\mathbb{R})\}$$

be the space of stopped paths. This is a complete metric space equipped with

$$d_\infty((t,\omega),(t',\omega')) := \sup_{s\in[0,T]} |\omega(s\wedge t) - \omega'(s\wedge t')| + |t-t'| = \|\omega_t - \omega_{t'}\|_\infty + |t-t'|.$$

We will also need to stop paths "right before" a given time, and set for $t > 0$

$$\omega_{t-}(s) := \begin{cases} \omega(s), & s < t, \\ \lim_{r\uparrow t} \omega(r), & s \geq t, \end{cases}$$

while $\omega_{0-} := \omega_0$. We first recall some concepts from the non-anticipative functional calculus [7, 8].

Definition 1.9. A *non-anticipative* functional is a map $F\colon \Lambda_T \to \mathbb{R}$. Let F be a non-anticipative functional.

i. We write $F \in \mathbb{C}_l^{0,0}(\Lambda_T)$ if for all $t \in [0,T]$ the map $F(t,\cdot)\colon D([0,T],\mathbb{R}) \to \mathbb{R}$ is continuous and if for all $(t,\omega) \in \Lambda_T$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(t',\omega') \in \Lambda_T$ with $t' < t$ and $d_\infty((t,\omega),(t',\omega')) < \delta$ we have $|F(t,\omega) - F(t',\omega')| < \varepsilon$.

ii. We write $F \in \mathbb{B}(\Lambda_T)$ if for every $t_0 \in [0,T)$ and every $K > 0$ there exists $C_{K,t_0} > 0$ such that for all $t \in [0,t_0]$ and all $\omega \in D([0,T],\mathbb{R})$ with $\sup_{s\in[0,t]} |\omega(s)| \le K$ we have $|F(t,\omega)| \le C_{K,t_0}$.

iii. F is *horizontally differentiable* at $(t,\omega) \in \Lambda_T$ if its *horizontal derivative*

$$\mathcal{D}F(t,\omega) := \lim_{h\downarrow 0} \frac{F(t+h,\omega_t) - F(t,\omega_t)}{h}$$

exists. If it exists for all $(t,\omega) \in \Lambda_T$, then $\mathcal{D}F$ is a non-anticipative functional.

iv. F is *vertically differentiable* at $(t,\omega) \in \Lambda_T$ if its *vertical derivative*

$$\nabla_\omega F(t,\omega) := \lim_{h\downarrow 0} \frac{F(t,\omega_t + h\mathbf{1}_{[t,T]}) - F(t,\omega_t)}{h}$$

exists. If it exists for all $(t,\omega) \in \Lambda_T$, then $\nabla_\omega F$ is a non-anticipative functional. In particular, we define recursively $\nabla_\omega^{k+1} F := \nabla_\omega \nabla_\omega^k F$ whenever this is well defined.

v. For $p \in \mathbb{N}_0$ we say that $F \in \mathbb{C}_b^{1,p}(\Lambda_T)$ if F is horizontally differentiable and p times vertically differentiable in every $(t,\omega) \in \Lambda_T$, and if $F, \mathcal{D}F, \nabla_\omega^k F \in \mathbb{C}_l^{0,0}(\Lambda_T) \cap \mathbb{B}(\Lambda_T)$ for $k = 1,\dots,p$.

Define the piecewise-constant approximation S^n to S along the partition π_n:

$$S^n(t) = \sum_{[t_j,t_{j+1}]\in\pi_n} S(t_{j+1})\mathbf{1}_{[t_j,t_{j+1})}(t) + S(T)\mathbf{1}_{\{T\}}(t). \tag{3}$$

Then $\lim_{n\to\infty} \|S^n - S\|_\infty = 0$ whenever $\operatorname{osc}(S,\pi_n) \to 0$.

Theorem 1.10 (Functional change of variable formula for paths with finite pth variation)**.** *Let p be an even integer, let $F \in \mathbb{C}_b^{1,p}(\Lambda_T)$, and let $S \in V_p(\pi)$ for a sequence of partitions (π_n) with vanishing mesh size $|\pi_n| \to 0$. Then the functional change of variable formula*

$$\begin{aligned} F(t,S_t) = F(0,S_0) &+ \int_0^t \mathcal{D}F(s,S_s)\mathrm{d}s + \int_0^t \\ &< \nabla F(s,S_s), \mathrm{d}S(s) > + \frac{1}{p!}\int_0^t \nabla_\omega^p F(s,S_s)\mathrm{d}[S]^p(s) \end{aligned}$$

holds, where

$$\begin{aligned} &\int_0^t < \nabla F(s,S_s), \mathrm{d}S(s) > \\ &\qquad := \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p-1} \frac{1}{k!} \nabla_\omega^k F(t_j, S^n_{t_j-})(S(t_{j+1}\wedge t) - S(t_j \wedge t))^k, \end{aligned}$$

with the piecewise constant approximation S^n as defined in (3).

Proof. Since the right hand side is a telescoping sum, we have

$$\begin{aligned} F(t,S_t^n)-F(0,S_0^n) &= \sum_{[t_j,t_{j+1}]\in\pi_n} (F(t_{j+1}\wedge t, S^n_{(t_{j+1}\wedge t)-}) - F(t_j\wedge t, S^n_{(t_j\wedge t)-})) \\ &\quad + F(t,S_t^n) - F(t,S_{t-}^n) \\ &= \sum_{[t_j,t_{j+1}]\in\pi_n} (F(t_{j+1}\wedge t, S^n_{(t_{j+1}\wedge t)-}) - F(t_j\wedge t, S^n_{(t_j\wedge t)-})) + o(1). \end{aligned}$$

Consider j with $t_{j+1}\leqslant t$ and split up the difference as follows:

$$F(t_{j+1},S^n_{t_{j+1}-})-F(t_j,S^n_{t_j-}) = (F(t_{j+1},S^n_{t_{j+1}-})-F(t_j,S^n_{t_j})) + (F(t_j,S^n_{t_j})-F(t_j,S^n_{t_j-})).$$

Now $S^n_{t_{j+1}-}(s) = S^n_{t_j}(s)$ for all $s\in[0,t_{j+1}]$, and therefore the first term on the right hand side is simply

$$F(t_{j+1},S^n_{t_{j+1}-}) - F(t_j,S^n_{t_j}) = \int_{t_j}^{t_{j+1}} \mathcal{D}F(r,S^n_{t_j})\mathrm{d}r,$$

from where we easily get (using that the mesh size of (π_n) converges to zero)

$$\lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} (F(t_{j+1}\wedge t, S^n_{(t_{j+1}\wedge t)-}) - F(t_j\wedge t, S^n_{(t_j\wedge t)})) = \int_0^t \mathcal{D}F(r,S_r)\mathrm{d}r.$$

It remains to consider the term

$$F(t_j,S^n_{t_j}) - F(t_j,S^n_{t_j-}) = F(t_j,S^{n,S_{t_j,t_{j+1}}}_{t_j-}) - F(t_j,S^n_{t_j-}),$$

where $S_{t_j,t_{j+1}} := S(t_{j+1}) - S(t_j)$ and $S^{n,x}_{t_j-}(s) := S^n_{t_j}(s) + \mathbf{1}_{[t_j,T]}(s)x$. By Taylor's formula and the definition of the vertical derivative, we have

$$\begin{aligned} &F(t_j,S^{n,S_{t_j,t_{j+1}}}_{t_j-}) - F(t_j,S^n_{t_j-}) \\ &\quad = \sum_{k=1}^p \frac{\nabla_\omega^k F(t_j,S^n_{t_j-})}{k!}(S(t_{j+1}\wedge t) - S(t_j\wedge t))^k \\ &\qquad + \frac{1}{(p-1)!}\int_0^1 \mathrm{d}\lambda(1-\lambda)^{p-1}(S(t_{j+1}\wedge t) - S(t_j\wedge t))^p \\ &\qquad\qquad \times \left(\nabla_\omega^p F(t_j,S^{n,\lambda S_{t_j,t_{j+1}}}_{t_j-}) - \nabla_\omega^p F(t_j,S^n_{t_j-})\right). \end{aligned}$$

Now we sum over $[t_j,t_{j+1}]\in\pi_n$ and see as in Theorem 1.5 that the correction term vanishes for $n\to\infty$. Moreover, since $S\in V_p(\pi)$ we have

$$\lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \frac{\nabla_\omega^p F(t_j,S^n_{t_j-})}{p!}(S(t_{j+1}\wedge t) - S(t_j\wedge t))^p = \frac{1}{p!}\int_0^t \nabla_\omega^p F(s,S_s)\mathrm{d}[S]^p(s);$$

see [7, Lemma 5.3.7]. Since $F\in\mathbb{C}_l^{0,0}(\Lambda_T)$, we have

$$\lim_{n\to\infty}(F(t,S_t^n) - F(0,S_0^n)) = F(t,S_t) - F(0,S_0),$$

which completes the proof. □

2. Isometry relation and rough-smooth decomposition

Given a path (or process) $S \in V_p(\pi)$ with finite pth variation along the sequence of partitions (π_n), the results above may be used to derive a decomposition of regular functionals of S into a rough component with non-zero pth variation along (π_n) and a smooth component with zero pth variation along (π_n). For $p = 2$ such a decomposition was obtained in [1] and is a pathwise analog of the decomposition of a Dirichlet process into a local martingale and a "zero energy" part [15].

For $\alpha \in (0,1)$ we write $C^\alpha([0,T],\mathbb{R})$ for the α-Hölder continuous paths from $[0,T]$ to $\mathbb{R}$, and $\|\cdot\|_\alpha$ denotes the α-Hölder semi-norm.

2.1. An "isometry" property of the pathwise integral.

Theorem 2.1 ("Isometry" formula)**.** *Let $p \in \mathbb{N}$ be an even integer, let $\alpha > ((1+\frac{4}{p})^{1/2}-1)/2$, let (π_n) be a sequence of partitions with mesh size going to zero, and let $S \in V_p(\pi) \cap C^\alpha([0,T],\mathbb{R})$. Let $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$ such that $\nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T)$. Assume furthermore that F is Lipschitz-continuous with respect to d_∞. Then $F(\cdot,S) \in V_p(\pi)$ and*

$$[F(\cdot,S)]^p(t) = \int_0^t |\nabla_\omega F(s,S_s)|^p \mathrm{d}[S]^p(s).$$

Proof. The proof is similar to the case $p = 2$ considered in [1]. Indeed, our assumptions allow us to apply [1, Lemma 2.2], which shows that there exists $C > 0$, only depending on T, F, and $\|S\|_\alpha$, such that for all $0 \le s \le t \le T$

$$|R_F(s,t)| := |F(t,S_t) - F(s,S_s) - \nabla_\omega F(s,S_s)(S(t)-S(s))| \le C|t-s|^{\alpha+\alpha^2}. \tag{4}$$

Writing also $\gamma_F(s,t) := \nabla_\omega F(s,S_s)(S(t)-S(s))$, we obtain

$$\begin{aligned}\sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |F(t_{j+1},S_{t_{j+1}}) - F(t_j,S_{t_j})|^p &= \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |R_F(t_j,t_{j+1}) + \gamma_F(t_j,t_{j+1})|^p \\ &= \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |\gamma_F(t_j,t_{j+1})|^p + \sum_{k=1}^p \binom{p}{k} \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} R_F(t_j,t_{j+1})^k \gamma_F(t_j,t_{j+1})^{p-k}.\end{aligned} \tag{5}$$

Since $S \in V_p(\pi)$ we have

$$\lim_{n\to\infty} \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |\gamma_F(t_j,t_{j+1})|^p = \int_0^t |\nabla_\omega F(s,S(s))|^p \mathrm{d}[S]^p(s). \tag{6}$$

Our result follows once we show that the double sum on the right hand side of (5) vanishes. For that purpose let $k \in \{1,\dots,p\}$ and write $q_k := p/(p-k) \in [1,\infty]$

and let $q_k' = p/k$ be its conjugate exponent. Hölder's inequality yields

$$\begin{aligned}
&\Big|\sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} R_F(t_j,t_{j+1})^k \gamma_F(t_j,t_{j+1})^{p-k}\Big| \\
&\le \Big(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |R_F(t_j,t_{j+1})|^{kq_k'}\Big)^{1/q_k'} \Big(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |\gamma_F(t_j,t_{j+1})|^{(p-k)q_k}\Big)^{1/q_k} \\
&= \Big(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |R_F(t_j,t_{j+1})|^{p}\Big)^{k/p} \Big(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |\gamma_F(t_j,t_{j+1})|^{p}\Big)^{(p-k)/p}.
\end{aligned}$$

By (4) the first sum on the right hand side is bounded by

$$\begin{aligned}
&\Big(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |R_F(t_j,t_{j+1})|^{p}\Big)^{k/p} \\
&\quad\lesssim \Big(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |t_{j+1}-t_j|^{p(\alpha+\alpha^2)}\Big)^{k/p} \\
&\quad\le (t\times \max\{|t_{j+1}-t_j|^{p(\alpha+\alpha^2)-1} : [t_j,t_{j+1}]\in\pi_n, t_{j+1}\le t\})^{k/p},
\end{aligned}$$

which converges to zero for $n\to\infty$ because $p(\alpha+\alpha^2)>1$ (which is equivalent to our assumption $\alpha > (\sqrt{1+\frac{4}{p}}-1)/2$) and because $k>0$. Moreover, by (6) the sum over $|\gamma_F(t_j,t_{j+1})|^p$ is bounded and this concludes the proof. □

Remark 2.2.

(1) Keeping the example of the (fractional) Brownian motion in mind, we would typically expect paths in $V_p(\pi)$ to be $(1/p-\kappa)$-Hölder continuous for any $\kappa>0$. Since for $f(x)=(1+x)^{1/2}$ we have

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2} < 0,$$

we have $f(x) < f(0)+f'(0)x$ for all $x>0$, and therefore

$$\frac{(1+\frac{4}{p})^{1/2}-1}{2} < \frac{\frac{1}{2}\frac{4}{p}}{2} = \frac{1}{p},$$

which means that in Theorem 2.1 we can take $\alpha<1/p$ and our constraint on the Hölder regularity is not unreasonable.

(2) In fact the constraint on α comes from inequality (4), which only gives us a control of order $|t-s|^{\alpha+\alpha^2}$ for $R_F(s,t)$, while $|t-s|^{2\alpha}$ might seem more natural (after all $R_F(s,t)$ is something like the remainder in a first-order Taylor expansion). The difficulty is that horizontal differentiability is a very weak notion and gives us no control on $R_F(s,t)$. To obtain any bounds at all we first need to approximate our path by piecewise linear or piecewise constant paths, and through this approximation procedure we lose a little bit of regularity; see [1, Lemma 2.2] for details. One can improve the estimate on $R_F(s,t)$ by taking a higher-order Taylor expansion (which would require more regularity from F), but we do not need this here.

2.2. **Pathwise rough-smooth decomposition.** Using the above result we may derive, as in [1], a pathwise "signal plus noise" decomposition for regular functionals of paths with strictly increasing pth variation. Let

$$\mathbb{C}_b^{1,p}(S) = \{F(\cdot, S), F \in \mathbb{C}_b^{1,p}(\Lambda_T)\} \subset V_p(\pi).$$

The following result extends the pathwise rough-smooth decomposition of paths in $\mathbb{C}_b^{1,p}(S)$, obtained in [1] for $p = 2$, to higher values of p.

Theorem 2.3 (Rough-smooth decomposition)**.** *Let $p \in \mathbb{N}$ be an even integer, let $\alpha > ((1 + \frac{4}{p})^{1/2} - 1)/2$, let (π_n) be a sequence of partitions with vanishing mesh size $|\pi_n| \to 0$ and let $S \in V_p(\pi) \cap C^\alpha([0, T], \mathbb{R})$ be a path with strictly increasing pth variation $[S]^p$ along (π_n). Then any $X \in \mathbb{C}_b^{1,p}(S)$ admits a unique decomposition*

$$X = X(0) + A + M \quad \textit{where} \quad [A]^p = 0 \quad \textit{and} \quad M(t) = \int_0^t \phi(s)\mathrm{d}S(s)$$

is a pathwise integral defined as in Theorem 1.10.

Proof. Existence of the decomposition is a consequence of Theorem 1.10. Consider two such decompositions $X - X_0 = A + M = \tilde{A} + \tilde{M}$. Since $[A]^p = [\tilde{A}]^p = 0$ and

$$|(A - \tilde{A})(t) - (A - \tilde{A})(s)|^p \lesssim |A(t) - A(s)|^p + |\tilde{A}(t) - \tilde{A}(s)|^p,$$

we get $A - \tilde{A} \in V_p(\pi)$ and $[A - \tilde{A}]^p \equiv 0$. But then also $[M - \tilde{M}]^p = [A - \tilde{A}]^p \equiv 0$. Now

$$M(t) = \int_0^t \nabla_\omega F(s, S_s)\mathrm{d}S(s), \qquad \tilde{M}(t) = \int_0^t \nabla_\omega \tilde{F}(s, S_s)\mathrm{d}S(s)$$

for some $F, \tilde{F} \in C_b^{1,p}(\Lambda_T)$, and by Theorem 2.1 we have

$$0 = [M - \tilde{M}]^p(T) = \int_0^T |\nabla_\omega(F - \tilde{F})(s, S_s)|^p \mathrm{d}[S]^p(s).$$

Since $(F - \tilde{F})(s, S_s)$ is continuous in s and $[S]^p$ is strictly increasing we have $\nabla_\omega(F - \tilde{F})(\cdot, S) \equiv 0$. This means that $M - \tilde{M} \equiv 0$, and then also $A - \tilde{A} \equiv 0$. □

Remark 2.4. If $t \mapsto [S]^p(t)$ is not strictly increasing, uniqueness of the decomposition still holds $d[S]^p$−almost everywhere.

3. Local times and higher-order Wuermli formula

An extension of Föllmer's pathwise Itô formula to less regular functions was given by Wuermli [30] in her (unpublished) thesis. Wuermli considered paths with finite quadratic variation which further admit a *local time* along a sequence of partitions, and derive a pathwise change of variable formula for more general functions that need not be C^2. Depending on the notion of convergence used to define the local time, one then obtains a Tanaka-type change of variable formulas for various classes of functions; convergence in stronger topologies leads to a formula valid for a larger class of functions. Wuermli [30] assumed weak convergence in L^2 in the space variable (see also [2]) and some recent works have extended the approach to other topologies, for example uniform convergence or weak convergence in L^q [10, 24]. To a certain extent Wuermli's approach can be generalized to our higher-order setting, but as we will discuss below in the higher-order case we do not expect to have convergence of the pathwise local times in strong topologies.

To derive the generalization of Wuermli's formula, we consider $f \in C^{p-2}$ with absolutely continuous $f^{(p-2)}$ and apply the Taylor expansion of order $p-2$ with integral remainder to obtain

$$f(b) - f(a) = \sum_{k=1}^{p-2} \frac{f^{(k)}(a)}{k!}(b-a)^k + \int_a^b \frac{f^{(p-1)}(x)}{(p-2)!}(b-x)^{p-2}\mathrm{d}x.$$

Assume now that $f^{(p-1)}$ is of bounded variation. Since every bounded variation function $f^{(p-1)}$ is regulated (làdlàg) and therefore has only countably many jumps, its càdlàg version is also a weak derivative of $f^{(p-2)}$, and from now on we only work with this version. Since $(b-\cdot)^{p-2}$ is continuous, the integration by parts rule for the Lebesgue-Stieltjes integral applies in the case $b \geq a$ and we obtain

$$\begin{aligned}\int_a^b \frac{f^{(p-1)}(x)}{(p-2)!}(b-x)^{p-2}\mathrm{d}x &= f^{(p-1)}(b)\frac{-(b-b)^{p-1}}{(p-1)!} - f^{(p-1)}(a)\frac{-(b-a)^{p-1}}{(p-1)!} \\ &\quad - \int_{(a,b]} \frac{-(b-x)^{p-1}}{(p-1)!}\mathrm{d}f^{(p-1)}(x) \\ &= f^{(p-1)}(a)\frac{(b-a)^{p-1}}{(p-1)!} + \int_{(a,b]} \frac{(b-x)^{p-1}}{(p-1)!}\mathrm{d}f^{(p-1)}(x).\end{aligned}$$

Similarly we get for $b < a$

$$\begin{aligned}\int_a^b \frac{f^{(p-1)}(x)}{(p-2)!}(b-x)^{p-2}\mathrm{d}x &= -\int_b^a \frac{f^{(p-1)}(x)}{(p-2)!}(b-x)^{p-2}\mathrm{d}x \\ &= f^{(p-1)}(a)\frac{(b-a)^{p-1}}{(p-1)!} - \int_{(b,a]} \frac{(b-x)^{p-1}}{(p-1)!}\mathrm{d}f^{(p-1)}(x),\end{aligned}$$

and therefore

$$\begin{aligned}f(b) - f(a) &= \sum_{k=1}^{p-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \operatorname{sign}(b-a)\int_{(\!|a,b|\!]} \frac{(b-x)^{p-1}}{(p-1)!}\mathrm{d}f^{(p-1)}(x) \\ &= \sum_{k=1}^{p-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \operatorname{sign}(b-a)^p\int_{(\!|a,b|\!]} \frac{|b-x|^{p-1}}{(p-1)!}\mathrm{d}f^{(p-1)}(x) \\ &= \sum_{k=1}^{p-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \int_{\mathbb{R}} \mathbf{1}_{(\!|a,b|\!]}(x)\frac{\operatorname{sign}(b-a)^p|b-x|^{p-1}}{(p-1)!}\mathrm{d}f^{(p-1)}(x),\end{aligned}$$

with the notation

$$(\!|a,b|\!] = \begin{cases}(a,b], & b \geq a, \\ (b,a], & a \leq b.\end{cases}$$

For any partition σ of $[0,T]$, we define

$$L_t^{\sigma,p-1}(x) := \sum_{t_j \in \sigma} \operatorname{sign}(S_{t_{j+1}\wedge t} - S(t_j \wedge t))^p \mathbf{1}_{(\!|S(t_j\wedge t), S_{t_{j+1}\wedge t}|\!]}(x)|S(t_{j+1}\wedge t) - x|^{p-1}.$$

To extend Theorem 1.5 to $S \in V_p(\pi)$, we first note that the following identity holds for any partition π_n:

$$f(S_t) - f(S_0) = \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S_{t_j})}{k!}(S(t_{j+1}\wedge t) - S(t_j \wedge t))^k + \frac{1}{(p-1)!}\int_{\mathbb{R}} L_t^{\pi_n,p-1}(x)\mathrm{d}f^{(p-1)}(x). \tag{7}$$

To obtain a change of variable formula for less regular functions, we need the last term to converge as the partition is refined. This motivates the following definition.

Definition 3.1 (Local time of order p)**.** Let $p \in \mathbb{N}$ be an even integer and let $q \in [1,\infty]$. A continuous path $S \in C([0,T],\mathbb{R})$ has an *L^q-local time of order $p-1$* along a sequence of partitions $\pi = (\pi_n)_{n\geq 1}$ if $\mathrm{osc}(S,\pi_n) \to 0$ and

$$L_t^{\pi_n,p-1}(\cdot) = \sum_{t_j\in\pi} \mathbf{1}_{(\!(S(t_j\wedge t),S_{t_{j+1}\wedge t}]\!]}(\cdot)|S(t_{j+1}\wedge t) - \cdot|^{p-1}$$

converges weakly in $L^q(\mathbb{R})$ to a weakly continuous map $L\colon [0,T] \to L^q(\mathbb{R})$ which we call the *order p local time* of S. We denote $\mathcal{L}_p^q(\pi)$ the set of continuous paths S with this property.

Intuitively, the limit $L_t(x)$ then measures the rate at which the path S accumulates pth order variation near x. This definition is further justified by the following result, which is a 'pathwise Tanaka formula' [30] for paths of arbitrary regularity.

Theorem 3.2 (Pathwise "Tanaka" formula for paths with finite pth order variation)**.** *Let $p \in 2\mathbb{N}$ be an even integer, $q \in [1,\infty]$ with conjugate exponent $q' = q/(q-1)$. Let $f \in C^{p-1}(\mathbb{R},\mathbb{R})$ and assume that $f^{(p-1)}$ is weakly differentiable with derivative in $L^{q'}(\mathbb{R})$. Then for any $S \in \mathcal{L}_p^q(\pi)$ the pointwise limit of compensated Riemann sums*

$$\int_0^t f'(S(s))\mathrm{d}S(s) := \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!}(S(t_{j+1}\wedge t) - S(t_j\wedge t))^k$$

exists and the following change of variable formula holds:

$$f(S(t)) - f(S(0)) = \int_0^t f'(S(s))\mathrm{d}S(s) + \frac{1}{(p-1)!}\int_{\mathbb{R}} f^{(p)}(x)L_t(x)\mathrm{d}x.$$

Proof. The formula (7) is exact and does not involve any error terms. Noting that $L^{q'}(\mathbb{R}) \subset (L^q)^*(\mathbb{R})$ also for $q = \infty$, our assumptions imply that the second term on the right hand side of (7) converges, so the result follows. □

To justify the name "local time" for L, we illustrate how L is related to classical definitions of local times by restricting our attention to a particular sequence of partitions [6, 20].

Definition 3.3. Let $S \in C([0,T],\mathbb{R})$. The *dyadic Lebesgue partition* generated by S is defined via $\tau_0^n := 0$ and

$$\tau_{j+1}^n := \inf\{t \geq \tau_j^n : S_t \in 2^{-n}\mathbb{Z} \setminus \{S_{\tau_j^n}\}\},$$

and then $\pi_n = (\{\tau_j^n : j \in \mathbb{N}_0\} \cap [0,T]) \cup \{T\}$.

Lemma 3.4. *Let $p \in \mathbb{N}$ be even, let $S \in C([0,T],\mathbb{R})$, and let (π_n) be the dyadic Lebesgue partition generated by S. Given an interval $[a,b]$ we write $U_t([a,b])$ for the number of upcrossings of $[a,b]$ that S performs until time t. Let $x \in \mathbb{R}$ and let $I_k^n = (k2^{-n},(k+1)2^{-n}]$ be the unique dyadic interval of generation n with $x \in I_k^n$. Then*

$$L_t^{\pi_n}(x) = (|(k+1)2^{-n} - x|^{p-1} + |x - k2^{-n}|^{p-1})U_t(I_k^n) + O(2^{-n(p-1)}).$$

Proof. We have $\mathbf{1}_{(\!(S_{\tau_j^n}, S_{\tau_{j+1}^n}]\!]}(x) \neq 0$ if either $S_{\tau_j^n} = k2^{-n}$ and $S_{\tau_{j+1}^n} = (k+1)2^{-n}$ (i.e., S performs an upcrossing of I_k^n), or $S_{\tau_j^n} = (k+1)2^{-n}$ and $S_{\tau_{j+1}^n} = k2^{-n}$ (i.e., S performs a downcrossing of I_k^n). In the first case we have to add $|(k+1)2^{-n} - x|^{p-1}$ to $L_t^{\pi_n}(x)$, and in the second case we add $(-1)^p|x - k2^{-n}|^{p-1} = |x - k2^{-n}|^{p-1}$. Therefore, we obtain

$$L_t^{\pi_n}(x) = |(k+1)2^{-n} - x|^{p-1}U_t(I_k^n) + |x - k2^{-n}|^{p-1}D_t(I_k^n) + O(2^{-n(p-1)}),$$

and since up- and downcrossings of I_k^n differ by at most one, our claim follows. □

Note that the expression for $L_t^{\pi_n}$ strongly fluctuates on I_k^n. For $x \simeq k2^{-n}$ and $x \simeq (k+1)2^{-n}$ the factor in front of $U_t(I_k^n)$ is $\simeq 2^{-n(p-1)}$, while for $x = (2k+1)2^{-n-1}$ we get the factor $2^{-n(p-1)}2^{p-2}$. Therefore, we do not expect $L_t^{\pi_n}(x)$ to converge uniformly or even pointwise in x as $n \to \infty$ (unless if $p = 2$).

Lemma 3.5. *In the setting of Lemma 3.4 set*

$$\tilde{L}_t^{\pi_n}(x) := \sum_{k\in\mathbb{Z}} 2^{-n(p-1)}U_t(I_k^n)\mathbf{1}_{I_k^n}(x).$$

Let $q \in (1,\infty)$. If $\tilde{L}_t^{\pi_n}$ converges weakly in $L^q(\mathbb{R})$ to a limit $\tilde{L}_t$, then $L_t^{\pi_n}$ converges weakly in $L^q(\mathbb{R})$ to $(2/p)\tilde{L}_t$.

Proof. Let us introduce an averaging operator,

$$(\mathcal{A}_n f)(x) := \sum_{k\in\mathbb{Z}} 2^n \int_{I_k^n} f(y)\mathrm{d}y\, \mathbf{1}_{I_k^n}(x).$$

Since

$$\int_{I_k^n} (|(k+1)2^{-n} - x|^{p-1} + |x - k2^{-n}|^{p-1})\mathrm{d}x = 2\int_0^{2^{-n}} x^{p-1}\mathrm{d}x = \frac{2}{p}2^{-np},$$

we have $\tilde{L}_t^{\pi_n} = \frac{p}{2}\mathcal{A}_n L_t^{\pi_n} + O(2^{-n(p-1)})$, with a compactly supported remainder $O(2^{-n(p-1)})$. We claim that if (f_n) is a sequence of functions for which $\mathcal{A}_n f_n$ converges weakly in $L^q(\mathbb{R})$ and for which $|f_n| \leq C|\mathcal{A}_n f_n|$, then also (f_n) converges weakly in $L^q(\mathbb{R})$ to the same limit, which will imply our claim. To show this, let f be the limit of $\mathcal{A}_n$ and let $g \in L^{q'}(\mathbb{R})$. We have $\langle \mathcal{A}_n\varphi, \psi\rangle = \langle \mathcal{A}_n\varphi, \mathcal{A}_n\psi\rangle = \langle \varphi, \mathcal{A}_n\psi\rangle$ for all φ, ψ, and therefore

$$\begin{aligned}|\langle f_n - f, g\rangle| &\leq |\langle f_n - \mathcal{A}_n f_n, g\rangle| + |\langle \mathcal{A}_n f_n - f, g\rangle| \\ &= |\langle f_n, g - \mathcal{A}_n g\rangle| + |\langle \mathcal{A}_n f_n - f, g\rangle| \\ &\leq \|f_n\|_{L^q}\|g - \mathcal{A}_n g\|_{L^{q'}} + |\langle \mathcal{A}_n f_n - f, g\rangle|.\end{aligned}$$

The second term on the right hand side converges to zero by assumption. For the first term we note that by assumption $\|f_n\|_{L^q} \leq \|\mathcal{A}_n f_n\|_{L^q}$, which is uniformly bounded in n because $(\mathcal{A}_n f_n)$ converges weakly in L^q. The proof is therefore

complete once we show that $\lim_{n\to\infty} \|g - \mathcal{A}_n g\|_{L^{q'}} = 0$ for all $g \in L^{q'}$. But this easily follows from the fact that the continuous and compactly supported functions are dense in $L^{q'}$. $\square$

In fact, we conjecture that, for fractional Brownian motion, this notion of local time defined along the dyadic Lebesge partition coincides, up to a constant, with the usual concept of local time defined as the density of the occupation measure.

Conjecture. *Let B be the fractional Brownian motion with Hurst parameter $H \in (0,1)$, and let (π_n) be the dyadic Lebesgue partition generated by B. Let I_k^n and U_t be as in Lemma 3.4 (where now we count the upcrossings of B instead of S). We conjecture that*

$$\tilde{L}_t^{\pi_n}(x) := \sum_{k\in\mathbb{Z}} 2^{-n(1/H-1)} U_t(I_k^n) \mathbf{1}_{I_k^n}(x)$$

almost-surely converges uniformly in $(t,x) \in [0,T] \times \mathbb{R}$ to $\ell_t(x)\mathbb{E}[|B_1|^{1/H}]/2$, where ℓ is the local time of B, i.e., the Radon-Nikodym derivative of the occupation measure $A \mapsto \int_0^t \mathbf{1}_A(B(s))\mathrm{d}s$ with respect to the Lebesgue measure; see, e.g., [3]. In particular, for any even integer $p \in 2\mathbb{N}$, $B \in \mathcal{L}_q^{p-1}(\pi_n)$ for any $q \in (1,\infty)$.

This result is well known for $H = 1/2$; see, e.g., [6, 24]. In the general case $H \in (0,1)$, it is natural to expect that

$$\mu^n([0,t]) := \sum_{j=0}^{\infty} 2^{-n/H} \mathbf{1}_{\tau_{j+1}^n \le t} \xrightarrow{n\to\infty} [B]_t^{1/H} = \mathbb{E}[|B_1|^{1/H}]t,$$

which would be an extension of the convergence result of [27] from deterministic partitions to the Lebesgue partition generated by B. Moreover, we know that the local time ℓ of the fractional Brownian motion satisfies

$$\ell_t(x) = \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} 2^n \int_0^t \mathbf{1}_{I_k^n}(B_s)\mathrm{d}s \mathbf{1}_{I_k^n}(x).$$

If we *formally* replace the Lebesgue measure in the integral by $\mathbb{E}[|B_1|^{1/H}]^{-1}\mu^n$, then we get

$$\begin{aligned}
\ell_t(x) &= \mathbb{E}[|B_1|^{1/H}]^{-1} \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} 2^n \int_0^t \mathbf{1}_{I_k^n}(B_s)\mu^n(\mathrm{d}s)\mathbf{1}_{I_k^n}(x) \\
&= \mathbb{E}[|B_1|^{1/H}]^{-1} \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} 2^{n-n/H} \sum_{j:\tau_{j+1}^n \le t} \mathbf{1}_{I_k^n}(B_{\tau_j^n})\mathbf{1}_{I_k^n}(x) \\
&= \mathbb{E}[|B_1|^{1/H}]^{-1} \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} 2^{n-n/H}(D_t(I_k^n) + U_t(I_{k+1}^n))\mathbf{1}_{I_k^n}(x),
\end{aligned}$$

and if we further assume that $2^{n-n/H}|U_t(I_{k+1}^n) - U_t(I_k^n)| \to 0$, then our conjecture formally follows.

If the conjecture holds, then for any $p \in 2\mathbb{N}$ and B a typical sample path of the fractional Brownian motion with Hurst index $1/p$ and $f \in C^{p-1}$ with weak pth derivative $f^{(p)} \in L^q$ for any $q \in (1,\infty)$:

$$f(B(t)) - f(B(0)) = \int_0^t f'(B(s))\mathrm{d}B(s) + \frac{\mathbb{E}[|B_1|^p]}{p!} \int_{\mathbb{R}} f^{(p)}(x)\ell_t(x)\mathrm{d}x, \tag{8}$$

where ℓ is the local time of B and

$$\int_0^t f'(B(s))\mathrm{d}B(s) := \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(B(t_j))}{k!}(B(t_{j+1}\wedge t) - B(t_j\wedge t))^k.$$

By Theorem 1.5 the formula holds for $f \in C^p$, because then

$$\frac{\mathbb{E}[|B_1|^p]}{p!}\int_{\mathbb{R}} f^{(p)}(x)\ell_t(x)\mathrm{d}x = \frac{\mathbb{E}[|B_1|^p]}{p!}\int_0^t f^{(p)}(S(s))\mathrm{d}s = \frac{1}{p!}\int_0^t f^{(p)}(S(s))\mathrm{d}[S]_s^p,$$

which adds further credibility to our conjecture.

4. Extension to multidimensional paths

As in the case $p = 2$, the set $V_p(\pi)$ is not stable under linear combinations: for $S_1, S_2 \in V_p(\pi)$, expanding $((S_1(t_{j+1}) - S_1(t_j) + S_2(t_{j+1}) - S_2(t_j))^p$ yields many cross terms whose sum cannot be controlled in general as the partition is refined. The extension of Definition 1.1 to vector-valued functions $S = (S_1, \ldots, S_d)$ therefore requires some care. The original approach of Föllmer [14] was to require that $S_i, S_i + S_j \in V_p(\pi)$. We propose here a slightly different formulation, which is equivalent to Föllmer's construction for $p = 2$ but easier to relate to other approaches, such as rough path integration.

4.1. Tensor formulation.

Define $T_p(\mathbb{R}^d) = \mathbb{R}^d \otimes \ldots \otimes \mathbb{R}^d$ as the space of p-tensors on $\mathbb{R}^d$. A *symmetric p-tensor* is a tensor $T \in T_p(\mathbb{R}^d)$ that is invariant under any permutation σ of its arguments:

$$\forall (v_1, v_2, \ldots, v_p) \in (\mathbb{R}^d)^p, \qquad T(v_1, v_2, \ldots, v_p) = T(v_{\sigma 1}, v_{\sigma 2}, \ldots, v_{\sigma p}).$$

The coordinates $(T_{i_1 i_2 \cdots i_p})$ of a symmetric tensor of order p satisfy

$$T_{i_1 i_2 \cdots i_p} = T_{i_{\sigma 1} i_{\sigma 2} \cdots i_{\sigma p}}.$$

The space $\mathrm{Sym}_p(\mathbb{R}^d)$ of symmetric tensors of order p on $\mathbb{R}^d$ is naturally isomorphic to the dual of the space $\mathbb{H}_p[X_1, \ldots, X_d]$ of symmetric homogeneous polynomials of degree p on $\mathbb{R}^d$. We set $\mathrm{Sym}_0(\mathbb{R}^d) := \mathbb{R}$.

An important example of a symmetric p-tensor on $\mathbb{R}^d$ is given by the pth order derivative of a smooth function:

$$\forall f \in C^p(\mathbb{R}^d, \mathbb{R}), \forall x \in \mathbb{R}^d : \qquad \nabla^p f(x) \in \mathrm{Sym}_p(\mathbb{R}^d).$$

The symmetry property is obtained by repeated application of Schwarz's lemma.

We define $\mathbb{S}_p(\mathbb{R}^d)$ as the direct sum of $\mathrm{Sym}_k(\mathbb{R}^d)$ for $k = 0, 1, 2, \ldots, p$:

$$\mathbb{S}_p(\mathbb{R}^d) = \bigoplus_{k=0}^{p} \mathrm{Sym}_k(\mathbb{R}^d).$$

The space $\mathbb{S}_p(\mathbb{R}^d)$ is naturally isomorphic to the dual of the space $\mathbb{R}_p[X_1, \ldots, X_d]$ of polynomials of degree $\leq p$ in d variables, which defines a bilinear product

$$\langle \cdot, \cdot \rangle \colon \mathbb{S}_p(\mathbb{R}^d) \times \mathbb{R}_p[X_1, \ldots, X_d] \to \mathbb{R}.$$

Slightly abusing notation, we also write $\langle \cdot, \cdot \rangle$ for the canonical inner product on $T_p(\mathbb{R}^d)$. Consider now a continuous $\mathbb{R}^d$-valued path $S \in C([0, T], \mathbb{R}^d)$ and a sequence of partitions $\pi_n = \{t_0^n, \ldots, t_{N(\pi_n)}^n\}$ with $t_0^n = 0 < \ldots < t_k^n < \ldots < t_{N(\pi_n)}^n = T$.

Then

$$\mu^n := \sum_{[t_j,t_{j+1}]\in\pi_n} \delta(\cdot - t_j)\underbrace{(S(t_{j+1}) - S(t_j)) \otimes \cdots \otimes (S(t_{j+1}) - S(t_j))}_{p \text{ times}}$$

defines a tensor-valued measure on $[0,T]$ with values in $\mathrm{Sym}_p(\mathbb{R}^d)$. This space of measures is in duality with the space $C([0,T], \mathbb{H}_p[X_1,\dots,X_d])$ of continuous functions taking values in homogeneous polynomials of degree p, i.e., homogeneous polynomials of degree p with continuous time-dependent coefficients.

Definition 4.1 (pth variation of a multidimensional function)**.** Let $p \in \mathbb{N}$ be even, let $S \in C([0,T],\mathbb{R}^d)$ be a continuous path, and let $\pi = (\pi_n)_{n\geq 1}$ be a sequence of partitions of $[0,T]$. Consider the sequence of tensor-valued measures

$$\mu^n := \sum_{[t_j,t_{j+1}]\in\pi_n} \delta(\cdot - t_j)(S(t_{j+1}) - S(t_j))^{\otimes p}.$$

We say that S has a pth variation along $\pi = (\pi_n)_{n\geq 1}$ if $\mathrm{osc}(S,\pi_n) \to 0$ and there exists a $\mathrm{Sym}_p(\mathbb{R}^d)$–valued measure μ_S without atoms such that for all $f \in C([0,T], \mathbb{H}_p[X_1,\dots,X_d])$

$$\lim_{n\to\infty} \int_0^T \langle f, \mathrm{d}\mu_n\rangle = \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \langle f(t_j), (S(t_{j+1}) - S(t_j))^{\otimes p}\rangle = \int_0^T \langle f, \mathrm{d}\mu_S\rangle.$$

In that case we write $S \in V_p(\pi)$ and we call $[S]^p \colon [0,T] \to \mathrm{Sym}_p(\mathbb{R}^d)$ defined by

$$[S]^p(t) := \mu([0,t])$$

the *pth variation* of S.

By analogy with the positivity property of symmetric matrices, we say that a symmetric p-tensor $T \in \mathrm{Sym}_p(\mathbb{R}^d)$ is positive if

$$\langle T, v \otimes \ldots \otimes v\rangle \geq 0 \qquad \forall v \in \mathbb{R}^d.$$

We denote the set of positive symmetric p-tensors by $\mathrm{Sym}_p^+(\mathbb{R}^d)$. For $T, \tilde{T} \in \mathrm{Sym}_p(\mathbb{R}^d)$ we write $T \geq \tilde{T}$ if $T - \tilde{T} \in \mathrm{Sym}_p^+(\mathbb{R}^d)$. This defines a partial order on $\mathrm{Sym}_p(\mathbb{R}^d)$.

Property 4.2. Let $S \in V_p(\pi) \cap C([0,T],\mathbb{R}^d)$. Then

(i) $[S]^p$ has finite variation and is increasing in the sense of the partial order on $\mathrm{Sym}_p(\mathbb{R}^d)$:

$$[S]^p(t+h) - [S]^p(t) \in \mathrm{Sym}_p^+(\mathbb{R}^d) \qquad \forall\, 0 \leq t \leq t+h \leq T.$$

(ii) $$\forall t \in [0,T], \qquad \sum_{\pi_n} (S(t_{j+1}\wedge t) - S(t_j \wedge t))^{\otimes p} \overset{n\to\infty}{\to} [S]^p(t).$$

Proof. Let $v \in \mathbb{R}^d$. Before passing to the limit, the function

$$\sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_j\leq t}} \langle v^{\otimes p}, (S(t_{j+1}) - S(t_j))^{\otimes p}\rangle = \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_j\leq t}} |v\cdot(S(t_{j+1}) - S(t_j))|^p$$

is increasing in t, and therefore it defines a finite (positive) measure. By assumption, this measure converges weakly to the measure defined by $(a,b] \mapsto \int_0^T \langle \mathbf{1}_{(a,b]} v^{\otimes p}, \mathrm{d}\mu_S \rangle$. In particular, we have

$$\langle v^{\otimes p}, [S]^p(t+h) - [S]^p(t) \rangle = \int_0^T \langle \mathbf{1}_{(t,t+h]} v^{\otimes p}, \mathrm{d}\mu_S \rangle \geq 0.$$

Thus, $\langle v^{\otimes p}, [S]^p \rangle$ is increasing for all $v \in \mathbb{R}^d$, and from here it is easy to see that $[S]^p$ has finite variation (apply, e.g., polarization to go from $v^{\otimes p}$ to $v_1 \otimes \cdots \otimes v_p$). $\square$

Theorem 4.3 (Change of variable formula for paths with finite pth variation)**.** *Let $p \in \mathbb{N}$ be even, let (π_n) be a sequence of partitions of $[0,T]$, and let $S \in V_p(\pi) \cap C([0,T], \mathbb{R}^d)$. Then for all $f \in C^p(\mathbb{R}^d, \mathbb{R})$ the limit of compensated Riemann sums*

$$\int_0^t \langle \nabla f(S(s)), \mathrm{d}S(s) \rangle := \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p-1} \frac{1}{k!} \langle \nabla^k f(S(t_j)), (S(t_{j+1}\wedge t) - S(t_j \wedge t))^{\otimes k} \rangle$$

exists for every $t \in [0,T]$ and satisfies the pathwise change of variable formula:

$$f(S(t)) - f(S(0)) = \int_0^t \langle \nabla f(S(s)), \mathrm{d}S(s) \rangle + \frac{1}{p!} \int_0^t \langle \nabla^p f(S(s)), \mathrm{d}[S]^p(s) \rangle.$$

Proof. The proof follows similar ideas to the case $p = 2$. By applying a Taylor expansion at order p to the increments of $f(S)$ along the partition, we obtain

(9)

$$\begin{aligned} & f(S(t)) - f(S(0)) \\ &\quad = \sum_{[t_j,t_{j+1}]\in\pi_n} (f(S(t_{j+1}\wedge t)) - f(S(t_j \wedge t))) \\ &\quad = \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p} \frac{1}{k!} \langle \nabla^k f(S(t_j)), (S(t_{j+1}\wedge t) - S(t_j \wedge t))^{\otimes k} \rangle \\ &\qquad + \sum_{[t_j,t_{j+1}]\in\pi_n} \int_0^1 \mathrm{d}\lambda \frac{(1-\lambda)^{p-1}}{(p-1)!} \\ &\qquad\qquad \times \big\langle \big(\nabla^p f(S(t_j) + \lambda(S(t_{j+1}\wedge t) - S(t_j \wedge t))) - \nabla^p f(S(t_j))\big), \\ &\qquad\qquad\qquad (S(t_{j+1}\wedge t) - S(t_j \wedge t))^{\otimes p} \big\rangle. \end{aligned}$$

As in the proof of Theorem 1.5 we assume that f is compactly supported and use this to show that the remainder on the right hand side vanishes as $n \to \infty$. Since $S \in V_p(\pi)$ we know that

$$\begin{aligned} &\lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \frac{1}{p!} \langle \nabla^k f(S(t_j)), (S(t_{j+1}\wedge t) - S(t_j \wedge t))^{\otimes p} \rangle \\ &\qquad = \frac{1}{p!} \int_0^t \langle \nabla^p f(S(s)), \mathrm{d}[S]^p(s) \rangle, \end{aligned}$$

and therefore we obtain from (9)

$$\lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p-1} \frac{1}{k!}\langle \nabla^k f(S(t_j)), (S(t_{j+1}\wedge t) - S(t_j \wedge t))^{\otimes k}\rangle$$
$$= f(S(t)) - f(S(0)) - \frac{1}{p!}\int_0^t f^{(p)}(S(s))\mathrm{d}[S]^p(s),$$

and we simply define $\int_0^t \langle \nabla f(S(s)), \mathrm{d}S(s)\rangle$ as the limit on the left hand side. □

4.2. **Relation with rough path integration.** To explain the link between Föllmer's pathwise Itô integral and rough path integration [21], Friz and Hairer [17, Chapter 5.3] introduced the notion of (second order) reduced rough path.

Definition 4.4. Let $\alpha \in (1/3, 1/2)$. We set $\Delta_T := \{(s,t) : 0 \le s \le t \le T\}$. A *reduced rough path* of regularity α is a pair $(X, \mathbb{X})\colon \Delta_T \to \mathbb{R}^d \oplus \mathrm{Sym}_2(\mathbb{R}^d)$, such that

(i) there exists $C > 0$ with

$$|X_{s,t}| + \sqrt{|\mathbb{X}_{s,t}|} \le C|t-s|^\alpha, \qquad (s,t) \in \Delta_T;$$

(ii) the *reduced Chen relation* holds

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = \mathrm{Sym}(X_{s,u} \otimes X_{u,t}), \qquad (s,u),(u,t) \in \Delta_T,$$

where $\mathrm{Sym}(\cdot)$ denotes the symmetric part.

Friz and Hairer [17] also show that, for any $S \in V_2(\pi)$, there is a canonical candidate for a reduced rough path. Indeed, the pair

$$X_{s,t} := S(t) - S(s), \qquad \mathbb{X}_{s,t} := \frac{1}{2}X_{s,t} \otimes X_{s,t} - \frac{1}{2}([S]^2(t) - [S]^2(s)),$$

satisfies the reduced Chen relation. But in general we do not know anything about the Hölder regularity of $S \in V_2(\pi)$, because for any continuous path S there exists a sequence of partitions (π_n) with $S \in V_2(\pi)$ and $[S]^2 \equiv 0$; see [16]. If, however, we take the dyadic Lebesgue partition (π_n) generated by S as in Definition 3.3 and if $S \in V_2(\pi)$, then it follows from [4, Lemme 1][1] that S has finite q-variation for any $q > 2$. So in that case every $S \in V_2(\pi)$ corresponds to a reduced rough path with p-variation regularity. Rather than adapting Definition 4.4 from Hölder to p-variation regularity, we directly introduce a concept of higher-order reduced rough paths. For that purpose we first define the concept of *control function*.

Definition 4.5. A *control function* is a continuous map $c\colon \Delta_T \to \mathbb{R}_+$ such that $c(t,t) = 0$ for all $t \in [0,T]$ and such that $c(s,u) + c(u,t) \le c(s,t)$ for all $0 \le s \le u \le t \le T$.

A function $f\colon [0,T] \to \mathbb{R}^d$ has finite p-variation if and only if there exists a control function c with $|f(t) - f(s)|^p \le c(s,t)$, and in that case $\|f\|_{\text{p-var}} \le c(0,T)^{1/p}$.

[1]Note that for $\lambda > 0$ the path S has finite q-variation if and only if $\lambda^{-1}S$ has finite q-variation, and therefore we can assume that $\lambda = 1$ in [4, Lemme 1].

Definition 4.6. Let $p \geq 1$. A *reduced rough path* of finite p-variation is a tuple

$$\mathbb{X} = (1, \mathbb{X}^1, \dots, \mathbb{X}^{\lfloor p \rfloor}) \colon \Delta_T \longrightarrow \mathbb{S}_{\lfloor p \rfloor}(\mathbb{R}^d),$$

such that

(i) there exists a control function c with

$$\sum_{k=1}^{\lfloor p \rfloor} |\mathbb{X}^k_{s,t}|^{p/k} \leq c(s,t), \qquad (s,t) \in \Delta_T;$$

(ii) the *reduced Chen relation* holds

$$\mathbb{X}_{s,t} = \mathrm{Sym}(\mathbb{X}_{s,u} \otimes \mathbb{X}_{u,t}), \qquad (s,u),(u,t) \in \Delta_T,$$

where the symmetric part of $T \in T_k(\mathbb{R}^d)$ is defined as

$$\mathrm{Sym}(T) := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma T, \qquad \sigma T(v_1, \dots, v_k) := T(v_{\sigma 1}, \dots, v_{\sigma k}),$$

where the sum is across the group of permutations $\mathfrak{S}_k$ of $\{1, \dots, k\}$.

Lemma 4.7. *Let $S \in C([0,T], \mathbb{R}^d)$ and let (π_n) be the dyadic Lebesgue partition generated by S. Let $p \geq 1$ and assume that $S \in V_p(\pi)$. Then for any $q > p$ with $\lfloor q \rfloor = \lfloor p \rfloor$ we obtain a reduced rough path of finite q-variation by setting $\mathbb{X}^0_{s,t} := 1$,*

$$\mathbb{X}^k_{s,t} := \frac{1}{k!}(S(t) - S(s))^{\otimes k}, \qquad k = 1, \dots, \lfloor p \rfloor - 1,$$
$$\mathbb{X}^{\lfloor p \rfloor}_{s,t} := \frac{1}{\lfloor p \rfloor!}(S(t) - S(s))^{\otimes \lfloor p \rfloor} - \frac{1}{\lfloor p \rfloor!}([S]^p(t) - [S]^p(s)).$$

Proof. Let $q > p$. As discussed above we know that S has finite q-variation, so let us start by setting

$$\tilde{c}(s,t) := \|S\|^q_{\text{q-var},[s,t]} := \sup_{\pi \in \Pi([s,t])} \sum_{[t_j, t_{j+1}] \in \pi} |S(t_{j+1}) - S(t_j)|^q, \qquad (s,t) \in \Delta_T,$$

which is a control function such that

$$\sum_{k=1}^{\lfloor p \rfloor} |\mathbb{X}^k_{s,t}|^{q/k} \leq C_{d,p}\big(\tilde{c}(s,t) + |[S]^p(t) - [S]^p(s)|^{q/\lfloor p \rfloor}\big),$$

with a constant $C_{d,p} > 0$ that only depends on the dimension d and on p. By Property 4.2 the path $[S]^p$ has finite variation and therefore it also has finite $q/\lfloor p \rfloor$-variation, so

$$\tilde{\tilde{c}}(s,t) := \|[S]^p\|^{q/\lfloor p \rfloor}_{\text{q}/\lfloor \text{p} \rfloor\text{-var},[s,t]}$$

defines another control function. Therefore, $c(s,t) := C_{d,p}(\tilde{c}(s,t) + \tilde{\tilde{c}}(s,t))$ is a control function for which the analytic property (i) in Definition 4.6 holds.

To show the reduced Chen relation let us write $\mathfrak{S}_{\ell,k}$ for $0 \leq \ell, k$ for the *shuffles* of words of length ℓ, k, i.e., for those permutations $\sigma \in \mathfrak{S}_{\ell+k}$ which satisfy $\sigma i < \sigma j$ for all $1 \leq i < j \leq \ell$, respectively, $\ell + 1 \leq i < j \leq k$. Note that there are $\binom{\ell+k}{\ell}$

shuffles in $\mathfrak{S}_{\ell,k}$. We have for $k < \lfloor p \rfloor$

$$\begin{aligned}\mathbb{X}^k_{s,t} &= \frac{1}{k!}(S(t)-S(s))^{\otimes k} = \frac{1}{k!}(S(t)-S(u)+S(u)-S(s))^{\otimes k}\\ &= \frac{1}{k!}\sum_{\ell=0}^{k}\sum_{\sigma\in\mathfrak{S}_{\ell,k-\ell}} \sigma\big((S(u)-S(s))^{\otimes \ell}\otimes (S(t)-S(u))^{\otimes(k-\ell)}\big),\end{aligned}$$

where we set $v^{\otimes 0} := 1$ for all $v \in R^d$. On the other hand, if $\mathcal{P}_k$ denotes the projection onto $T_k(\mathbb{R}^d)$, then for $k < \lfloor p \rfloor$

$$\begin{aligned}&\mathcal{P}_k(\mathrm{Sym}(\mathbb{X}_{s,u}\otimes \mathbb{X}_{u,t}))\\ &\quad= \sum_{\ell=0}^{k}\mathrm{Sym}(\mathbb{X}^{\ell}_{s,u}\otimes \mathbb{X}^{k-\ell}_{u,t})\\ &\quad= \sum_{\ell=0}^{k}\frac{1}{\ell!(k-\ell)!}\mathrm{Sym}\big((S(u)-S(s))^{\otimes\ell}\otimes (S(t)-S(u))^{\otimes(k-\ell)}\big)\\ &\quad= \sum_{\ell=0}^{k}\frac{1}{\ell!(k-\ell)!}\binom{k}{\ell}^{-1}\sum_{\sigma\in\mathfrak{S}_{\ell,k-\ell}}\sigma((S(u)-S(s))^{\otimes\ell}\otimes (S(t)-S(u))^{\otimes(k-\ell)})\\ &\quad= \mathbb{X}^k_{s,t},\end{aligned}$$

which proves the reduced Chen relation for $k < \lfloor p \rfloor$. For $k = \lfloor p \rfloor$ we get the same relation by noting that $[S]^p$ is already symmetric and therefore

$$\mathrm{Sym}([S]^p(t)-[S]^p(s)) = [S]^p(t)-[S]^p(s).$$

□

The following space of (higher-order) controlled paths in the sense of Gubinelli [19] is defined for example in [17, Chapter 4.5]. We adapt the definition to paths that are controlled in the p-variation sense by a reduced rough path. If $\ell < k$ and $T \in T_\ell$, $\tilde{T} \in T_k$, then we interpret

$$\langle T, \tilde{T}\rangle \in T_{k-\ell}, \qquad \langle T,\tilde{T}\rangle(v_1,\dots,v_{k-\ell}) := \langle T\otimes(v_1\otimes\cdots\otimes v_{k-\ell}),\tilde{T}\rangle,$$

and similarly for $\langle \tilde{T}, T\rangle$.

Definition 4.8. Let $p \geq 1$ and let $\mathbb{X}$ be a reduced rough path of finite p-variation. A path

$$Y = (Y^1,\dots,Y^{\lfloor p\rfloor}) \in C([0,T],\mathbb{S}_{\lfloor p\rfloor}(\mathbb{R}^d))$$

is *controlled* by $\mathbb{X}$ if there exists a control function c such that

$$\sum_{\ell=1}^{\lfloor p\rfloor}\Big|Y^\ell(t)-\sum_{k=\ell}^{\lfloor p\rfloor}\langle Y^k(s),\mathbb{X}^{k-\ell}_{s,t}\rangle\Big|^{\frac{p}{\lfloor p\rfloor-\ell+1}} \leq c(s,t), \qquad (s,t)\in\Delta_T.$$

In that case we write $Y \in \mathcal{D}^{\lfloor p\rfloor/p}_{\mathbb{X}}([0,T])$.

Example 4.9. Let $p \geq 1$, let S, $\mathbb{X}$ and q be as in Lemma 4.7, and let $f \in C^{\lfloor q\rfloor}(\mathbb{R}^d,\mathbb{R})$. Then $Y^0 := 1$,

$$Y^k(s) := \nabla^k f(S(s)), \qquad k = 1,\dots,\lfloor q\rfloor,$$

defines a controlled path in $\mathcal{D}_{\mathbb{X}}^{\lfloor q\rfloor/q}([0,T])$. Indeed, as we discussed above $\nabla^k f(S(s)) \in \mathrm{Sym}_k(\mathbb{R}^d)$ for all $k = 1,\dots,\lfloor q\rfloor$, and by Taylor's formula we have for $\ell \in \{1,\dots,\lfloor q\rfloor\}$

$$\begin{aligned} Y^\ell(t) &= \nabla^\ell f(S(t)) \\ &= \sum_{k=\ell}^{\lfloor q\rfloor} \frac{1}{(k-\ell)!}\langle \nabla^k f(S(s)), (S(t)-S(s))^{\otimes(k-\ell)}\rangle + O(c(s,t)^{(\lfloor q\rfloor-\ell+1)/q}) \\ &= \sum_{k=\ell}^{\lfloor q\rfloor} \langle Y^k(s), \mathbb{X}_{s,t}^{k-\ell}\rangle + O(c(s,t)^{(\lfloor q\rfloor-\ell+1)/q}). \end{aligned}$$

Proposition 4.10. *Let $p \geq 1$, let $\mathbb{X}$ be a reduced rough path of finite p-variation, and let $Y \in \mathcal{D}_{\mathbb{X}}^{\lfloor p\rfloor/p}([0,T])$. Then the rough path integral*

$$I_{\mathbb{X}}(Y)(t) = \int_0^t \langle Y(s), \mathrm{d}\mathbb{X}(s)\rangle = \lim_{\substack{\pi\in\Pi([0,t])\\ |\pi|\to 0}} \sum_{[t_j,t_{j+1}]\in\pi} \sum_{k=1}^{\lfloor p\rfloor} \langle Y^k(t_j), \mathbb{X}_{t_j,t_{j+1}}^k\rangle, \qquad t\in[0,T],$$

defines a function in $C([0,T],\mathbb{R})$, and it is the unique function with $I_{\mathbb{X}}(Y)(0) = 0$ for which there exists a control function c with

$$\Big| \int_s^t \langle Y(r), \mathrm{d}\mathbb{X}(r)\rangle - \sum_{k=1}^{\lfloor p\rfloor} \langle Y^k(s), \mathbb{X}_{s,t}^k\rangle \Big| \lesssim c(s,t)^{\frac{\lfloor p\rfloor+1}{p}}, \qquad (s,t)\in\Delta_T.$$

Proof. This follows from classical arguments (Theorem 4.3 in [22]; see also [19]) once we show that for $0 \leq s \leq u \leq t \leq T$

$$\sum_{k=1}^{\lfloor p\rfloor} \langle Y^k(s), \mathbb{X}_{s,t}^k\rangle - \sum_{k=1}^{\lfloor p\rfloor} \langle Y^k(s), \mathbb{X}_{s,u}^k\rangle - \sum_{k=1}^{\lfloor p\rfloor} \langle Y^k(u), \mathbb{X}_{u,t}^k\rangle = O(c(s,t)^{\frac{\lfloor p\rfloor+1}{p}}),$$

where c is a control function such that the estimates in Definitions 4.6 and 4.8 hold. But

$$\begin{aligned} \sum_{k=1}^{\lfloor p\rfloor} \langle Y^k(u), \mathbb{X}_{u,t}^k\rangle &= \sum_{k=1}^{\lfloor p\rfloor} \Big(\sum_{\ell=k}^{\lfloor p\rfloor} \langle Y^\ell(s), \mathbb{X}_{s,u}^{\ell-k} \otimes \mathbb{X}_{u,t}^k\rangle + O\big(c(s,u)^{\frac{\lfloor p\rfloor-k+1}{p}} c(u,t)^{\frac{k}{p}}\big)\Big) \\ &= \sum_{k=1}^{\lfloor p\rfloor} \sum_{\ell=1}^{k} \langle Y^k(s), \mathbb{X}_{s,u}^{k-\ell} \otimes \mathbb{X}_{u,t}^\ell\rangle + O\big(c(s,t)^{\frac{\lfloor p\rfloor+1}{p}}\big) \\ &= \sum_{k=1}^{\lfloor p\rfloor} \langle Y^k(s), \mathcal{P}_k(\mathrm{Sym}(\mathbb{X}_{s,u}\otimes\mathbb{X}_{u,t})) - \mathbb{X}_{s,u}^k\rangle + O\big(c(s,t)^{\frac{\lfloor p\rfloor+1}{p}}\big), \end{aligned}$$

where in the last step we used that $Y^k(s)$ is symmetric. Therefore, the reduced Chen relation gives

$$\begin{aligned}
&\sum_{k=1}^{\lfloor p\rfloor}\langle Y^k(s),\mathbb{X}^k_{s,t}\rangle-\sum_{k=1}^{\lfloor p\rfloor}\langle Y^k(s),\mathbb{X}^k_{s,u}\rangle-\sum_{k=1}^{\lfloor p\rfloor}\langle Y^k(u),\mathbb{X}^k_{u,t}\rangle\\
&\quad=\sum_{k=1}^{\lfloor p\rfloor}\langle Y^k(s),\mathbb{X}^k_{s,t}-\mathbb{X}^k_{s,u}-\mathcal{P}_k(\mathrm{Sym}(\mathbb{X}_{s,u}\otimes\mathbb{X}_{u,t}))+\mathbb{X}^k_{s,u}\rangle+O\big(c(s,t)^{\frac{\lfloor p\rfloor+1}{p}}\big)\\
&\quad=\sum_{k=1}^{\lfloor p\rfloor}\langle Y^k(s),\mathbb{X}^k_{s,t}-\mathbb{X}^k_{s,t}\rangle+O\big(c(s,t)^{\frac{\lfloor p\rfloor+1}{p}}\big)=O\big(c(s,t)^{\frac{\lfloor p\rfloor+1}{p}}\big),
\end{aligned}$$

which concludes the proof. □

Corollary 4.11. *Let $p\in\mathbb{N}$ be an even integer and let $q,S,\mathbb{X},f$ be as in Example 4.9. Then*

$$\int_0^t\langle\nabla f(S(s)),\mathrm{d}\mathbb{X}(s)\rangle=\int_0^t\langle\nabla f(S(s)),\mathrm{d}S(s)\rangle,\qquad t\in[0,T],$$

where the left hand side denotes the rough path integral of Proposition 4.10 and the right hand side is the integral of Theorem 4.3.

Proof. It suffices to show that

$$\int_0^t\langle\nabla f(S(s)),\mathrm{d}\mathbb{X}(s)\rangle=f(S(t))-f(S(0))-\frac{1}{p!}\int_0^t\langle\nabla^p f(S(s)),\mathrm{d}[S]^p(s)\rangle,$$

and since

$$\lim_{\substack{\pi\in\Pi([0,t])\\|\pi|\to0}}\sum_{[t_j,t_{j+1}]\in\pi}\langle\nabla^p f(S(t_j)),[S]^p(t_{j+1})-[S]^p(t_j)\rangle=\int_0^t\langle\nabla^p f(S(s)),\mathrm{d}[S]^p(s)\rangle,$$

this is equivalent to

$$\lim_{\substack{\pi\in\Pi([0,t])\\|\pi|\to0}}\sum_{[t_j,t_{j+1}]\in\pi}\sum_{k=1}^{p}\langle\nabla^k f(S(t_j)),\frac{1}{k!}(S(t_{j+1})-S(t_j))^{\otimes k}\rangle=f(S(t))-f(S(0)).$$

The last identity can be shown by writing $f(S(t))-f(S(0))$ as a telescoping sum and by performing a Taylor expansion up to order p and controlling the remainder term as in the proof of Theorem 4.3. □

Appendix: pth variation for odd integer values of p

Lemma .12. *Let $p>1$ be an odd integer and let π_n be the dyadic Lebesgue partition generated by $S\in C([0,T],\mathbb{R})$. Assume that $\nu^n:=\sum_{[t_j,t_{j+1}]\in\pi_n}\delta(\cdot-t_j)|S(t_{j+1})-S(t_j)|^p$ converges weakly to a signed measure ν without atoms. Then we have for all $f\in C(\mathbb{R},\mathbb{R})$*

$$\lim_{n\to\infty}\sum_{[t_j,t_{j+1}]\in\pi_n}f(S(t_j))(S(t_{j+1}\wedge t)-S(t_j\wedge t))^p=0,\qquad t\in[0,T].$$

Proof. We can assume without loss of generality that f has compact support, since the image of S on $[0,T]$ is compact. Let $k\in\mathbb{Z}$ and note that whenever S completes an upcrossing of $I^n_k=[k2^{-n},(k+1)2^{-n}]$ we have to add $f(k2^{-n})2^{-np}$ to the sum. On the other hand, if S completes a downcrossing of I^n_k before t, then we have to

add $-f((k+1)2^{-n})2^{-np}$ to the sum. Let $U_t(I_k^n)$ (resp., $D_t(I_k^n)$) denote the number of up- (resp., down-) crossings of I_k^n by S on $[0,t]$. Since $U_t(I_k^n)$ and $D_t(I_k^n)$ differ by at most 1, we get

$$\begin{aligned}
&\left|\sum_{[t_j,t_{j+1}]\in\pi_n:t_{j+1}\leqslant t} f(S(t_j))(S(t_{j+1})-S(t_j))^p\right| \\
&\quad= \left|\sum_{k\in\mathbb{Z}} 2^{-np}(f(k2^{-n})U_t(I_k^n) - f((k+1)2^{-n})D_t(I_k^n))\right| \\
&\quad\leqslant \left|\sum_{k\in\mathbb{Z}} 2^{-np} f(k2^{-n})(U_t(I_k^n) - D_t(I_k^n))\right| \\
&\qquad + \left|\sum_{k\in\mathbb{Z}} 2^{-np}(f(k2^{-n}) - f((k+1)2^{-n}))D_t(I_k^n)\right| \\
&\quad\leqslant \sum_{k\in\mathbb{Z}} 2^{-np}|f(k2^{-n})| + \sum_{k\in\mathbb{Z}} 2^{-np}|(f(k2^{-n}) - f((k+1)2^{-n}))|N_t(I_k^n) \\
&\quad\leqslant \sum_{k\in\mathbb{Z}} 2^{-np}|f(k2^{-n})| + \omega_f(2^{-n})\sum_{k\in\mathbb{Z}} 2^{-np}N_t(I_k^n),
\end{aligned}$$

where we wrote $N_t(I_k^n) = U_t(I_k^n) + D_t(I_k^n)$ for the total number of interval crossings and where ω_f is the modulus of continuity of f, i.e., $\lim_{n\to\infty}\omega_f(2^{-n}) = 0$. By assumption,

$$\lim_{n\to\infty}\sum_{k\in\mathbb{Z}} 2^{-np}N_t(I_k^n) = \nu([0,t]) \in \mathbb{R},$$

and since $f(k2^{-n}) \neq 0$ for at most $O(2^n)$ values of k and $p > 1$ the claim follows. □

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