KERNEL THEOREMS IN COORBIT THEORY

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Abstract. We prove general kernel theorems for operators acting between coorbit spaces. These are Banach spaces associated to an integrable representation of a locally compact group and contain most of the usual function spaces (Besov spaces, modulation spaces, etc.). A kernel theorem describes the form of every bounded operator between a coorbit space of test functions and distributions by means of a kernel in a coorbit space associated to the tensor product representation. As special cases we recover Feichtinger's kernel theorem for modulation spaces and the recent generalizations by Cordero and Nicola. We also obtain a kernel theorem for operators between the Besov spaces $\dot{B}^0_{1,1}$ and $\dot{B}^0_{\infty,\infty}$.

1. INTRODUCTION

Kernel theorems assert that every "reasonable" operator can be written as a "generalized" integral operator. For instance, the Schwartz kernel theorem states that a continuous linear operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ possesses a unique distributional kernel $K \in \mathcal{S}'(\mathbb{R}^{2d})$ such that

$$\langle Af, g \rangle = \langle K, g \otimes f \rangle, \qquad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{1}$$

If K is a locally integrable function, then

$$\langle Af, g \rangle = \int_{\mathbb{R}^d} K(x, y) f(y) \overline{g(x)} dy dx, \qquad f, g \in \mathcal{S}(\mathbb{R}^d),$$

and thus A has indeed the form of an integral operator. Similar kernel theorems hold for continuous operators from $\mathcal{D}(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ [24, Theorem 5.2] and for Gelfand-Shilov spaces and their distribution spaces [21]. The importance of these kernel theorems stems from the fact that they offer a general formalism for the description of linear operators.

In the context of time-frequency analysis, Feichtinger's kernel theorem [12] (see also [18] and [23, Theorem 14.4.1]) states that every bounded linear operator from the modulation space $M^1(\mathbb{R}^d)$ to the modulation space $M^\infty(\mathbb{R}^d)$ can be represented in the form (1) with a kernel in $M^\infty(\mathbb{R}^{2d})$. The advantage of this kernel theorem is

that both the space of test functions $M^1(\mathbb{R}^d)$ and the distribution space $M^\infty(\mathbb{R}^d) = M^1(\mathbb{R}^d)^*$ are Banach spaces and thus technically easier than the locally convex spaces $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$.

Recently, Cordero and Nicola [8] revisited Feichtinger's kernel theorem and proved several new kernel theorems that "do not have a counterpart in distribution theory". They argue that "this reveals the superiority, in some respects, of the modulation space formalism upon distribution theory." While we agree full-heartedly with this claim, we would like to add a more abstract point of view and argue that the deeper reason for this superiority lies in the theory of coorbit spaces and in the convenience of Schur's test for integral operators. Indeed, we will prove kernel theorems similar to Feichtinger's kernel theorem for many coorbit spaces.

The main idea is to investigate operators in a transform domain after taking a short-time Fourier transform, a wavelet transform, or an abstract wavelet transform, i.e., a continuous transform with respect to a unitary group representation. In this new representation every operator between a suitable space of test functions and distributions is an integral operator. The standard boundedness conditions of Schur's test then yield strong kernel theorems.

The technical framework for this idea is coorbit theory, which was introduced and studied in [15–17, 22] for the construction and analysis of function spaces by means of a generalized wavelet transform. The main idea is that functions in the standard function spaces, such as Besov spaces and modulation spaces, can be characterized by the decay or integrability properties of an associated transform (the wavelet transform or the short-time Fourier transform). In the abstract setting, G is a locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H})$ is an irreducible, unitary, integrable representation of G. Leaving technical details aside, the coorbit space $Co_\pi L_w^p(G)$ consists of all distributions f in a suitable distribution space such that the representation coefficient $g \mapsto \langle f, \pi(g)\psi \rangle$ is in the weighted space $L_w^p(G)$.

Next, let G_1 and G_2 be two locally compact groups, and let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be irreducible, unitary, integrable representations of G_1 and G_2, respectively.

Let A be a bounded linear operator between $Co_{\pi_1} L_{w_1}^1(G_1)$ and $Co_{\pi_2} L_{1/w_2}^\infty(G_2)$. Our main insight is that such an operator can be described by a kernel in a coorbit space that is related to the tensor product representation $\pi = \pi_2 \otimes \pi_1$ of $G = G_1 \times G_2$ on the tensor product space $\mathcal{H}_2 \otimes \mathcal{H}_1$. The following non-technical formulation offers a flavor of our main result in Theorem 3:

A linear operator A is bounded from $Co_{\pi_1} L_{w_1}^1(G_1)$ to $Co_{\pi_2} L_{1/w_2}^\infty(G_2)$ if and only if there exists a kernel $K \in Co_\pi L_{w_1^{-1} \otimes w_2^{-1}}^\infty(G_1 \times G_2)$ such that

$$(2) \qquad\qquad\qquad \langle Av, \varphi \rangle = \langle K, \varphi \otimes v \rangle$$

for all $v \in Co_{\pi_1} L_{w_1}^1(G_1)$, $\varphi \in Co_{\pi_2} L_{w_2}^1(G_2)$.

This statement is not just a mere abstraction and generalization of the classical kernel theorem. With the choice of a specific group and representation one obtains explicit kernel theorems. For instance, using the Schrödinger representation of the Heisenberg group, one recovers Feichtinger's original kernel theorem. The added value is our insight that the conditions on the kernel of [8] in terms of mixed modulation spaces [4] amount to coorbit spaces with respect to the tensor product representation. Choosing the $ax + b$-group and the continuous wavelet representation, one obtains a kernel theorem for all bounded operators between the Besov spaces $\dot{B}_{1,1}^0$ and $\dot{B}_{\infty,\infty}^0$ with a kernel in a space of dominating mixed smoothness.

This class of function spaces has been studied extensively [31,32] and is by no means artificial.

By using suitable versions of Schur's test, it is then possible to derive characterizations for the boundedness of operators between other coorbit spaces. For example, in Theorem 7 we will prove the following, with $\frac{1}{p} + \frac{1}{q} = 1$:

(i) $A : \mathcal{Co}_{\pi_1} L^1_{w_1}(G_1) \to \mathcal{Co}_{\pi_2} L^p_{w_2}(G_2)$ bounded $\Leftrightarrow$ $K \in \mathcal{Co}_\pi \mathcal{L}^{p,\infty}_{1/w_1 \otimes w_2}(G_1 \times G_2)$,

(ii) $A : \mathcal{Co}_{\pi_1} L^p_{w_1}(G_1) \to \mathcal{Co}_{\pi_2} L^\infty_{w_2}(G_2)$ bounded $\Leftrightarrow$ $K \in \mathcal{Co}_\pi L^{q,\infty}_{1/w_1 \otimes w_2}(G_1 \times G_2)$,

where the mixed-norm Lebesgue spaces $\mathcal{L}^{p,q}$ and $L^{p,q}$ on $G_1 \times G_2$ are defined in (23) and (24), respectively.

The paper is organized as follows. In Section 2 we present the basics of tensor products and coorbit space theory. The theory of coorbit spaces of kernels with respect to products of integrable representations is developed in Section 3. Our main results, the kernel theorems, are proved in Section 4 and applied to particular examples of group representations in Section 5.

We note that our proofs require a meaningful formulation of coorbit theory. One can therefore prove kernel theorems also in the context of other coorbit space theories [6,9], e.g., for certain reducible representations.

2. Preliminaries on tensor products and coorbit spaces

2.1. **Tensor products and Hilbert-Schmidt operators.** The theory of tensor products is at the heart of kernel theorems for operators. Algebraically, a simple tensor of two vectors (in two possibly different Hilbert spaces) is a formal product of two vectors $f_1 \otimes f_2$, and the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is obtained by taking the completion of all linear combinations of simple tensors with respect to the inner product

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle := \langle f_1, g_1 \rangle \langle g_2, f_2 \rangle .$$

This tensor product is homogeneous in the following sense: $\alpha \cdot (f_1 \otimes f_2) = (\alpha f_1) \otimes f_2 = f_1 \otimes (\overline{\alpha} f_2)$. Note explicitly that the product $f_1 \otimes f_2$ is anti-linear in the second factor. In some books this is done by introducing the dual Hilbert space $\mathcal{H}'_2$ [25].

If each Hilbert space is an L^2-space $\mathcal{H}_1 = L^2(X, \mu)$, $\mathcal{H}_2 = L^2(Y, \nu)$, then the simple tensor $f \otimes g$ is just the product $(x, y) \mapsto f(x) \cdot \overline{g(y)}$, and the tensor product becomes the product space $\mathcal{H}_1 \otimes \mathcal{H}_2 = L^2(X, \mu) \otimes L^2(Y, \nu) = L^2(X \times Y, \mu \times \nu)$.

The connection between functions and operators arises in the analytic approach to tensor products. We interpret a function of *two* variables as an integral kernel for an operator. Thus a simple tensor $f_1 \otimes f_2$ of two functions becomes the rank one operator $f \mapsto \langle f, f_2 \rangle f_1$ with integral kernel $f_1(x)\overline{f_2(y)}$, and a general $k \in L^2(X \times Y, \mu \times \nu)$ becomes a Hilbert-Schmidt operator from $L^2(Y, \nu)$ to $L^2(X, \mu)$. The systematic, analytic treatment of general tensor products of two Hilbert spaces often defines the tensor product as a space of Hilbert-Schmidt operators between $\mathcal{H}_2$ and $\mathcal{H}_1$. We note that his definition is already based on the characterization of Hilbert-Schmidt operators and thus represents a non-trivial kernel theorem [7]. Whereas the working mathematician habitually identifies an operator with its distributional kernel, we will make the conceptual distinction between tensor products and operators for our study of kernel theorems.

In the sequel we will denote the (distributional) kernel of an integral operator by k and the abstract kernel in a tensor product by K.

2.2. **Coorbit space theory.** Let G be a locally compact group with left Haar measure $\int_G \ldots dg$, let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{U}(\mathcal{H})$ be the group of unitary operators acting on $\mathcal{H}$. A continuous unitary group representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is called *square integrable* [1,11] if it is irreducible and there exist $\psi \in \mathcal{H}$ such that

$$(3) \qquad \int_G |\langle \psi, \pi(g)\psi \rangle|^2 dg < \infty\,.$$

A non-zero vector ψ satisfying (3) is called *admissible*. For every square integrable representation there exists a densely defined operator T such that $\forall f_1, f_2 \in \mathcal{H}$, $\psi_1, \psi_2 \in \mathsf{Dom}\,(T)$, one has

$$(4) \qquad \int_G \langle f_1, \pi(g)\psi_1 \rangle \langle \pi(g)\psi_2, f_2 \rangle dg = \langle T\psi_2, T\psi_1 \rangle \langle f_1, f_2 \rangle.$$

For fixed $\psi_1 = \psi_2 = \psi$ the representation coefficient $f \mapsto V_\psi f(g) := \langle f, \pi(g)\psi \rangle$ is interpreted as a *generalized wavelet transform*. The orthogonality relation (4) then implies that V_ψ is a multiple of an isometry from $\mathcal{H}$ to $L^2(G)$. By using a weak interpretation of vector-valued integrals, (4) can also be recast as the inversion formula

$$(5) \qquad f = \frac{1}{\|T\psi\|^2} \int_G \langle f, \pi(g)\psi \rangle \pi(g)\psi dg\,.$$

For the rest of this paper we assume without loss of generality that the chosen admissible vectors ψ are normalized, i.e., $\|T\psi\| = 1$.

The adjoint operator $V_\psi^* : L^2(G) \to \mathcal{H}$ is formally defined by

$$V_\psi^* F := \int_G F(g)\pi(g)\psi dg.$$

Other domains and convergence properties will be discussed later.

With this notation (5) says that $V_\psi^* V_\psi = I_\mathcal{H}$ for all admissible and normalized vectors ψ, which in the language of recent frame theory means that $\{\pi(g)\psi\}_{g \in G}$ is a continuous Parseval frame. By [5, Proposition 2.1] one can always assume that G is σ-finite since we assume $\mathcal{H}$ to be separable.

In coorbit theory one needs much stronger hypotheses on π. The representation π is called *integrable* with respect to a weight w if there exists an admissible vector $\psi \in \mathcal{H}$ such that

$$(6) \qquad \int_G |\langle \psi, \pi(g)\psi \rangle| w(g)\, dg < \infty.$$

Let $g_1, g_2, g_3 \in G$. We call a weight $w : G \to \mathbb{R}^+$ *submultiplicative* if $w(g_1 g_2) \leqslant w(g_1)w(g_2)$ and a function $m : G \to \mathbb{R}^+$ *w-moderate* if it satisfies $m(g_1 g_2 g_3) \leqslant w(g_1)m(g_2)w(g_3)$. If m is w-moderate, the weighted Lebesgue space $L_m^p(G)$ is then invariant under left translation $L_x f(y) = f(x^{-1}y)$ and under the right translation $R_x f(y) = f(yx)$. Throughout this paper, we will assume that the weight w satisfies

$$(7) \qquad w(x) \geqslant C \max\left\{\alpha(x), \alpha(x^{-1}), \beta(x), \Delta(x^{-1})\beta(x^{-1})\right\},$$

where $\alpha(x) := \|L_x\|_{L_m^p(G) \to L_m^p(G)}$, $\beta(x) := \|R_x\|_{L_m^p(G) \to L_m^p(G)}$, and Δ denotes the modular function of G.

Our standing assumption is that the representation π of G possesses an admissible vector ψ such that $V_\psi \psi \in L^1_w(G)$. We denote the corresponding set by

$$\mathcal{A}_w(G) := \left\{ \psi \in \mathcal{H}, \psi \neq 0 : \ V_\psi \psi \in L^1_w(G) \right\}.$$

For fixed $\psi \in \mathcal{A}_w(G)$ the linear version of $\mathcal{A}_w(G)$,

$$(8) \qquad\qquad \mathcal{H}^1_w := \left\{ f \in \mathcal{H} : \ V_\psi f \in L^1_w(G) \right\},$$

is dense in $\mathcal{H}$. Let $(\mathcal{H}^1_w)^\sim$ denote the anti-dual of $\mathcal{H}^1_w$, i.e., the space of anti-linear continuous functionals on $\mathcal{H}^1_w$. As $\mathcal{H}^1_w$ is dense in $\mathcal{H}$, it follows that the inner product on $\mathcal{H} \times \mathcal{H}$ extends to $(\mathcal{H}^1_w)^\sim \times \mathcal{H}^1_w$ and so does the generalized wavelet transform.

The coorbit space with respect to $L^p_m(G)$ is then defined by

$$\mathcal{C}o_\pi\, L^p_m(G) := \left\{ f \in (\mathcal{H}^1_w)^\sim : \ V_\psi f \in L^p_m(G) \right\}$$

and is equipped with the natural norm

$$\|f\|_{\mathcal{C}o_\pi\, L^p_m(G)} := \|V_\psi f\|_{L^p_m(G)}.$$

With our assumptions on π, ψ, m, the coorbit space $\mathcal{C}o_\pi\, L^p_m(G)$ is a Banach space [16]. Alternatively, $\mathcal{C}o_\pi\, L^p_m(G)$ for $p < \infty$ can be defined as the completion of H^1_w with respect to this norm. Moreover,

$$(9) \quad \mathcal{C}o_\pi\, L^2(G) = \mathcal{H}, \ \ \mathcal{C}o_\pi\, L^1_w(G) = \mathcal{H}^1_w, \quad \text{and} \quad \mathcal{C}o_\pi\, L^\infty_{1/w}(G) = (\mathcal{H}^1_w)^\sim = \mathcal{H}^\infty_{1/w},$$

and

$$(10) \qquad\qquad \mathcal{H}^1_w \subseteq \mathcal{C}o_\pi\, L^p_m(G) \subseteq \mathcal{H}^\infty_{1/w}$$

for $1 \leqslant p \leqslant \infty$ and w-moderate weight m. In the context of coorbit space theory the space $\mathcal{H}^1_w$ serves as a space of test functions, and $\mathcal{H}^\infty_{1/w}$ is the corresponding distribution space.

We quickly recall some of the fundamental properties of coorbit spaces; see for example [16, Theorems 4.1 and 4.2 and Proposition 4.3].

Proposition 1. *Let $\psi, \phi \in \mathcal{A}_w(G)$, $f \in \mathcal{C}o_\pi\, L^p_m(G)$, $g \in \mathcal{C}o_\pi\, L^q_{1/m}(G)$, and $F \in L^p_m(G)$. Then the following properties hold:*

    (i) *$V_\psi : \mathcal{C}o_\pi\, L^p_m(G) \to L^p_m(G)$ is an isometry.*

    (ii) *$\mathcal{H}^p_m$ is invariant with respect to π and*

$$\|\pi(g)f\|_{\mathcal{C}o_\pi\, L^p_m(G)} \leqslant w(g)\|f\|_{\mathcal{C}o_\pi\, L^p_m(G)} \text{ for all } g \in G, f \in \mathcal{C}o_\pi\, L^p_m(G).$$

    (iii) *$V^*_\psi : L^p_m(G) \to \mathcal{C}o_\pi\, L^p_m(G)$ is continuous.*

    (iv) *$V^*_\psi V_\psi = I_{\mathcal{C}o_\pi\, L^p_m(G)}.$*

    (v) *Correspondence principle: Let $F \in L^p_m(G)$. There exists $f \in \mathcal{C}o_\pi\, L^p_m(G)$ such that $F = V_\psi f$ if and only if $F = F * V_\psi \psi$, where $*$ denotes convolution on G.*

    (vi) *Duality: For $1 \leqslant p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, we have $(\mathcal{C}o_\pi\, L^p_m(G))^* = \mathcal{C}o_\pi\, L^q_{1/m}(G)$, where the duality is given by*

$$\langle f, g \rangle_{\mathcal{C}o_\pi\, L^p_m(G), \mathcal{C}o_\pi\, L^q_m(G)} = \langle V_\psi f, V_\psi g \rangle_{L^p_m(G), L^q_{1/m}(G)}.$$

    (vii) *The definition of $\mathcal{C}o_\pi\, L^p_m(G)$ is independent of the particular choice of the window function from $\mathcal{A}_w(G)$. In particular, $\|V_\psi f\|_{L^p_m(G)} \asymp \|V_\phi f\|_{L^p_m(G)}$ for arbitrary non-zero $\phi, \psi \in \mathcal{A}_w(G)$.*

We furthermore need a result on the existence of atomic decompositions for the space $\mathcal{C}o_\pi L_w^1(G)$; see [15, Theorem 4.7].

Theorem 2. *Let $\psi \in \mathcal{A}_w(G)$. There exists a discrete subset $\{g_i\}_{i \in \mathcal{I}} \subset G$ and a collection of linear functionals $\lambda_i : \mathcal{C}o_\pi L_w^1(G) \to \mathbb{C}$, $i \in \mathcal{I}$, such that*

$$(11) \qquad f = \sum_{i \in \mathcal{I}} \lambda_i(f) \pi(g_i)\psi, \quad with \sum_{i \in \mathcal{I}} |\lambda_i(f)| w(g_i) \asymp \|f\|_{\mathcal{C}o_\pi L_w^1(G)},$$

and the sum converges absolutely in $\mathcal{C}o_\pi L_w^1(G)$.

3. Frames and coorbit spaces via tensor products

Let G_1, G_2 be two locally compact groups with unitary square integrable representations $\pi_1 : G_1 \to \mathcal{U}(\mathcal{H}_1)$ and $\pi_2 : G_2 \to \mathcal{U}(\mathcal{H}_2)$. For $g := (g_1, g_2) \in G := G_1 \times G_2$ the tensor representation $\pi : G \to \mathcal{U}(\mathcal{H}_2 \otimes \mathcal{H}_1)$,

$$\pi(g) := \pi_2(g_2) \otimes \pi_1(g_1),$$

acts on a simple tensor $\Psi := \psi_2 \otimes \psi_1 \in \mathcal{H}_2 \otimes \mathcal{H}_1$ by

$$(12) \qquad \pi(g)(\psi_2 \otimes \psi_1) = \pi_2(g_2)\psi_2 \otimes \pi_1(g_1)\psi_1.$$

It follows immediately that π is a unitary representation of G on $\mathcal{H}_2 \otimes \mathcal{H}_1$. Moreover, π is irreducible (e.g., by [34, Section 4.4, Theorem 6]). Note that the order of indices is in agreement with the formulation of the kernel theorem in Theorem 3.

If we interpret the simple tensor $\Psi = \psi_2 \otimes \psi_1$ as the rank-one operator $f \mapsto \psi_1(f)\psi_2$ with $\psi_1 \in \mathcal{H}_1'$, then we can write (12) as

$$\pi(g)(\Psi)(f) = (\pi_1'(g_1)\psi_1)(f) \cdot \pi_2(g_2)\psi_2 = \big(\pi_2(g_2)\psi_2 \otimes \pi_1'(g_1)\psi_1\big)(f),$$

where the contragredient representation $\pi_1' : G_1 \to GL(\mathcal{H}_1')$ of π_1 is defined as $(\pi_1'(g_1)\psi_1)(f) = \psi_1(\pi_1(g_1^{-1})f)$; see [34, Section 3.1].

In case we treat the tensor product as a space of Hilbert-Schmidt operators, π acts on $A \in \mathcal{HS}(\mathcal{H}_1, \mathcal{H}_2)$ as

$$\pi(g)A = \pi_2(g_2)A\pi_1(g_1)^*.$$

The generalized wavelet transform of a simple tensor $f_2 \otimes f_1$ with respect to a "wavelet" $\Psi = \psi_2 \otimes \psi_1$ is given by

$$(13) \qquad \begin{aligned} V_\Psi(f_2 \otimes f_1)(g) &= \langle f_2 \otimes f_1, (\pi_2(g_2) \otimes \pi_1(g_1))(\psi_2 \otimes \psi_1) \rangle \\ &= \langle f_2, \pi_2(g_2)\psi_2 \rangle \overline{\langle f_1, \pi_1(g_1)\psi_2 \rangle} \\ &= V_{\psi_2}f_2(g_2)\overline{V_{\psi_1}f_1(g_1)}. \end{aligned}$$

Thus, the wavelet transform of the tensor product representation factors into the product of wavelet transforms on G_1 and G_2. Strictly speaking, we would have to write $V_{\psi_i}^{\pi_i} f_i$ to indicate the underlying representation, but we omit the reference to the group to keep notation simple.

Throughout this paper we consider only separable weights $w : G \to \mathbb{R}_+$ with $w(g) = (w_1 \otimes w_2)(g) = w_1(g_1)w_2(g_2)$ and $m(g) = (m_1 \otimes m_2)(g) = m_1(g_1)m_2(g_2)$, where w_i is submultiplicative and m_i is w_i-moderate. Moreover we write $(1/w)(g) = (w_1 \otimes w_2)(g)^{-1}$. It follows from (13) that the tensor representation $\pi_2 \otimes \pi_1$ of two square-integrable representations is again square-integrable and that the tensor $\Psi = \psi_2 \otimes \psi_1$ of two admissible vectors ψ_2 and ψ_1 is admissible for π. Likewise, if $w = w_1 \otimes w_2$ and $\psi_1 \in \mathcal{A}_{w_1}(G_1)$, $\psi_2 \in \mathcal{A}_{w_2}(G_2)$, then $\psi_2 \otimes \psi_1 \in \mathcal{A}_w(G_1 \times G_2)$

(where we assume that w_i, $i = 1, 2$, satisfies (7)). Therefore all definitions and results of Section 2.2 hold for the representation $\pi = \pi_2 \otimes \pi_1$ and $\Psi = \psi_2 \otimes \psi_1$. In particular, the orthogonality relation (4), the inversion formula (5), Proposition 1, and Theorem 2 hold for suitable admissible vectors $\Psi = \psi_2 \otimes \psi_1$.

4. KERNEL THEOREMS

In this section we derive the general kernel theorems for operators between coorbit spaces. The basic idea comes from linear algebra, where a linear operator is identified with its matrix with respect to a basis. In coorbit theory the basic structure consists of the vectors $\pi(g)\psi$. Thus in analogy to linear algebra we try to describe an operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ by the kernel (= continuous matrix)

$$(14) \qquad k_A(g_1, g_2) = \langle A\pi_1(g_1)\psi_1, \pi_2(g_2)\psi_2 \rangle.$$

This can be seen as a continuous Galerkin-like representation of the operator A [2, 3]. The idea goes back to coherent state theory [30, Ch. 1.6]. One of its goals is to associate to every operator A a function or symbol k_A, and (14) is one of the many possibilities to do so.

Assume that $A : Co_{\pi_1} L^1_{w_1}(G_1) \to Co_{\pi_2} L^\infty_{1/w_2}(G_2)$ and $f \in Co_{\pi_1} L^1_{w_1}(G_1)$, i.e., A maps "test functions" to "distributions". By using the inversion formula (5) for f and applying A to it, it follows formally that

$$Af = \int_{G_1} \langle f, \pi_1(g_1)\psi_1 \rangle A\pi_1(g_1)\psi_1 dg_1,$$

and furthermore that

$$V_{\psi_2}(Af)(g_2) = \langle Af, \pi_2(g_2)\psi_2 \rangle = \int_{G_1} \langle f, \pi_1(g_1)\psi_1 \rangle \langle A\pi_1(g_1)\psi_1, \pi_2(g_2)\psi_2 \rangle dg_1$$

$$(15) \qquad = \int_G \langle f, \pi_1(g_1)\psi_1 \rangle k_A(g_1, g_2) dg_1 .$$

Let

$$(16) \qquad \mathfrak{A}F(g_2) = \int_{G_1} F(g_1) k_A(g_1, g_2) dg_1$$

be the integral operator with the kernel k_A. Then (15) can be written as

$$(17) \qquad V_{\psi_2} Af = \mathfrak{A} V_{\psi_1} f$$

or, equivalently,

$$(18) \qquad A = V_{\psi_2}^* \mathfrak{A} V_{\psi_1} .$$

Using this factorization, the computation in (15) can be given a precise meaning on coorbit spaces. Identity (18) is the heart of the kernel theorems. The combination of the properties of the generalized wavelet transform (Proposition 1) and boundedness properties of integral operators yields powerful and very general kernel theorems.

We will first show the existence of a generalized kernel for operators mapping the space of test functions $\mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)$ into the distribution space $\mathcal{C}o_{\pi_2} L^\infty_{1/w_2}(G_2)$. Subsequently, we will characterize continuous operators in certain subclasses.

Theorem 3. *Let G_1 and G_2 be two locally compact groups, and let $(\pi_j, \mathcal{H}_j)$ be integrable, unitary, irreducible representations of G_j, such that $\mathcal{A}_{w_j}(G_j) \neq \emptyset$ for $j = 1, 2$.*

(i) *Every kernel $K \in \mathcal{C}o_\pi L^\infty_{1/w}(G_1 \times G_2)$ defines a unique linear operator $A : \mathcal{C}o_{\pi_1} L^1_{w_1}(G_1) \to \mathcal{C}o_{\pi_2} L^\infty_{1/w_2}(G_2)$ by means of*

$$\langle Av, \varphi \rangle = \langle K, \varphi \otimes v \rangle \tag{19}$$

for all $v \in \mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)$ and $\varphi \in \mathcal{C}o_{\pi_2} L^1_{w_2}(G_2)$. The operator norm satisfies

$$\|A\|_{Op} \asymp \|K\|_{\mathcal{C}o_\pi L^\infty_{1/w}(G)} \tag{20}$$

and

$$k_A = V_\Psi K. \tag{21}$$

(ii) *Kernel theorem: Conversely, if $A : \mathcal{C}o_{\pi_1} L^1_{w_1}(G_1) \to \mathcal{C}o_{\pi_2} L^\infty_{1/w_2}(G_2)$ is bounded, then there exists a unique kernel $K \in \mathcal{C}o_\pi L^\infty_{1/w}(G_1 \times G_2)$ such that (19) holds.*

Proof. (i) Fix $K \in \mathcal{C}o_\pi L^\infty_{1/w}(G)$ with $G = G_1 \times G_2$, and let $v \in \mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)$, $\varphi \in \mathcal{C}o_{\pi_2} L^1_{w_2}(G_2)$ be arbitrary. By (13) it follows that $\varphi \otimes v \in \mathcal{C}o_\pi L^1_w(G)$. Therefore, the duality in (19) is well-defined and

$$|\langle K, \varphi \otimes v \rangle| \leqslant \|K\|_{\mathcal{C}o_\pi L^\infty_{1/w}(G)} \|\varphi \otimes v\|_{\mathcal{C}o_\pi L^1_w(G)}$$
$$= \|K\|_{\mathcal{C}o_\pi L^\infty_{1/w}(G)} \|\varphi\|_{\mathcal{C}o_{\pi_2} L^1_{w_2}(G_2)} \|v\|_{\mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)}. \tag{22}$$

Therefore, if we fix v, the mapping $\varphi \mapsto \langle K, \varphi \otimes v \rangle$ is a bounded, anti-linear functional on $\mathcal{C}o_{\pi_2} L^1_{w_2}(G_2)$, which we call $Av \in \mathcal{C}o_{\pi_2} L^\infty_{1/w_2}(G_2)$. The map $v \mapsto Av$ is clearly linear, and (19) defines a linear operator $A : \mathcal{C}o_{\pi_1} L^1_{w_1}(G_1) \to \mathcal{C}o_{\pi_2} L^\infty_{1/w_2}(G_2)$. The estimate (22) implies that

$$\|Av\|_{\mathcal{C}o_{\pi_2} L^\infty_{1/w_2}(G_2)} \leqslant \|K\|_{\mathcal{C}o_\pi L^\infty_{1/w}(G)} \|v\|_{\mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)},$$

and thus

$$\|A\|_{Op} \leqslant \|K\|_{\mathcal{C}o_\pi L^\infty_{1/w}(G)}.$$

(ii) To prove the converse, we need to show that the mapping $K \mapsto A$ is one-to-one and onto.

Uniqueness: Let us assume that the kernel $\mathcal{K} \in \mathcal{C}o_\pi L^\infty_{1/w}(G)$ also satisfies

$$\langle Av, \varphi \rangle = \langle K, \varphi \otimes v \rangle = \langle \mathcal{K}, \varphi \otimes v \rangle$$

for every $v \in \mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)$, $\varphi \in \mathcal{C}o_{\pi_2} L^1_{w_2}(G_2)$. By Theorem 2, there exists a discrete set $\{\gamma_i\}_{i \in \mathcal{I}} \subset G$ such that every $F \in \mathcal{C}o_\pi L^1_w(G)$ can be written as

$$F = \sum_{i \in \mathcal{I}} \lambda_i(F) \pi(\gamma_i)(\psi_2 \otimes \psi_1),$$

with unconditional convergence in $\mathcal{C}o_\pi L^1_w(G)$ and

$$\sum_i |\lambda_i(F)| w(\gamma_i) \leqslant C \|F\|_{\mathcal{C}o_\pi L^1_w(G_1 \times G_2)}.$$

Since $\pi(\gamma_i)(\psi_2 \otimes \psi_1) = \pi(\gamma_{i,2})\psi_2 \otimes \pi(\gamma_{i,1})\psi_1$, we conclude that

$$\begin{aligned}
\langle K, F \rangle &= \sum_{i \in \mathcal{I}} \overline{\lambda_i(F)} \langle K, \pi(\gamma_{i,2})\psi_2 \otimes \pi(\gamma_{i,1})\psi_1 \rangle \\
&= \sum_{i \in \mathcal{I}} \overline{\lambda_i(F)} \langle \mathcal{K}, \pi(\gamma_{i,2})\psi_2 \otimes \pi(\gamma_{i,1})\psi_1 \rangle \\
&= \langle \mathcal{K}, F \rangle.
\end{aligned}$$

As this equality holds for every $F \in \mathcal{C}o_\pi L^1_w(G)$, it follows that $K = \mathcal{K}$.

Surjectivity: Let us assume that $A : \mathcal{C}o_{\pi_1} L^1_{w_1}(G_1) \to \mathcal{C}o_{\pi_2} L^\infty_{1/w_2}(G_2)$ is bounded. Then the kernel k_A defined in (14) is an element of $L^\infty_{1/w}(G_1 \times G_2)$, because

$$\begin{aligned}
|k_A(g)| &= |\langle A\pi_1(g_1)\phi, \pi_2(g_2)\psi \rangle| \\
&\leqslant \|A\|_{Op} \|\pi_1(g_1)\phi\|_{\mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)} \|\pi_2(g_2)\psi\|_{\mathcal{C}o_{\pi_2} L^1_{w_2}(G_2)} \\
&\leqslant \|A\|_{Op} \, w_1(g_1) \, \|\phi\|_{\mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)} \, w_2(g_2) \, \|\psi\|_{\mathcal{C}o_{\pi_2} L^1_{w_2}(G_2)}.
\end{aligned}$$

We claim that k_A is a generalized wavelet transform. Precisely, there exists $K \in \mathcal{C}o_\pi L^\infty_{1/w}(G_1 \times G_2)$ such that $k_A = V_\Psi K$. To prove this claim, we use Proposition 1(v), which asserts that $k_A = V_\psi K$ for some $K \in \mathcal{C}o_\pi L^\infty_{1/w}(G)$ if and only if $k_A = k_A * V_\Psi \Psi$.

As $k_A \cdot V_\Psi(\pi(g)\Psi) \in L^1(G_1 \times G_2)$, we may choose the most convenient order of integration and apply the reproducing formula of Proposition 1(v) consecutively to the representations π_1 and π_2. Using (13) we obtain

$$\begin{aligned}
(k_A * V_\Psi \Psi)(g) &= \int_G k_A(h) V_\Psi \Psi(h^{-1}g) \, dh \\
&= \int_{G_1} \int_{G_2} V_{\psi_2}\big(A\pi_1(h_1)\psi_1\big)(h_2) V_{\psi_2}\psi_2(h_2^{-1}g_2) dh_2 \, \overline{V_{\psi_1}\psi_1(h_1^{-1}g_1)} dh_1 \\
&= \int_{G_1} \Big(V_{\psi_2}\big(A\pi_1(h_1)\psi_1\big) * V_{\psi_2}\psi_2\Big)(g_2) \, \overline{V_{\psi_1}\psi_1(h_1^{-1}g_1)} dh_1 \\
&= \int_{G_1} \langle A\pi_1(h_1)\psi_1, \pi_2(g_2)\psi_2 \rangle \, \overline{V_{\psi_1}\psi_1(h_1^{-1}g_1)} dh_1 = (*).
\end{aligned}$$

At this point we note that by the assumption on A there exists a unique operator $A' : \mathcal{C}o_{\pi_2} L^1_{w_2}(G_2) \to \mathcal{C}o_{\pi_1} L^\infty_{1/w_1}(G_1)$ that satisfies

$$\langle Av, \varphi \rangle = \langle v, A' \varphi \rangle$$

for every $\upsilon \in Co_{\pi_1} L_{w_1}^1(G_1)$ and $\varphi \in Co_{\pi_2} L_{w_2}^1(G_2)$. By its definition, A' is weak*-continuous. We continue with the integration over G_1 and obtain

$$(*) = \overline{\int_{G_1} \langle A'\pi_2(g_2)\psi_2, \pi_1(h_1)\psi_1\rangle \; V_{\psi_1}\psi_1(h_1^{-1}g_1)dh_1}$$

$$= \overline{\Big(V_{\psi_1}\big(A'\pi_2(g_2)\psi_2\big) * V_{\psi_1}\psi_1\Big)(g_1)} = \overline{\langle A'\pi_2(g_2)\psi_2, \pi_1(g_1)\psi_1\rangle}$$

$$= \langle A\pi_1(g_1)\psi_1, \pi_2(g_2)\psi_2\rangle = k_A(g).$$

By Proposition 1(v) there exists a kernel $K \in Co_\pi L_{1/w}^\infty(G_1 \times G_2)$ such that $k_A(g_1, g_2) = V_\Psi K(g_1, g_2)$. By the first part of the proof K defines an operator $B: Co_{\pi_1}L_{w_1}^1(G_1) \to Co_{\pi_2}L_{1/w_2}^\infty(G_2)$ by means of $\langle Bv, \phi\rangle = \langle K, \phi \otimes v\rangle$. In particular,

$$\langle B\pi_1(g_1)\psi_1, \pi_2(g_2)\psi_2\rangle = \langle K, \pi_2(g_2)\psi_2 \otimes \pi_1(g_1)\psi_1\rangle = V_\Psi K(g_1, g_2)$$

$$= k_A(g_1, g_2) = \langle A\pi_1(g_1)\psi_1, \pi_2(g_2)\psi_2\rangle.$$

Consequently, $B\pi_1(g_1)\psi_1 = A\pi_1(g_1)\psi_1$ for all $g_1 \in G_1$. This identity extends to all finite linear combinations of vectors $\pi_1(g_1)\psi_1$ and by Theorem 2 to $Co_{\pi_1}L_{w_1}^1(G_1)$. Thus $B = A$, and we have shown that the map from kernels to operators is onto.

The map $K \mapsto A$ is bounded and invertible. By the inverse mapping theorem we obtain that $\|K\|_{Co_\pi L_{1/w}^\infty(G)} \leqslant C\|A\|_{\mathrm{Op}}$, which proves (20). $\qquad \square$

Remark 4. It is crucial to interpret the brackets in (19) correctly. For utmost precision, we would have to write

$$\langle A\upsilon, \varphi\rangle_{Co_{\pi_2} L_{1/w_2}^\infty(G_2), \, Co_{\pi_2} L_{w_2}^1(G_2)} = \langle K, \varphi \otimes \upsilon\rangle_{Co_\pi L_{1/w}^\infty(G), \, Co_\pi L_w^1(G)},$$

but we feel that this notation would distract from the analogy to distribution theory.

The injectivity of the mapping $K \mapsto A$ from kernels to operators is closely related to an important property of the coorbit spaces $Co_{\pi_1}L_{w_1}^1(G_1)$. This so-called tensor product property has gained considerable importance in certain special cases [13, Theorem 7D] and [26]. We therefore state and prove a general version. Recall that the projective tensor product of two Banach spaces B_1 and B_2 is defined to be

$$B_1 \widehat{\otimes} B_2 = \{f = \sum_{i \in \mathcal{I}} \phi_i \otimes \psi_i : \phi_i \in B_1, \psi_i \in B_2 \quad \text{and} \quad \sum_{i \in \mathcal{I}} \|\phi_i\|_{B_1}\|\psi_i\|_{B_2} < \infty\}.$$

The norm is given as $\|f\|_{\widehat{\otimes}} = \inf \sum_{i \in \mathcal{I}} \|\phi_i\|_{B_1}\|\psi_i\|_{B_2}$ over all representations of $f = \sum_{i \in \mathcal{I}} \phi_i \otimes \psi_i$.

The following identification of the projective tensor product of $Co_{\pi_1}L_{w_1}^1(G_1)$ and $Co_{\pi_2}L_{w_2}^1(G_2)$ with the coorbit space $Co_\pi L_w^1(G_1 \times G_2)$ is a generalization of Feichtinger's original result for modulation spaces [13, Theorem 7D].

Theorem 5. *Under the general assumptions on the groups G_i and the representations $(\pi_i, \mathcal{H}_i)$ we have*

$$Co_\pi L_w^1(G_1 \times G_2) = Co_{\pi_2}L_{w_2}^1(G_2) \, \widehat{\otimes} \, Co_{\pi_1}L_{w_1}^1(G_1).$$

Proof. Let $F \in \mathcal{C}o_\pi L^1_w(G)$. Then by Theorem 2 applied to $\pi = \pi_2 \otimes \pi_1$, F possesses the representation $F = \sum_{i \in \mathcal{I}} \lambda_i(F) \pi(\gamma_i) \Psi \in \mathcal{C}o_\pi L^1_w(G)$ with $\gamma_i = (\gamma_{i,1}, \gamma_{i,2}) \in G_1 \times G_2$ and $\sum_{i \in \mathcal{I}} |\lambda_i(F)| w(\gamma_i) \leqslant C \|F\|_{\mathcal{C}o_\pi L^1_w(G)}$. Using Proposition 1(ii) we obtain that

$$\sum_{i \in \mathcal{I}} \|\lambda_i(F) \pi_1(\gamma_{i,1}) \psi_1\|_{\mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)} \|\pi_2(\gamma_{i,2}) \psi_2\|_{\mathcal{C}o_{\pi_2} L^1_{w_2}(G_2)}$$

$$\leqslant \sum_{i \in \mathcal{I}} |\lambda_i(F)| w_1(\gamma_{i,1}) w_2(\gamma_{i,2}) \|\psi_1\|_{\mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)} \|\psi_2\|_{\mathcal{C}o_{\pi_2} L^1_{w_2}(G_2)}$$

$$\leqslant C \|F\|_{\mathcal{C}o_\pi L^1_w(G)}.$$

Thus $F \in \mathcal{C}o_{\pi_2} L^1_{w_2}(G_2) \widehat{\otimes} \mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)$, and $\mathcal{C}o_\pi L^1_w(G)$ is continuously embedded into $\mathcal{C}o_{\pi_2} L^1_{w_2}(G_2) \widehat{\otimes} \mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)$.

Conversely, let $F \in \mathcal{C}o_{\pi_2} L^1_{w_2}(G_2) \widehat{\otimes} \mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)$. Choose a representation $F = \sum_{i \in \mathcal{I}} f_{i,2} \otimes f_{i,1}$ with $\sum_{i \in \mathcal{I}} \|f_{i,1}\|_{\mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)} \|f_{i,2}\|_{\mathcal{C}o_{\pi_2} L^1_{w_2}(G_2)} < \infty$. Using Fubini's theorem and Proposition 1(ii) yields

$$\|F\|_{\mathcal{C}o_\pi L^1_w(G)} = \int_G |V_\Psi F(g)| \, w(g) dg$$

$$\leqslant \sum_{i \in \mathcal{I}} \left(\int_{G_1} |V_{\psi_1} f_{i,1}(g_1)| \, w_1(g_1) dg_1 \right) \cdot \left(\int_{G_2} |V_{\psi_2} f_{i,2}(g_2)| \, w_2(g_2) dg_2 \right)$$

$$= \sum_{i \in \mathcal{I}} \|f_{i,1}\|_{\mathcal{C}o_{\pi_1} L^1_{w_1}(G_1)} \|f_{i,2}\|_{\mathcal{C}o_{\pi_2} L^1_{w_2}(G_2)} < \infty.$$

Thus, $\mathcal{C}o_{\pi_2} L^1_{w_2}(G_2) \widehat{\otimes} \mathcal{C}o_{\pi_1} L^1_{w_1}(G_1) \subseteq \mathcal{C}o_\pi L^1_w(G)$. The equivalence of the norms follows from the inverse mapping theorem. $\square$

Once the kernel theorem provides a general description of operators between test functions and distributions, we may try to characterize certain classes of operators by properties of their kernel. Since on the level of the generalized wavelet transform such operators correspond to integral operators (see diagram in Figure 1), we may translate the various versions of Schur's test to kernel theorems for operators between coorbit spaces. Following the procedure in [8, Theorem 3.3], we first formulate a general version of Schur's test and then derive the abstract kernel theorem.

We introduce two classes of mixed norm spaces. For two σ-finite measure spaces (X, μ) and (Y, ν), $1 \leqslant p \leqslant \infty$, and $m : X \times Y \to \mathbb{R}_+$, we define the spaces $L^{p,\infty}_m(X \times Y)$ and $\mathcal{L}^{p,\infty}_m(X \times Y)$ by the norms

$$(23) \qquad \|F\|_{L^{p,\infty}_m(X \times Y)} := \operatorname*{ess\,sup}_{y \in Y} \left(\int_X |F(x,y)|^p m(x,y)^p d\mu(x) \right)^{1/p}$$

and

$$(24) \qquad \|F\|_{\mathcal{L}^{p,\infty}_m(X \times Y)} := \operatorname*{ess\,sup}_{x \in X} \left(\int_Y |F(x,y)|^p m(x,y)^p d\nu(y) \right)^{1/p}.$$

The following version of Schur's test is folklore and can be found in [33, Propositions 5.2 and 5.4] or [27].

Proposition 6. *Let (X, μ) and (Y, ν) be σ-finite measure spaces, let $1 \leqslant p \leqslant \infty$, let $\frac{1}{p} + \frac{1}{q} = 1$, and let T be the integral operator $Tf(y) = \int_X f(x)k_T(x,y)d\mu(x)$ with kernel $k_T : X \times Y \to \mathbb{C}$.*

    (i) *The operator T is bounded from $L^1_{m_1}(X)$ to $L^p_{m_2}(Y)$ if and only if $k_T \in \mathcal{L}^{p,\infty}_{m_1^{-1} \otimes m_2}(X \times Y)$. In that case*

$$(25) \qquad \|T\|_{L^1_{m_1}(X) \to L^p_{m_2}(Y)} = \|k_T\|_{\mathcal{L}^{p,\infty}_{m_1^{-1} \otimes m_2}(X \times Y)}.$$

    (ii) *The operator T is bounded from $L^p_{m_1}(X)$ to $L^\infty_{m_2}(Y)$ if and only if $k_T \in L^{q,\infty}_{m_1^{-1} \otimes m_2}(X \times Y)$. In this case*

$$(26) \qquad \|T\|_{L^p_{m_1}(X) \to L^\infty_{m_2}(Y)} = \|k_T\|_{L^{q,\infty}_{m_1^{-1} \otimes m_2}(X \times Y)}.$$

We now characterize the boundedness of operators between certain coorbit spaces.

Theorem 7. *Let $1 \leqslant p, q \leqslant \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let m_j be w_j-moderate weights on G_j. If A is a bounded operator from $Co_{\pi_1} L^1_{w_1}(G_1)$ to $Co_{\pi_2} L^\infty_{1/w_2}(G_2)$ with kernel K, then the following hold:*

    (i) *A is bounded from $Co_{\pi_1} L^1_{m_1}(G_1)$ to $Co_{\pi_2} L^p_{m_2}(G_2)$ if and only if its kernel K is in $Co_\pi \mathcal{L}^{p,\infty}_{m_1^{-1} \otimes m_2}(G_1 \times G_2)$. Its operator norm satisfies*

$$\|A\|_{Op} \asymp \|K\|_{Co_\pi \mathcal{L}^{p,\infty}_{m_1^{-1} \otimes m_2}(G)}.$$

    (ii) *A is bounded from $Co_{\pi_1} L^p_{m_1}(G_1)$ to $Co_{\pi_2} L^\infty_{m_2}(G_2)$ if and only if its kernel K is in $Co_\pi L^{q,\infty}_{m_1^{-1} \times m_2}(G_1 \times G_2)$. Its operator norm satisfies*

$$\|A\|_{Op} \asymp \|K\|_{Co_\pi L^{q,\infty}_{m_1^{-1} \otimes m_2}(G)}.$$

Proof. Since $Co_{\pi_1} L^1_{w_1}(G_1) \subseteq Co_{\pi_1} L^1_{m_1}(G_1)$ and $Co_{\pi_2} L^p_{m_2}(G_2) \subseteq Co_{\pi_2} L^\infty_{1/w_2}(G_2)$ by (10), the kernel theorem is applicable to the operator A, and there exists a kernel $K \in Co_\pi L^\infty_{1/w}(G_1 \otimes G_2)$ such that

$$V_\Psi K(g_1, g_2) = k_A(g_1, g_2) = \langle A\pi_1(g_1)\psi_1, \pi_2(g_2)\psi_2 \rangle.$$

Assume first that $K \in Co_\pi \mathcal{L}^{p,\infty}_{m_1^{-1} \otimes m_2}(G)$, which means that $V_\Psi K \in \mathcal{L}^{p,\infty}_{1/m_1 \otimes m_2}(G)$. By Proposition 6, the integral operator $\mathfrak{A}$ defined by the integral kernel k_A is bounded from $L^1_{m_1}(G_1)$ to $L^p_{m_2}(G_2)$. According to (18), A factors as $A = V^*_{\psi_2} \mathfrak{A} V_{\psi_1}$, where V_{ψ_1} is an isometry from $Co_{\pi_1} L^1_{m_1}(G_1)$ to $L^1_{m_1}(G_1)$, and $V^*_{\psi_2}$ is bounded from $L^p_{m_2}(G_2)$ to $Co_{\pi_2} L^p_{m_2}(G_2)$ by Proposition 1. Consequently A is bounded from $Co_{\pi_1} L^1_{m_1}(G_1)$ to $Co_{\pi_2} L^p_{m_2}(G_2)$. The boundedness estimate follows from

$$\|A\|_{Op} \leqslant \|V^*_{\psi_2}\|_{Op} \|\mathfrak{A}\|_{L^1_{m_1}(G_1) \to L^p_{m_2}(G_2)} \|V_{\psi_1}\|_{Op}$$
$$\leqslant C\|k_A\|_{\mathcal{L}^{p,\infty}_{m_1^{-1} \otimes m_2}(G)} = C\|K\|_{Co_\pi \mathcal{L}^{p,\infty}_{m_1^{-1} \otimes m_2}(G)}.$$

Conversely, let A be bounded from $\mathcal{C}o_{\pi_1}L^1_{m_1}(G_1)$ to $\mathcal{C}o_{\pi_2}L^p_{m_2}(G_2)$. Then $A\pi_1(g_1)\psi_1 \in \mathcal{C}o_{\pi_2}L^p_{m_2}(G_2)$ and the following estimates make sense:

$$\|K\|_{\mathcal{C}o_\pi\,\mathcal{L}^{p,\infty}_{m_1^{-1}\otimes m_2}(G)} = \|V_\Psi K\|_{\mathcal{L}^{p,\infty}_{m_1^{-1}\otimes m_2}(G)} = \|k_A\|_{\mathcal{L}^{p,\infty}_{m_1^{-1}\otimes m_2}(G)}$$

$$= \sup_{g_1\in G_1}\left(\int_{G_2}|\langle A\pi_1(g_1)\psi_1,\pi_2(g_2)\psi_2\rangle m_2(g_2)|^p dg_2\right)^{1/p} m_1(g_1)^{-1}$$

$$= \sup_{g_1\in G_1}\|A\pi_1(g_1)\psi_1\|_{\mathcal{C}o_{\pi_2}L^p_{m_2}(G_2)}\,m_1(g_1)^{-1}$$

$$\leqslant \|A\|_{Op}\sup_{g_1\in G_1}\|\pi_1(g_1)\psi_1\|_{\mathcal{C}o_{\pi_1}L^1_{m_1}(G_1)}\,m_1(g_1)^{-1}.$$

Since $V_{\psi_1}\psi_1 \in L^1_{w_1}(G_1)$ and m_1 is w_1-moderate and thus satisfies $m_1(g_1h)m_1(g_1)^{-1} \leqslant w_1(h)$, the last expression is bounded by

$$\sup_{g_1\in G_1}\|\pi_1(g_1)\psi_1\|_{\mathcal{C}o_{\pi_1}L^1_{m_1}(G_1)}\cdot m_1(g_1)^{-1} = \sup_{g_1\in G_1}\int_{G_1}|\langle\psi_1,\pi_1(g_1^{-1}h)\psi_1\rangle|\frac{m_1(h)}{m_1(g_1)}dh$$

$$= \sup_{g_1\in G_1}\int_{G_1}|\langle\psi_1,\pi_1(h)\psi_1\rangle\frac{m_1(g_1h)}{m_1(g_1)}dh \leqslant \|V_{\psi_1}\psi_1\|_{L^1_{w_1}(G_1)}.$$

Thus $K \in \mathcal{C}o_\pi\,\mathcal{L}^{p,\infty}_{m_1^{-1}\otimes m_2}(G)$.

Part (ii) follows by using Proposition 6(ii) instead of (i) and is proved similarly. □

The following diagram (Figure 1) shows the connection between the different operators and spaces.
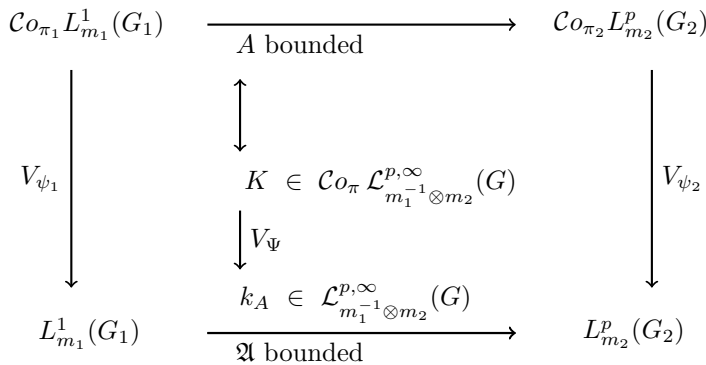

FIGURE 1

Using interpolation between L^p-spaces, Schur's test can also be formulated as saying that an integral operator is bounded on all L^p simultaneously if and only if its kernel belongs to $L^{1,\infty} \cap \mathcal{L}^{1,\infty}$. The corresponding version for coorbit spaces is a consequence of Theorem 7 and an interpolation argument.

Corollary 8. *The following conditions are equivalent:*

(i) $A : Co_{\pi_1} L^p_{m_1}(G_1) \to Co_{\pi_2} L^p_{m_2}(G_2)$ *is bounded for every* $1 \leqslant p \leqslant \infty$.

(ii) *Both* $A : Co_{\pi_1} L^1_{m_1}(G_1) \to Co_{\pi_2} L^1_{m_2}(G_2)$ *and* $A : Co_{\pi_1} L^\infty_{m_1}(G_1) \to Co_{\pi_2} L^\infty_{m_2}(G_2)$ *are bounded.*

(iii) $K \in Co_\pi \mathcal{L}^{1,\infty}_{m_1^{-1} \otimes m_2}(G) \bigcap Co_\pi L^{1,\infty}_{m_1^{-1} \otimes m_2}(G)$.

Clearly one can now translate every boundedness result for an integral operator into a kernel theorem for coorbit spaces. As a simple but important example we offer a sufficient condition for regularizing operators, i.e., operators that map distributions to test functions.

Theorem 9. *Under the assumptions of Theorem 3, if the unique kernel of the operator A satisfies $K \in Co_\pi L^1_w(G)$, then A is bounded from $Co_{\pi_1} L^\infty_{1/w_1}(G_1)$ to $Co_{\pi_2} L^1_{w_2}(G_2)$.*

Proof. Consider the integral operator $\mathfrak{A}$ as in the proof of Theorem 7 and observe that $V_\Psi K = k_A \in L^1_w(G)$ is a sufficient condition for $\mathfrak{A} : L^\infty_{1/w_1}(G_1) \to L^1_{w_2}(G_2)$ to be bounded by Schur's test. $\qquad\square$

4.1. **Discretization.** Coorbit theory guarantees the discretization of the coorbit spaces via atomic decompositions and Banach frames. For our purposes, it is sufficient to state a shortened and simplified version of [22, Theorem 5.3]. Let Y be one of the function spaces $L^p_m(G), L^{p,\infty}_m(G)$, or $\mathcal{L}^{p,\infty}_m(G)$, and let Y_d be the natural sequence space associated to Y.

Proposition 10. *If ψ satisfies*

$$(27) \qquad \int_G \sup_{h \in gQ} |V_\psi \psi(h)| w(g) dg < \infty,$$

for a compact neighborhood Q of e, then there exist a discrete subset $\Lambda \subset G$ and constants $C_1, C_2 > 0$ such that

$$(28) \qquad C_1 \|f\|_{Co_\pi Y} \leqslant \|V_\psi f\|_{Y_d} \leqslant C_2 \|f\|_{Co_\pi Y} \qquad \text{for every } f \in Co_\pi Y.$$

Corollary 11. *Let $\Lambda = \Lambda_1 \times \Lambda_2 \subset G$ be a discrete set such that $\{\pi(\lambda)\Psi\}_{\lambda \in \Lambda}$ satisfies (28) for $Co_\pi L^{p,\infty}_{m_1^{-1} \otimes m_2}(G)$ and $Co_\pi \mathcal{L}^{p,\infty}_{m_1^{-1} \otimes m_2}(G)$. If A is a bounded operator from $Co_{\pi_1} L^1_{w_1}(G_1)$ to $Co_{\pi_2} L^\infty_{1/w_2}(G_2)$ with kernel K, then the following hold:*

(i) $A : Co_{\pi_1} L^1_{m_1}(G_1) \to Co_{\pi_2} L^p_{m_2}(G_2)$ *is bounded if and only if*

$$(29) \qquad \sup_{\lambda_1 \in \Lambda_1} \left(\sum_{\lambda_2 \in \Lambda_2} |V_\Psi K(\lambda)(m_1^{-1} \otimes m_2)(\lambda)|^p \right)^{1/p} < \infty.$$

(ii) *Likewise* $A : Co_{\pi_1} L^p_{m_1}(G_1) \to Co_{\pi_2} L^\infty_{m_2}(G_2)$ *is bounded if and only if*

$$(30) \qquad \sup_{\lambda_2 \in \Lambda_2} \left(\sum_{\lambda_1 \in \Lambda_1} |V_\Psi K(\lambda)(m_1^{-1} \otimes m_2)(\lambda)|^q \right)^{1/q} < \infty.$$

Proof. (i) By Theorem 7 A has a kernel in $Co_\pi \mathcal{L}^{p,\infty}(G)$, and

$$\|A\|_{Op} \asymp \|K\|_{Co_\pi \mathcal{L}^{p,\infty}(G)}.$$

By (28), the expression in (29) is an equivalent norm for $\|K\|_{Co_\pi \mathcal{L}^{p,\infty}(G)}$. The proof of (ii) works in exactly the same way. $\qquad\square$

5. EXAMPLES

5.1. **Modulation spaces.** The Weyl-Heisenberg group $G_{WH} = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ is defined by the group law

$$(x, \omega, e^{2\pi i \tau}) \cdot (x', \omega', e^{2\pi i \tau'}) = (x + x', \omega + \omega', e^{2\pi i (\tau + \tau' - x \cdot \omega')}).$$

Let $T_x f(t) := f(t - x)$ denote the translation, and let $M_\omega f(t) := e^{2\pi i \omega t} f(t)$ be the modulation operator. The operator $\pi_{WH}(x, \omega, \tau) = e^{2\pi i \tau} M_\omega T_x$ for $(x, \omega, \tau) \in G_{WH}$ defines a unitary square-integrable representation of G_{WH} acting on $L^2(\mathbb{R}^d)$, for which every non-zero vector in $L^2(\mathbb{R}^d)$ is admissible. Since the phase factor $e^{2\pi i \tau}$ is irrelevant for the definition of coorbit spaces, it is convenient to drop the trivial third component and consider the time-frequency shift $\pi(x, \omega) = \pi_{WH}(x, \omega, 1) = M_\omega T_x$. Formally, we treat the projective representation π of $\mathbb{R}^{2d}$ instead of the unitary representation π_{WH} of G_{WH}. The transform corresponding to π is nothing else but the short-time Fourier transform

$$V_\psi f(x, \omega) = \langle f, M_\omega T_x \psi \rangle = \int_{\mathbb{R}^d} f(t) \overline{\psi(t - x)} e^{-2\pi i \omega \cdot t} dt, \qquad f, \psi \in L^2(\mathbb{R}^d).$$

The coorbit spaces associated to π_{WH} coincide therefore with the coorbit spaces associated to π. These are the modulation spaces $M_m^p(\mathbb{R}^d)$ which were first introduced by Feichtinger in [14] as certain decomposition spaces and subsequently were identified with the coorbit spaces of the Heisenberg group $\mathcal{C}o_{\pi_{WH}} L_m^p(G_{WH}) = \mathcal{C}o_\pi L_m^p(\mathbb{R}^{2d}) = M_m^p(\mathbb{R}^d)$ [18]. We refer to the standard textbooks [20, 23] for more information on time-frequency analysis.

Theorem 3 asserts that every bounded operator from $M_w^1(\mathbb{R}^d) = \mathcal{C}o_{\pi_{WH}} L_w^1(G_{WH}) = \mathcal{C}o_\pi L_w^1(\mathbb{R}^{2d})$ to $M_{1/w}^\infty(\mathbb{R}^d) = \mathcal{C}o_{\pi_{WH}} L_{1/w}^\infty(G_{WH}) = \mathcal{C}o_\pi L_{1/w}^\infty(\mathbb{R}^{2d})$ possesses a kernel $K \in \mathcal{C}o_{\pi_{WH} \otimes \pi_{WH}} L_{w^{-1} \otimes w^{-1}}^\infty(G_{WH} \times G_{WH})$ such that $\langle Af, g \rangle = \langle K, g \otimes f \rangle$ for $f, g \in M_w^1(\mathbb{R}^d)$. Let us elaborate in detail what the kernel theorem asserts in this case: for $g_i = (x_i, \omega_i, \tau_i) \in G_{WH}, i = 1, 2$, the tensor representation $\pi_{WH} \otimes \pi_{WH}$ acts on the simple tensor $(\psi_2 \otimes \psi_1)(t_2, t_1) = \psi_2(t_2) \overline{\psi_1(t_1)} \in L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \cong L^2(\mathbb{R}^{2d})$ as

$$\pi_{WH} \otimes \pi_{WH}(g_2, g_1)(\psi_2 \otimes \psi_1)(t_2, t_1) = e^{2\pi i (\tau_1 - \tau_2)} M_{\omega_2} T_{x_2} \psi_2(t_2) \overline{M_{\omega_1} T_{x_1} \psi_1(t_1)}$$
$$= e^{2\pi i (\tau_1 - \tau_2)} M_{(\omega_2, -\omega_1)} T_{(x_2, x_1)} (\psi_2 \otimes \psi_1)(t_2, t_1).$$

Thus except for the phase factor $e^{2\pi i (\tau_1 - \tau_2)}$ the tensor representation $\pi_{WH} \otimes \pi_{WH}$ is just the time-frequency shift $M_{(\omega_2, -\omega_1)} T_{(x_2, x_1)}$ acting on $L^2(\mathbb{R}^{2d})$. Consequently, the coorbit spaces with respect to the product group $G_{WH} \otimes G_{WH}$ are again modulation spaces, this time on $\mathbb{R}^{2d}$. For the coorbit of L^∞ we compare the norms

$$\|K\|_{M^\infty(\mathbb{R}^{2d})} = \sup_{(x_1, x_2, \omega_1, \omega_2) \in \mathbb{R}^{4d}} \left| \langle K, M_{(\omega_1, \omega_2)} T_{(x_1, x_2)} (\psi_2 \otimes \psi_1) \rangle \right|$$

and

$$\|K\|_{\mathcal{C}o_{\pi \otimes \pi} L^\infty(\mathbb{R}^{4d})} = \sup_{\left((x_1, \omega_1), (x_2, \omega_2)\right) \in \mathbb{R}^{4d}} \left| \langle K, \left(\pi(x_1, \omega_1) \otimes \pi(x_2, \omega_2)\right)(\psi_2 \otimes \psi_1) \rangle \right|,$$

which are obviously equal. In this case Theorem 3 is therefore just Feichtinger's kernel theorem: *For $A : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ there exists a unique kernel $K \in M^\infty(\mathbb{R}^{2d})$ such that $\langle Af, g \rangle = \langle K, g \otimes f \rangle$.*

The recent extension of Feichtinger's kernel theorem by Cordero and Nicola [8] can be seen in the same light. Let us explain the difference in the formulations. Our approach considers the generalized wavelet transform

$$V_\Psi K(x_1, \omega_1, x_2, \omega_2) = \langle K, \pi_{WH}(x_2, \omega_2, 1) \otimes \pi_{WH}(x_1, \omega_1, 1)(\psi_2 \otimes \psi_1) \rangle$$

of the kernel. The conditions of Theorem 7 are formulated by mixed norms acting simultaneously on the variables (x_2, ω_2) and (x_1, ω_1). The treatment in [8] uses the short-time Fourier transform on $\mathbb{R}^{2d}$

$$V_\Psi K(x_1, x_2, \omega_1, \omega_2) = \langle K, M_{(\omega_1, \omega_2)} T_{(x_1, x_2)} \Psi \rangle,$$

which is the same transform, except for the order of the variables. In [8] it was therefore necessary to reshuffle the order of integration of time-frequency shifts and to use the notion of mixed modulation spaces, which were studied in [4, 29]. The new insight of our formulation is that the mixed modulation spaces are simply the coorbit spaces with respect to the tensor product representation.

The special case of Theorem 7 for the Weyl-Heisenberg group and the weights $m_s(x, \omega, \tau) = (1 + |x| + |\omega|)^s$ for $s \in \mathbb{R}$ states the following: Fix $\sigma > 0$ and let A be an operator from $M^1_{m_\sigma}(\mathbb{R}^d)$ to $M^\infty_{m_{-\sigma}}(\mathbb{R}^d)$. Then for $|r|, |s| \leqslant \sigma$, $1 \leqslant p, q \leqslant \infty$, and $1/p + 1/q = 1$ we have

(i) $A : M^1_{m_s}(\mathbb{R}^d) \to M^p_{m_r}(\mathbb{R}^d)$ bounded $\Leftrightarrow$ $K \in \mathcal{C}o_\pi \mathcal{L}^{p,\infty}_{m_{-s} \otimes w_r}(\mathbb{R}^{4d})$,

(ii) $A : M^p_{m_s}(\mathbb{R}^d) \to M^\infty_{m_r}(\mathbb{R}^d)$ bounded $\Leftrightarrow$ $K \in \mathcal{C}o_\pi L^{q,\infty}_{m_{-s} \otimes m_r}(\mathbb{R}^{4d})$.

Regularizing operators from M^∞ to M^1 were recently studied by Feichtinger and Jakobsen [19]: they characterized a subclass of this space of operators by an integral kernel in $M^1(\mathbb{R}^{2d})$. The sufficiency of this result in a coorbit version is contained in Theorem 9.

5.2. Wavelet coorbit spaces and Besov spaces.

The affine group $G_{\text{aff}} = \mathbb{R} \times \mathbb{R}^*$ is given by the group law $(x, a) \cdot (y, b) = (x + ay, ab)$, where $x, y \in \mathbb{R}$ and $a, b \in \mathbb{R} \setminus \{0\}$. Its left Haar measure is given by $\frac{dx\,da}{a^2}$. Let $D_a f(t) = |a|^{-1/2} f(t/a)$ denote the dilation operator. Then $(x, a) \to \pi_{\text{aff}}(x, a) = T_x D_a$ defines a unitary, square-integrable representation of G_{aff} on $L^2(\mathbb{R})$.

Now let $f, \psi \in L^2(\mathbb{R})$. The continuous wavelet transform is defined as

$$W_\psi f(x, a) := \langle f, \pi_{\text{aff}}(x, a)\psi \rangle = |a|^{-1/2} \int_\mathbb{R} f(t)\overline{\psi(a^{-1}(t - x))}dt,$$

and the admissibility condition (3) reads as

$$\int_{\mathbb{R}^*} |\widehat{\psi}(\omega)|^2 \frac{d\omega}{|\omega|} < \infty.$$

It is well-known that the coorbit spaces associated to the representation π_{aff} are the homogeneous Besov spaces. See the textbooks [10, 28] for details and further expositions of wavelet theory. For brevity, we consider only the coorbit spaces with respect to the weighted $L^p(G_{\text{aff}})$-spaces with the weight function $\nu_s(x, a) = \nu_s(a) = |a|^{-s}$ for $s \in \mathbb{R}$. Note that $\nu_{-s} = 1/\nu_s$. Then $\mathcal{C}o_{\pi_{\text{aff}}} L^p_{\nu_s}(G_{\text{aff}}) = \dot{B}^{s-1/2+1/p}_{p,p}(\mathbb{R})$ by [15, Section 7.2]. In particular $\mathcal{C}o_{\pi_{\text{aff}}} L^1_{\nu_s}(G_a) = \dot{B}^{s+1/2}_{1,1}(\mathbb{R})$ and $\mathcal{C}o_{\pi_{\text{aff}}} L^\infty_{\nu_s}(G_{\text{aff}}) = \dot{B}^{s-1/2}_{\infty,\infty}(\mathbb{R})$. In this example Theorem 3 states that *an operator $A : \dot{B}^s_{1,1}(\mathbb{R}) \to \dot{B}^{-r}_{\infty,\infty}(\mathbb{R})$ is bounded if and only if its associated kernel K is in $\mathcal{C}o_{\pi_{\text{aff}} \otimes \pi_{\text{aff}}} L^\infty_{\nu_{-s-1/2} \otimes \nu_{-r-1/2}}(G^2_{\text{aff}})$.* At first glance not much seems to have been

gained by this formulation, but it turns out that the coorbit spaces of the tensor product $\pi_{\mathrm{aff}} \otimes \pi_{\mathrm{aff}}$ of G_{aff}^2 are well understood in the theory of function spaces under the name of *Besov spaces of dominating mixed smoothness.* In particular, $\mathcal{C}o_{\pi_{\mathrm{aff}} \otimes \pi_{\mathrm{aff}}} L^\infty_{\nu_{-s-1/2} \otimes \nu_{-r-1/2}}(G_{\mathrm{aff}}^2)$ can be identified with the Besov space of dominating mixed smoothness $S^{-s,-r}_{\infty,\infty} B(\mathbb{R}^2)$. See [32, Definition A.4] and [31]. Moreover, Theorem 7 yields a characterization of continuous operators between certain Besov spaces:

(i) $A : \dot{B}^s_{1,1}(\mathbb{R}) \to \dot{B}^r_{p,p}(\mathbb{R})$ bounded $\Leftrightarrow$ $K \in \mathcal{C}o_{\pi_{\mathrm{aff}} \otimes \pi_{\mathrm{aff}}} \mathcal{L}^{p,\infty}_{\nu_{-s+1/2} \otimes \nu_{r+1/2-1/p}}(G_{\mathrm{aff}}^2)$,

(ii) $A : \dot{B}^s_{p,p}(\mathbb{R}) \to \dot{B}^r_{\infty,\infty}(\mathbb{R})$ bounded $\Leftrightarrow$ $K \in \mathcal{C}o_{\pi_{\mathrm{aff}} \otimes \pi_{\mathrm{aff}}} L^{q,\infty}_{\nu_{-s-1/2+1/p} \otimes \nu_{r+1/2}}(G_{\mathrm{aff}}^2)$.

The case (i) for $p = 1$ was already formulated in a discrete version by Meyer [28, Section 6.9, Proposition 6].

Theorem 12. *Let* $\{\psi_{k,j}\}_{(k,j) \in \mathbb{Z}^2}$ *be a wavelet basis with* $\psi_{k,j}(t) = 2^{j/2} \psi(2^j t - k)$, *and assume that* ψ *has compact support and satisfies sufficiently many moment conditions so that the assumption of Proposition* 10 *is satisfied. An operator* $A : \dot{B}^0_{1,1}(\mathbb{R}) \to \dot{B}^0_{1,1}(\mathbb{R})$ *is bounded if and only if*

$$\sup_{(k\prime,j\prime) \in \mathbb{Z}^2} \sum_{(k,j) \in \mathbb{Z}^2} \left| \langle A\psi_{k\prime,j\prime}, \psi_{k,j} \rangle \right| 2^{-j/2+j\prime/2} \leqslant C.$$

Proof. Set $p = 1$, $s = -1/2$, recall that $k_A = V_\Psi K$, and apply Corollary 11. $\square$

5.3. The case of two distinct representations.

For most applications it suffices to consider a single group G and its product group $G \times G$. Our formulation with two different groups allows us to study operators acting between coorbit spaces associated with different group representations. Using the representations of the Weyl-Heisenberg group and the affine group of Sections 5.1 and 5.2, one can characterize the boundedness of operators between certain modulation spaces and Besov spaces by properties of their associated kernels. Theorem 7 now reads as follows:

(i) $A : M^1_{m_s}(\mathbb{R}^d) \to \dot{B}^r_{p,p}(\mathbb{R})$ bdd.
$$\Leftrightarrow K \in \mathcal{C}o_{\pi_{\mathrm{aff}} \otimes \pi_{WH}} \mathcal{L}^{p,\infty}_{\widetilde{m}_{-s} \otimes \nu_{r+1/2-1/p}}(G_{WH} \times G_{\mathrm{aff}}),$$

(ii) $A : M^p_{m_s}(\mathbb{R}^d) \to \dot{B}^r_{\infty,\infty}(\mathbb{R})$ bdd.
$$\Leftrightarrow K \in \mathcal{C}o_{\pi_{\mathrm{aff}} \otimes \pi_{WH}} L^{q,\infty}_{m_{-s} \otimes \nu_{r+1/2}}(G_{WH} \times G_{\mathrm{aff}}),$$

(iii) $A : \dot{B}^r_{1,1}(\mathbb{R}) \to M^p_{m_s}(\mathbb{R}^d)$ bdd.
$$\Leftrightarrow K \in \mathcal{C}o_{\pi_{WH} \otimes \pi_{\mathrm{aff}}} \mathcal{L}^{p,\infty}_{m_{-r+1/2} \otimes \nu_s}(G_a \times G_{WH}),$$

(iv) $A : \dot{B}^r_{p,p}(\mathbb{R}) \to M^\infty_{m_s}(\mathbb{R}^d)$ bdd.
$$\Leftrightarrow K \in \mathcal{C}o_{\pi_{WH} \otimes \pi_{\mathrm{aff}}} L^{q,\infty}_{\nu_{-r-1/2+1/p} \otimes m_s}(G_a \times G_{WH}).$$

As a special case one obtains a characterization of the bounded operators A from $\dot{B}^r_{1,1}(\mathbb{R})$ to $L^2(\mathbb{R}^d)$. Since $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, they are completely characterized by the membership of their kernel in $\mathcal{C}o_{\pi_{\mathrm{aff}} \otimes \pi_{WH}} \mathcal{L}^{2,\infty}_{1 \otimes m_{-r+1/2}}(G_{\mathrm{aff}} \times G_{WH})$.

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