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PARTIAL DATA INVERSE PROBLEM WITH $L^{n/2}$ POTENTIALS

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Abstract. We construct an explicit Green's function for the conjugated Laplacian $e^{-\omega\cdot x/h}\Delta e^{-\omega\cdot x/h}$, which lets us control our solutions on roughly half of the boundary. We apply the Green's function to solve a partial data inverse problem for the Schrödinger equation with potential $q \in L^{n/2}$. Separately, we also use this Green's function to derive L^p Carleman estimates similar to the ones in Kenig-Ruiz-Sogge [Duke Math. J. 55 (1987), pp. 329–347], but for functions with support up to part of the boundary. Unlike many previous results, we did not obtain the partial data result from the boundary Carleman estimate—rather, both results stem from the same explicit construction of the Green's function. This explicit Green's function has potential future applications in obtaining direct numerical reconstruction algorithms for partial data Calderón problems which is presently only accessible with full data [Inverse Problems 27 (2011)].

1. Introduction

In this article we give an explicit construction of a "Dirichlet Green's function" for the conjugated Laplacian $e^{-x\cdot\omega/h}h^2\Delta e^{x\cdot\omega/h}$ on a bounded smooth domain $\Omega \subset \mathbb{R}^n$ for $n \geq 3$. We apply the Green's function to solve the longstanding partial data Calderón problem with *unbounded* Schrödinger potential in $L^{n/2}(\Omega)$ for $n \geq 3$.

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a smooth domain contained in $\mathbb{R}^n$ with outward pointing normal ν along the boundary and let $\omega_0 \in \mathbb{R}^n$ be a unit vector. Define

$$\Gamma^0_\pm := \{x \in \partial\Omega \mid \pm\nu(x)\cdot\omega_0 \geq 0\}$$

and let $\mathbf{F} \subset \partial\Omega$ be an open neighbourhood containing Γ^0_+ and $\mathbf{B} \subset \partial\Omega$ be an open neighbourhood containing Γ^0_-. We make the additional assumption that in the coordinate system given by $(x', x_n) \in \omega_0^\perp \oplus \mathbb{R}\omega_0$, the complements of $\mathbf{B}$ and $\mathbf{F}$ are disjoint unions of an open subset of $\partial\Omega$ so that the components Γ_j of the disjoint union are compactly contained in the graph $x_n = f_j(x')$ for some smooth function f_j.

If zero is not an eigenvalue of the operator $-\Delta + q$, then $q \in L^{n/2}(\Omega)$ gives rise to a well-defined Dirichlet-to-Neumann map

$$\Lambda_q : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega).$$

(We refer the reader to the appendix of [13] for the definition of the Dirichlet-to-Neumann map for $q \in L^{n/2}(\Omega)$.) We have the following theorem.

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Theorem 1.1. *Let $q_1, q_2 \in L^{n/2}(\Omega)$ be such that $\Lambda_{q_1} f \mid_{\mathbf{F}} = \Lambda_{q_2} f \mid_{\mathbf{F}}$ for all $f \in C_0^\infty(\mathbf{B})$. Then $q_1 = q_2$.*

To date this is the only partial data Calderón problem result for unbounded potentials. The integrability assumption that $q_j \in L^{n/2}$ is optimal in the context of well-posedness theory for the Dirichlet problem for L^p potentials; $L^{n/2}$ is also the optimal Lebesgue space for the strong unique continuation principle to hold (see [17] for more).

Using a well-known argument Theorem 1.1 leads directly to identifying scalar conductivities $\gamma \in W^{2,n/2}$ from partial data. This comes from the fact that $\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}} \in L^{n/2}$ if $\gamma \in W^{2,n/2}$ and $\gamma \geq c > 0$. One can then proceed as in Corollary 0.2 of [3] to show the following.

Corollary 1.2. *Let Λ_γ be the Dirichlet-to-Neumann map of the conductivity operator $\nabla \cdot \gamma \nabla$ for scalar conductivities $\gamma \in W^{2,n/2}(\Omega)$. The operator $f \mapsto (\Lambda_\gamma f) \mid_{\mathbf{F}}$ acting on $f \in C_0^\infty(\mathbf{B})$ uniquely determines γ provided that one knows $\gamma \mid_{\partial\Omega}$ and $\partial_\nu \gamma \mid_{\partial\Omega}$.*

Note that there are conductivities in $W^{2,n/2}$ which are not contained in the cases considered by [23]. In fact, since $W^{1,n} \subset \mathrm{BMO}$ but not in L^∞ this result allows one to consider partial data problems for some conductivities which are not Lipschitz. So even in the special case of the conductivity equation this gives a new result.

Traditionally the study of partial data problems are limited to bounded potentials due to their reliance on L^2 Carleman estimates on bounded domains. We circumvent this difficulty by constructing instead an explicit (conjugated) Green's function which has good L^p estimates in addition to desirable boundary conditions. Let $\omega \in \mathbb{R}^n$ be a unit vector and let $\Gamma \subset \partial\Omega$ be an open subset which is compactly contained in $\{x \in \partial\Omega \mid \nu(x) \cdot \omega > 0\}$. If $p' = \frac{2n}{n+2} < 2 < p = \frac{2n}{n-2}$, we have the following theorem, proved by an explicit construction via heat flow.

Theorem 1.3. *Suppose $h > 0$ is sufficently small. Then there exists an operator $G_\Gamma : L^{p'}(\Omega) \to L^p(\Omega)$ which satisfies*

$$e^{-x\cdot\omega/h} h^2 \Delta e^{x\cdot\omega/h} G_\Gamma = I$$

and the estimates

$$\|G_\Gamma\|_{L^2 \to H^1} \leq Ch^{-1}, \quad \|G_\Gamma\|_{L^{p'} \to L^p} \leq Ch^{-2}.$$

Furthermore, for all $f \in L^{p'}$, $G_\Gamma f \in H^1(\Omega)$ and $G_\Gamma f \mid_\Gamma = 0$.

This Green's function possesses several new features which makes it of potential use for studying a broad range of questions. First, note that in addition to desirable asymptotic L^p and L^2 estimates, this Green's function also allows us to impose the Dirichlet boundary condition on Γ. Secondly, we will see that its construction is by explicit integral kernels in contrast to the functional analysis based approach of [3, 21, 29]. The combination of these two features can inspire future progress in numerical algorithms for partial data reconstruction which are currently only available in the full data case [1, 11]. Furthermore, this Green's function gives new Carleman estimates which may be of interest on their own (see Theorem 1.4 and the ensuing discussions).

We will provide some brief historical context for Theorems 1.1 and 1.3. The construction of the Green's function for the conjugated Laplace operator was established by Sylvester-Uhlmann [33] using Fourier multipliers with characteristic sets. They proved an L^2 estimate for their Green's function and used it to solve the Calderón problem in dimensions $n \geq 3$ for bounded potentials. Chanillo in [4] showed that the Sylvester-Uhlmann Green's function also satisfies an $L^p \to L^{p'}$ estimate by applying using the result of Kenig-Ruiz-Sogge [20]. This allowed Chanillo to solve the inverse Schrödinger problem with full data for small potentials in the Fefferman-Phong class (which contains $L^{n/2}$). Related full data results were also proved by Lavine-Nachman [24] and Dos Santos Ferreira-Kenig-Salo [13]. We will follow some of the techniques developed by these authors in Section 7.1.

The drawback to the Fourier multiplier construction of the Green's function is that boundary conditions cannot be imposed. Bukhgeim-Uhlmann [3] and Kenig-Sjöstrand-Uhlmann [21] found a way to use Carleman estimates to overcome this problem and prove results for the Calderón problem with *partial* boundary data. Due to its versatility and robustness, this technique has since become the standard tool for solving partial data elliptic inverse problems. The review article [19] contains an excellent overview of recent work in partial data Calderón-type problems; examples for other elliptic inverse problems can be found in [31], [32], [22], [9], and [8].

This standard technique turns out to be insufficient for our purpose. The Carleman estimates in these papers are typically proved via an integration-by-parts procedure so that boundary conditions can be kept in check. The limitation of this approach is that only L^2-type estimates can be derived; none of the available techniques adapt well to L^p setting for functions with boundary conditions. Thus for $q \notin L^\infty$, there are no partial data results for the Calderón problem for Schrödinger equations—although using a different method [23] obtained a partial data result for low regularity conductivity equations.

The [3, 21] approach has the additional drawback that the Green's function one "constructs" is an abstract object arising from general statements in functional analysis, like the Hahn-Banach or Riesz representation theorems. This makes partial data reconstruction procedures like the ones in [29] much more difficult to implement in a concrete setting than equivalent ones like [28] for full data.

The Green's function we construct in Theorem 1.3 has the explicit representation of the Fourier multiplier Green's function of Sylvester-Uhlmann while at the same time allowing the boundary control of the existing methods. Due to its explicit representation as a parametrix, one can easily deduce L^p-type estimates as well as L^2-type estimates. In a forthcoming article the authors intend to apply the Green's function constructed here to the problem of reconstruction. One expects that in the context of computational algorithms this Green's function would open the door to direct inversion methods for partial data Calderón problems in $n \geq 3$ which is parallel to the full data case examined in [1, 10–12].

Theorem 1.3 also directly implies the following boundary Carleman estimates for the conjugated Laplacian. Let $H^1(\Omega)$ denote the semiclassical Sobolev space. Define $H^1_\Gamma(\Omega) \subset H^1(\Omega)$ to be the space of functions with vanishing trace along Γ and let $H^{-1}_\Gamma(\Omega)$ be its dual.

Theorem 1.4. *Let $u \in C^2(\bar\Omega)$ be a function which vanishes along $\partial\Omega$ and $\partial_\nu u \mid_{\Gamma^c} = 0$. One then has the Carleman estimates*

$$\|u\|_{L^2(\Omega)} \leq \frac{C}{h}\|h^2\Delta^*_\phi u\|_{H^{-1}_\Gamma(\Omega)}, \;\; \|u\|_{L^p(\Omega)} \leq C\|\Delta^*_\phi u\|_{L^{p'}(\Omega)}$$

for all $h > 0$ sufficiently small.

Remark 1.5. A modification of the argument presented here can also yield a boundary term of $h^{-1/p}\|\partial_\nu u\|_{L^{p'}(\Gamma)}$ on the left side of the L^p inequality.

The L^p inequality differs from other L^p Carleman estimates like the ones in Kenig-Ruiz-Sogge [20] in that it allows for u with *nontrivial boundary conditions*. The solution to the inverse problem does not use Theorem 1.4. We only state the theorem here because it may be of interest to those studying unique continuations in the future. To see why traditional methods do not yield the type of L^p Carleman estimates we obtain with boundary terms, the reader can compare our approach to [2, 20, 25–27].

In the remainder of the introduction we give a brief exposition of our approach to the proof of Theorem 1.3. The key observation is that there is a global ΨDO factorization of the conjugated Laplacian $h^2\Delta_\phi := e^{-\omega\cdot x/h}h^2\Delta e^{\omega\cdot x/h}$ into an elliptic operator J resembling a heat flow and a first-order operator Q which has the same characteristic set as $h^2\Delta_\phi$. One can then construct an inverse for J (and thus $h^2\Delta_\phi$) with Dirichlet boundary conditions by solving the heat flow with zero initial condition.

This way of factoring $h^2\Delta_\phi$ is in the spirit of [5]. However, in our case the factorization is global and occurs on the level of symbols so there will be error terms and they pose a challenge in the construction of the parametrix. As such this necessitates a modified factorization which differs from that of [5] (see (4.7) and the discussions which follow) to obtain the suitable estimates for the remainders of the parametrix.

This article is organized in the following way. In Section 2 we develop a ΨDO calculus which is compatible with our symbol class–proofs are given in the appendix. In Section 3 we invert a heat flow in the context of this ΨDO calculus and solve the Dirichlet problem for this heat flow. In Section 4 we restate some facts about the Sylvester-Uhlmann Green's function in the semiclassical setting and derive a factorization for the operator $h^2\Delta_\phi$ involving the heat operator described in the previous section. In Section 5 we use this factorization to construct a parametrix with Dirichlet boundary conditions, and in Section 6 we turn the parametrix into a Dirichlet Green's function G_Γ and prove Theorem 1.4. Section 7 is devoted to proving Theorem 1.1 using complex geometric optics solutions constructed with the help of G_Γ.

2. Elementary semiclassical ΨDO theory

We collect a set of facts about semiclassical pseudodifferential operators and also use this opportunity to establish some notation and conventions which we will use throughout. Proofs are contained in the appendix.

2.1. Mixed Sobolev spaces.

In this article we define the semiclassical Sobolev spaces with the norm

$$\|u\|_{W^{k,r}_{scl}(\mathbb{R}^n)} := \|\langle hD\rangle^k u\|_{L^r}. \tag{2.1}$$

For $k \in \mathbb{N}$ it turns out that this definition is equivalent to the one involving derivatives:

$$\|u\|^r_{W^{k,r}_{scl}} = \sum_{|\alpha|\le k} \|(hD)^\alpha u\|^r_{L^r}.$$

(Hereafter we will drop the "scl" subscript: unless otherwise stated, all of our Sobolev spaces will be semiclassical.) Choose coordinates (x', x_n) on $\mathbb{R}^n$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$, and let (ξ', ξ_n) be the corresponding coordinates on the cotangent space. An immediate consequence of the norm equivalence stated above is that $\langle \xi' \rangle$ is a multiplier from $W^{1,r}(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$. Indeed,

(2.2)
$$\begin{aligned}\|\langle hD'\rangle u\|^r_{L^r(\mathbb{R}^n)} &= \int_{-\infty}^{\infty} \|\langle hD'\rangle u(x', x_n)\|^r_{L^r_{x'}} dx_n \\ &\lesssim \int_{-\infty}^{\infty} \sum_{|\alpha|\leq 1} \|(hD')^\alpha u(x', x_n)\|^r_{L^r_{x'}} dx_n \leq \sum_{|\alpha|=1} \|(hD)^\alpha u\|^r_{L^r}.\end{aligned}$$

Now define the mixed Sobolev norms for $u \in C_0^\infty$ by

$$\|u\|_{W^{k,\ell,r}(\mathbb{R}^{n-1},\mathbb{R}^n)} := \|\langle hD'\rangle^k \langle hD\rangle^\ell u\|_{L^r} \tag{2.3}$$

and use these to define the mixed norm spaces $W^{k,\ell,r}(\mathbb{R}^{n-1}, \mathbb{R}^n)$. For convenience we will drop the $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$ in this notation and use the convention that the first superscript of $W^{k,\ell,r}$ denotes multiplication by $\langle hD'\rangle^k$ and the second denotes multiplication by $\langle hD\rangle^\ell$.

With this definition we have that for $k \geq 0$,

$$W^{-k,\ell,r} \subset W^{l-k,r}(\mathbb{R}^n). \tag{2.4}$$

Indeed, one can write

$$u = \langle hD'\rangle^k \langle hD\rangle^{-k} \langle hD\rangle^{-\ell+k} \langle hD'\rangle^{-k} \langle hD\rangle^\ell u$$

and use the fact that $\langle hD'\rangle^k \langle hD\rangle^{-k}$ is a multiplier on L^r by (2.2) and that

$$\langle hD'\rangle^{-k} \langle hD\rangle^\ell u \in L^r \iff u \in W^{-k,\ell,r}.$$

2.2. **Tangential calculus.** We denote the Hörmander symbols by $S_1^\ell(\mathbb{R}^n)$. We also consider symbols in the class $S_0^k(\mathbb{R}^n)$. We say that a belongs to $S_j^\ell(\mathbb{R}^n)$ for $j = 0, 1$ if

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\beta,\alpha} \langle \xi \rangle^{\ell - j|\beta|}$$

for all multi-indices α and β. In this article we will work with product symbols of the form $ba(x', \xi) \in S_1^k(\mathbb{R}^{n-1}) S_j^\ell(\mathbb{R}^n) := S_1^k S_j^\ell$ where $b(x', \xi') \in S_1^k(\mathbb{R}^{n-1})$ and $a(x', \xi) \in S_j^\ell(\mathbb{R}^n)$ for $j = 0, 1$. Observe that if $a(x', \xi) \in S_1^k S_j^\ell$, then derivatives with respect to either x' or ξ are a finite sum of symbols in $S_1^k S_j^\ell$:

$$\partial_x^\alpha : S_1^k S_j^\ell \to \mathrm{span}(S_1^k S_j^\ell) \quad \partial_\xi^\alpha : S_1^k S_j^\ell \to \mathrm{span}(S_1^k S_j^\ell). \tag{2.5}$$

We begin with the following Calderón-Vaillancourt-type estimate for (classical) ΨDO with symbols in $S_1^0(\mathbb{R}^n)$ which can be obtained by following the argument of Theorem 9.7 in [34].

Proposition 2.1. i) *Let $a(x, \xi)$ be a symbol in $S_1^0(\mathbb{R}^n)$. Then for all $1 < r < \infty$*

$$\|a(x, D)u\|_{L^r} \leq C_{r,n} \sum_{|\alpha|\leq k(n)-1, |\beta|\leq k(n)-1} p_{\alpha,\beta}(a) \|u\|_{L^r}, \tag{2.6}$$

where $p_{\alpha,\beta}$ is the seminorm defined by $p_{\alpha,\beta}(a) := \sup_{x,\xi} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \langle \xi \rangle^{|\beta|}$ and $k(n) \in \mathbb{N}$ depends on the dimension only.

ii) *Denote by $k(n)$ to be the smallest integer for which* (2.6) *holds. Let $a(x,\xi)$ be a symbol in $S_0^{-k(n)}(\mathbb{R}^n)$. Then for all $1<r<\infty$*

$$\|a(x,D)u\|_{L^r} \leq C_{r,n} \sum_{|\alpha|\leq k(n)-1, |\beta|\leq k(n)-1} p_{\alpha,\beta}(a)\|u\|_{L^r}. \tag{2.7}$$

Proof. The estimate (2.6) is Theorem 9.7 of [34]. For (2.7) we observe that if $a \in S_0^{-k(n)}$, then there exists a sequence $\{a_j\} \subset S^{-\infty}$ such that $p_{\alpha,\beta}(a-a_j) \to 0$ for $|\beta| \leq k(n)-1$. For $u \in \mathcal{S}$ we can conclude by using (2.6) that $\{a_j(x,D)u\}$ is a Cauchy sequence in L^r and therefore $a_j(x,D)u \to v$ in L^r. On the other hand, by standard L^2 estimates $a_j(x,D)u \to a(x,D)u$ in L^2. Therefore $v = a(x,D)u$. Applying estimate (2.6) to $a_j(x,D)u$ and taking the limit we have (2.7). □

Note that in $\mathbb{R}^n$ there is a relation between classical and semiclassical quantization of a symbol $a \in S^\infty$ given by

$$Op_h(a)u(x) = \sqrt{h}^{-n} A_h u_h(x/\sqrt{h}),$$

where u_h is defined by $(\mathcal{F}u_h)(\xi) = (\mathcal{F}u)(\xi/\sqrt{h})$ and $A_h = a_h(x,D)$ for $a_h(x,\xi) := a(\sqrt{h}x, \sqrt{h}\xi)$ ($\mathcal{F}$ denotes the **classical** Fourier transform). This identity combined with estimate (2.6) and (2.7) gives us a semiclassical version of Calderón-Vaillancourt: for all $1<r<\infty$, $h>0$ sufficiently small, and $a \in S_1^0 \cup S_0^{-k(n)}$

$$\|Op_h(a)u\|_{L^r} \leq \sum_{|\alpha|,|\beta|\leq k(n)-1} p_{\alpha,\beta}(a)\|u\|_{L^r}. \tag{2.8}$$

For symbols in $S_1^k S_1^{-\ell} \cup S_1^k S_0^{-k(n)-\ell}$, we have the following mapping properties.

Proposition 2.2. *If $b(x',\xi') \in S_1^k$ and $a(x',\xi) \in S_1^\ell \cup S_0^{-k(n)+\ell}$, then*

$$ab(x',hD) : W^{m,l,r} \to W^{m-k,l-\ell,r}$$

with norm uniformly bounded in $h>0$.

In addition, we have the following compositional calculus result.

Proposition 2.3. *If $a \in S_1^{k_1} S_1^{\ell_1} \cup S_1^{k_1} S_0^{-k(n)+\ell_1}$ and $b \in S_1^{k_2} S_1^{\ell_2} \cup S_1^{k_2} S_0^{-k(n)+\ell_2}$, then*

$$b(x',hD)a(x',hD) = ab(x',hD) + h\sum_{|\alpha|=1} (\partial_\xi^\alpha b \partial_{x'}^\alpha a)(x',hD) + h^2 m(x',hD),$$

where $m(x',hD) : W^{k,\ell,r} \to W^{k-k_1-k_2,\ell-\ell_1-\ell_2,r}$.

For proofs of Propositions 2.2 and 2.3, see the appendix.

Remark 2.4. We have omitted stating the mapping properties on H_δ^k spaces since $S_0^k S_0^\ell \subset S_0^{k+\ell}(\mathbb{R}^n)$ and the calculus for these symbols on weighted L^2 Sobolev spaces are well documented. See for example [30, Prop 2.2] for these results and for definition of weighted semiclassical Sobolev spaces.

3. Heat flow

Define coordinates on $\mathbb{R}^n$ and let $\mathbb{R}^n_+$ denote the upper half space $\{x_n > 0\}$. Let $F(x', \xi') \in S^1_1(\mathbb{R}^{n-1})$, and define the semiclassical pseudodifferential operator

$$j(x', hD) = h\partial_{x_n} + F(x', hD') \tag{3.1}$$

on $\mathbb{R}^n$. It follows by considering the ξ' and ξ_n direction separately and applying the semiclassical Calderón-Vaillancourt theorem that $j(x', hD)$ is a bounded operator $j(x', hD) : W^{1,r}(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ for $1 < r < \infty$. As we will see in the following section, one of the factors of the conjugated Laplacian has this form. In this section we will prove some basic facts about the existence and L^p mapping properties of the inverse of such an operator. This extends the L^2 theory explained in [6].

To obtain an inverse, we will assume that F obeys the ellipticity condition

$$c\langle\xi'\rangle \leq \mathrm{Re}F(x', \xi') \leq C\langle\xi'\rangle \tag{3.2}$$

uniformly in x' for some constants $c, C > 0$. This ensures that the principal symbol

$$j(x, \xi) := i\xi_n + F(x', \xi')$$

is uniformly elliptic. We will also assume a finiteness condition on F: that there exists $X' > 0$ such that for $|x'| > X'$,

$$\nabla_{x'} F(x', \xi') = 0. \tag{3.3}$$

We need an extra condition to ensure that the symbol j^{-1} is in the suitable calculus. We assume that there exists a first-order symbol $i\xi_n + F_-(x', \xi')$ with compact characteristic set, such that $D_{x'}F_-(x', \xi')$ is supported in $|x'| < X'$, and

$$(i\xi_n + F)(i\xi_n + F_-) = p(x', \xi) + a_0,$$

where $p(x', \xi)$ is a second-order polynomial in ξ with compact characteristic set and $a_0 \in S^{-\infty}(\mathbb{R}^{n-1})$.

The reason why we need this extra assumption is that $(i\xi_n + F)^{-1}$ is not in the class $S^{-1}_1(\mathbb{R}^n)$ (for example if $F = \langle\xi'\rangle$, then differentiating multiple times in ξ' does not yield additonal decay in the ξ_n direction). However, if $\chi \in C^\infty_0(\mathbb{R}^n)$ is identically 1 on a neighbourhood containing the characteristic sets of $i\xi_n + F_-$ and p, then we can derive the following expansion:

$$(1 - \chi(\xi))j^{-1} = (1 - \chi(\xi))(i\xi_n + F_-)\left(\frac{1}{p(x', \xi)} - \frac{a_0}{p(p + a_0)}\right).$$

Since χ is identically one on the characteristic set of p, it follows $(1 - \chi(\xi))/p(x', \xi)$ is a symbol in $S^{-2}_1(\mathbb{R}^n)$, and so

$$(1 - \chi(\xi))j^{-1} = (i\xi_n + F_-)\left(S^{-2}_1 - \frac{a_0(1 - \chi)}{p(p + a_0)}\right). \tag{3.4}$$

Using the fact that p is elliptic on support of $1 - \chi$ we can expand for all $N > 0$

$$\frac{1}{p + a_0} = \frac{1}{p}\sum_{j=0}^{N}\left(\frac{-a_0}{p}\right)^j + \frac{1}{p + a_0}\left(\frac{-a_0}{p}\right)^{N+1}.$$

Substituting this representation for $\frac{1}{p+a_0}$ into (3.4) we have

$$(1-\chi(\xi))j^{-1} = (i\xi_n + F_-)\Big(S_1^{-2} + a_0 S_1^{-4} + \cdots + a_0^m S_1^{-k(n)-1} + a_0^{m+1} S_0^{-k(n)-2}\Big),$$

where we are using S_j^k to represent a symbol from the class $S_j^k(\mathbb{R}^n)$. Now

$$(i\xi_n + F_-)S_1^{-2} \in S_1^{-1} + S_1^1 S_1^{-2}$$

and the same holds for

$$(i\xi_n + F_-)(a_0 S_1^{-4} + \cdots + a_0^m S_1^{-k(n)-1}).$$

Finally,

$$(i\xi_n + F_-)a_0^{m+1} S_0^{-k(n)-2} \in \operatorname{span}(S^{-\infty} S_0^{-1-k(n)})$$

so

$$(1-\chi(\xi))j^{-1} \in \operatorname{span}(S_1^0 S_1^{-1} + S^{-\infty} S_0^{-1-k(n)} + S_1^1 S_1^{-2}). \tag{3.5}$$

Meanwhile $\chi(\xi)j^{-1}(x',\xi) \in S^{-\infty}(\mathbb{R}^n)$, so we can use (3.5) in conjunction with Proposition 2.2 to get that

$$j^{-1}(x',hD) : L^2_\delta \to H^1_\delta, \quad \delta \in \mathbb{R}, \quad j^{-1}(x',hD) : L^r \to W^{1,r}, \quad 1 < r < \infty. \tag{3.6}$$

The operator $j^{-1}(x',hD)$ also turns out to have desirable support properties.

Lemma 3.1. *If $u \in L^r(\mathbb{R}^n)$ is supported only in $\{x_n \geq 0\}$, then $j^{-1}(x',hD)u \in W^{1,r}(\mathbb{R}^n)$ has trace zero along $\{x_n = 0\}$ and vanishes identically on the set $\{x_n < 0\}$.*

Proof. For $u \in C_c^\infty(\mathbb{R}^n)$, we can write

$$j^{-1}(x',hD)u(x) = h^{-n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} \frac{\mathcal{F}^h u(\xi)}{i\xi_n + F(x',\xi')} e^{\frac{i}{h}x\cdot\xi} d\xi_n\, d\xi',$$

where $\mathcal{F}^h$ is the semiclassical Fourier transform. We can write out the Fourier transform in the x_n variable to get

$$j^{-1}(x',hD)u(x) = h^{-n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\mathcal{F}^h_{x'} u(\xi',t)}{i\xi_n + F(x',\xi')} e^{\frac{i}{h}(x_n - t)\xi_n} d\xi_n\, dt\, e^{\frac{i}{h}x'\cdot\xi'} d\xi'.$$

Now we can use the residue theorem to evaluate the $d\xi_n$ integral explicitly. For $t \geq x_n$ we need to take a semicircular contour in the lower half plane. Since by assumption (3.2) ReF is positive this contour yields no residue. For $t \leq x_n$ we take a semicircular contour in the upper half plane. In this case the contour contains a pole at $\xi_n = iF(x',\xi')$. Therefore we get

$$j^{-1}(x',hD)u(x) = h^{-n}\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{x_n} \mathcal{F}^h_{x'} u(\xi',t) e^{\frac{t-x_n}{h}F(x',\xi')} dt\, e^{\frac{i}{h}x'\cdot\xi'} d\xi'.$$

For $u \in C_c^\infty(\mathbb{R}^n)$, the lemma follows immediately from this representation. The lemma follows for general $u \in L^r(\mathbb{R}^n)$ by (3.6) and density. □

Henceforth we will refer to the support property given in Lemma 3.1 as "preserving support in the x_n direction".

We can turn $j^{-1}(x',hD)$ into a proper inverse. We first prove a composition-type lemma for the operator $j^{-1}(x',hD)$.

Lemma 3.2. *Let $a(x',\xi') \in S^1_1(\mathbb{R}^{n-1})$. Then*

$$a(x',hD')j^{-1}(x',hD) = (aj^{-1})(x',hD) + h\sum_{|\alpha|=1}(j^{-2}\partial^\alpha_{\xi'}a\partial^\alpha_{x'}F)(x',hD) + h^2m(x',hD),$$

where $m(x',hD)$ and $\sum_{|\alpha|=1}(j^{-2}\partial^\alpha_{\xi'}a\partial^\alpha_{x'}F)(x',hD)$ map $L^r \to L^r$ with norm bounded by a constant independent of h. Furthermore, the commutator

$$[a(x',hD'),j^{-1}(x',hD)] = hm(x,hD)$$

with

$$m(x,hD) : L^r \to L^r, \quad m(x,hD) : L^2_\delta \to L^2_\delta.$$

Proof. The expansion (3.5) allows us to write $j^{-1}(x',\xi)$ as the span of elements in

$$S^0_1 S^{-1}_1 + S^{-\infty}S^{-1-k(n)}_0 + S^1_1 S^{-2}_1.$$

We can therefore apply Proposition 2.3 to each term to obtain

$$a(x',hD')j^{-1}(x',hD) = aj^{-1}(x',hD) + hm_1(x',hD) + h^2m_2(x',hD),$$

where

$$m_1(x',\xi) = \sum_{|\alpha|=1}(j^{-2}\partial^\alpha_{\xi'}a\partial^\alpha_{x'}F)(x',\xi) \quad\text{and}\quad m_2(x',hD) : L^r \to L^r.$$

Using expansion (3.5) again we see that $m_1(x',\xi)$ is a symbol in the span of

$$S^1_1 S^{-1}_1 + S^{-\infty}S^{-k(n)-1}_0 + S^2_1 S^{-2}_1.$$

Therefore, it maps $L^r \to L^r$ by Proposition 2.2 and the fact that $W^{-k,\ell,r} \subset W^{l-k,r}(\mathbb{R}^n)$. To obtain the commutator statement, we can repeat the argument for the composition $j^{-1}(x',hD)a(x',hD')$. □

Now we can use j^{-1} to build a proper inverse which preserves support in the x_n direction. More generally the inversion can still be carried out even if j is perturbed by a small tangential operator hF_0.

Proposition 3.3. *Suppose $F_0(x',\xi') \in S^0_1(\mathbb{R}^{n-1})$ obeys the same finiteness condition* (3.3) *as F, and consider the operator*

$$J := j(x',hD) + hF_0(x',hD).$$

For $h > 0$ sufficiently small there exists an inverse $J^{-1} : L^r \to W^{1,r}$ of the form

$$J^{-1} = j^{-1}(x',hD)(1 + hm_1(x',hD) + h^2m_2(x',hD))^{-1},$$

where $m_1(x',hD), m_2(x'hD) : L^r \to L^r$.

Furthermore, J^{-1} preserves support in the x_n direction. The same holds for J^{-1} acting on H^k_δ spaces.

Proof. We write

$$Jj^{-1}(x',hD) = h\partial_n j^{-1}(x',hD) + F(x',hD')j^{-1}(x',hD) + hF_0(x',hD')j^{-1}(x',hD).$$

We can apply Proposition 2.3 to the first term, using the expansion (3.5) for j^{-1}, and Lemma 3.2 to the second and third terms to obtain

$$Jj^{-1}(x',hD) = 1 + hm_1(x',hD) + h^2m_2(x',hD), \tag{3.7}$$

where

$$m_1(x',\xi) = \sum_{|\alpha|=1}(j^{-2}\partial^\alpha_{\xi'}F\partial^\alpha_{x'}F)(x',\xi) + (j^{-1}F_0)(x',\xi') \tag{3.8}$$

and $m_2(x', hD) : L^r \to L^r$. Using expansion (3.5) again we see that $m_1(x', \xi)$ is a symbol in the span of

$$S_1^1 S_1^{-1} + S^{-\infty} S_0^{-k(n)-1} + S_1^2 S_1^{-2}.$$

Therefore, it maps $L^r \to L^r$ by Proposition 2.2 and the fact that $W^{-k,\ell,r} \subset W^{l-k,r}(\mathbb{R}^n)$.

Observe that in equation (3.7) since J is a differential operator in the x_n direction, it preserves support in the x_n direction when acting on $W^{1,r}$. The operator $j^{-1}(x', hD) : L^r \to W^{1,r}$ preserves support in x_n by Lemma 3.1 and thus the left side preserves support in the x_n direction. We may conclude from this that the right side preserves x_n support as well and in particular $hm_1(x', hD) + h^2 m_2(x', hD)$ preserves x_n support. This means that inverting the right side by Neumann series preserves support in the x_n direction. ☐

One final consequence of the structure of J^{-1} we obtained in Proposition 3.3 is the following disjoint support property.

Lemma 3.4. *Let $\mathbf{1}_{\mathbb{R}^n_-}$ be the indicator function for $x_n \leq 0$ and $\epsilon > 0$. Then for all $f \in L^r(\mathbb{R}^n)$,*

$$\|J^{-1}\mathbf{1}_{\mathbb{R}^n_-} f\|_{W^{1,r}(\{x_n \geq \epsilon\})} \leq C_\epsilon h^2 \|f\|_{L^r}.$$

The analogous estimate holds as a map from weighted $L^2_\delta \to H^1_\delta$ Sobolev spaces.

Proof. Let $\zeta_\epsilon(x_n)$ be a smooth cutoff function which is identically one on $\{x_n \geq \epsilon\}$ and identically zero on an open set containing $\{x_n \leq 0\}$. Then

$$\|J^{-1}\mathbf{1}_{\mathbb{R}^n_-} f\|_{W^{1,r}(\{x_n \geq \epsilon\})} \leq \|\zeta_\epsilon J^{-1}\mathbf{1}_{\mathbb{R}^n_-} f\|_{W^{1,r}(\mathbb{R}^n)}.$$

Therefore it suffices to show that

$$\|\zeta_\epsilon J^{-1}\mathbf{1}_{\mathbb{R}^n_-} f\|_{W^{1,r}(\mathbb{R}^n)} \leq C_\epsilon h^2 \|f\|_{L^r}.$$

From Proposition 3.3, we have that

$$J^{-1} = j^{-1}(x'hD)(1 + hm_1(x', hD) + h^2 m_2(x', hD))^{-1},$$

where $(1 + hm_1(x', hD) + h^2 m_2(x', hD))^{-1}$ is given by the Neumann series

$$(1 + hm_1(x', hD) + h^2 m_2(x', hD))^{-1} = 1 + \sum_{k=1}^{\infty} (hm_1(x', hD) + h^2 m_2(x', hD))^k.$$

Therefore, by (3.6) we can write

$$J^{-1} = j^{-1}(x', hD)(1 + hm_1(x', hD)) + h^2 M(x', hD),$$

where $M : L^r \to W^{1,r}$ is bounded uniformly in h. Using this expression for J^{-1} it suffices to show that

$$\zeta_\epsilon j^{-1}(x', hD)(1 + hm_1(x', hD))\mathbf{1}_{\mathbb{R}^n_-} : L^r(\mathbb{R}^n) \to W^{1,r}(\mathbb{R}^n) \tag{3.9}$$

with norm bounded by $O(h^2)$. We will only show (3.9) for the principal part $\zeta_\epsilon j^{-1}(x', hD)\mathbf{1}_{\mathbb{R}^n_-}$ and leave the lower-order term, which can be written out explicitly using (3.8), to the reader. By using (3.5) we see that the symbol j^{-1} belongs to

$$j^{-1} \in \mathrm{span}(S_1^0 S_1^{-1} + S^{-\infty} S_0^{-1-k(n)} + S_1^1 S_1^{-2}).$$

We will only show (3.9) for $\zeta_\epsilon Op_h(S^1_1 S^{-2}_1)\mathbf{1}_{\mathbb{R}^n_-}$ and the others are treated in the same way. Suppose $b \in S^1_1(\mathbb{R}^{n-1})$ and $a \in S^{-2}_1(\mathbb{R}^n)$, by Proposition 2.3 we see that

$$\begin{aligned}\zeta_\epsilon ba(x', hD)\mathbf{1}_{\mathbb{R}^n_-} &= \zeta_\epsilon b(x', hD')a(x', hD)\mathbf{1}_{\mathbb{R}^n_-}\\ &\quad + h\zeta_\epsilon \sum_{|\alpha|=1} (\partial_{\xi'} b)(x', hD')(\partial_{x'} a)(x', hD)\mathbf{1}_{\mathbb{R}^n_-}\\ &\quad + h^2 m(x, hD),\end{aligned}$$

where $m(x', hD) : L^r \to W^{-1,2,r} \subset W^{1,r}(\mathbb{R}^n)$ by (2.4).

Since ζ_ϵ is a function of x_n only, it commutes with operators from $S^k_1(\mathbb{R}^{n-1})$, and thus estimating $\zeta_\epsilon ba(x', hD)\mathbf{1}_{\mathbb{R}^n_-}$ with $b \in S^1_1(\mathbb{R}^{n-1})$ and $a \in S^{-2}_1(\mathbb{R}^n)$ amounts to estimating terms of the form $\zeta_\epsilon Op_h(S^{-2}_1(\mathbb{R}^n))\mathbf{1}_{\mathbb{R}^n_-}$. Standard disjoint support properties of ΨDO then give the desired estimates. □

4. Green's functions on $\mathbb{R}^n$

The purpose of this discussion is to find a way to invert

$$h^2\Delta_\phi := h^2 e^{-\phi/h}\Delta e^{\phi/h}, \quad \phi(x) := x_n$$

with a suitable boundary condition and good $L^{p'} \to L^p$ estimates. We begin with the operator on $\mathbb{R}^n$ given by the Fourier multiplier $\frac{1}{|\xi|^2+2i\xi_n-1}$. We give a semiclassical formulation of an estimate established in Sylvester-Uhlmann [33].

Lemma 4.1. *The Fourier multiplier $\frac{1}{|\xi|^2+2i\xi_n-1}$ maps $L^2_\delta \to H^2_{\delta-1}$ for $\delta > 0$ with norm bounded by h^{-1}.*

Proof. Consider the multiplier given by $\frac{1}{|\xi|^2+2i\xi_n+2\xi_1}$. By the result of [33],

$$Op_h\left(\frac{1}{|\xi|^2 + 2i\xi_n + 2\xi_1}\right) : L^2_\delta \to_{h^{-1}} H^2_{\delta-1}.$$

Observe that $|\xi|^2 + 2i\xi_n + 2\xi_1 = |\xi_1 + 1|^2 + \sum_{j=2}^n |\xi_j|^2 + 2i\xi_n - 1$. Since shifting in the Fourier coordinate is equivalent to multiplying by a complex linear phase,

$$Op_h\left(\frac{1}{|\xi|^2 + 2i\xi_n - 1}\right) = e^{-ix_1/h} Op_h\left(\frac{1}{|\xi|^2 + 2i\xi_n + 2\xi_1}\right) e^{ix_1/h}$$

and the proof is complete. □

It turns out that the Fourier multiplier $\frac{1}{|\xi|^2+2i\xi_n-1}$ also satisfies $L^{p'} \to L^p$ estimates for $p = \frac{2n}{n-2}$ and $p' = \frac{2n}{n+2}$. We state below the semiclassical formulation of a result by Kenig-Ruiz-Sogge [20] and Chanillo [4].

Lemma 4.2. *The Fourier multiplier satisfies the estimate*

$$\left\| Op_h\left(\frac{1}{|\xi|^2 + 2i\xi_n - 1}\right) u \right\|_{L^p} \le \frac{C}{h^2}\|u\|_{L^{p'}}$$

for all $u \in L^{p'}(\mathbb{R}^n)$.

In order to deal with domains with nonflat boundaries we will "flatten" boundary pieces by a coordinate change of the type

$$\gamma : (y_n, y') \mapsto (x_n, x') = (y_n - f(y'), y'), \tag{4.1}$$

where $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is a smooth function which is constant outside of a compact set. Under this change of variables, the differential operator defined by

$$\tilde{u}(x) \mapsto -\left(e^{-y_n/h}\sum_{j=1}^{n} h^2\partial_{y_j}^2\left(e^{y_n/h}\tilde{u}(\gamma(y))\right)\right)\Big|_{y=\gamma^{-1}(x)}$$

can be written explicitly as

$$\begin{aligned} h^2\tilde{\Delta}_\phi &:= -(1+|K|^2)h^2\partial_n^2 - 2(1+\frac{h\Delta_{x'}f}{2})h\partial_n - 2K\cdot h\nabla_{x'}h\partial_n + h^2\Delta_{x'} - 1 \\ &= Op_h\left((1+|K|^2)\xi_n^2 - 2i\xi_n\left(\left(1+\frac{h\Delta_{x'}f}{2}\right)+i\xi'\cdot K\right) - (1-|\xi'|^2)\right), \end{aligned} \tag{4.2}$$

where $K(x') := \nabla f(x')$ and for convenience we will later denote $1+\frac{h\Delta_{x'}f}{2}$ by 1_h as it is 1 in the semiclassical limit. The next proposition concerns the Green's function for $h^2\tilde{\Delta}_\phi$. More specifically we define $\tilde{G}_\phi := \gamma^* G_\phi$ by

$$\tilde{u} \mapsto \left(G_\phi(\tilde{u}(\gamma(y)))\right)\big|_{y=\gamma^{-1}(x)}$$

which is equivalent to conjugating by the operator given by pulling back by γ.

Proposition 4.3. *The Green's function $\tilde{G}_\phi$ satisfies $h^2\tilde{\Delta}_\phi\tilde{G}_\phi = \mathrm{Id}$ and has the bounds*

$$\|\tilde{G}_\phi\|_{L^2_\delta\to H^2_{\delta-1}} \le Ch^{-1}, \quad \|\tilde{G}_\phi\|_{L^{p'}\to L^p} \le Ch^{-2}.$$

Furthermore, we can split $\tilde{G}_\phi = \tilde{G}^c_\phi + (\tilde{G}_\phi - \tilde{G}^c_\phi)$ such that $(\tilde{G}_\phi - \tilde{G}^c_\phi)$ is a ΨDO with symbol in $S^{-2}_1(\mathbb{R}^n)$ and

$$\|\tilde{G}^c_\phi\|_{L^{p'}\to W^{k,p}} \le Ch^{-2}, \quad \|\tilde{G}^c_\phi\|_{L^2_\delta\to H^k_{\delta-1}} \le Ch^{-1}\ \forall k\in\mathbb{N}.$$

Proof. The fact that $h^2\tilde{\Delta}_\phi\tilde{G}_\phi = \mathrm{Id}$ comes directly from the definition of $\tilde{\Delta}_\phi$ and $\tilde{G}_\phi$.

Since $\tilde{G}_\phi = (\gamma^{-1})^*\circ G_\phi\circ\gamma^*$ it suffices to prove the estimates for G_ϕ and show that G_ϕ can be split into $G_\phi = G^c_\phi + (G_\phi - G^c_\phi)$. Once this is done the corresponding statements for $\tilde{G}_\phi$ follow via conjugation by the diffeomorphism γ whose Jacobian is identity outside of a compact set.

By Lemmas 4.2 and 4.1 the operator G_ϕ satisfies $G_\phi : L^2_\delta(\mathbb{R}^n) \to H^2_{\delta-1}(\mathbb{R}^n)$ with $O(h^{-1})$ norm and $L^{p'}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ with $O(h^{-2})$ norm.

The multiplier of G_ϕ is constant coefficient so one can write $G_\phi = G^c_\phi + (G_\phi - G^c_\phi)$ where

$$G^c_\phi = G_\phi\chi_0(hD) = \chi_0(hD)G_\phi = \chi_1(hD)G_\phi\chi_0(hD),$$

where $\chi_0, \chi_1 \in C^\infty_c(\mathbb{R}^n)$, with χ_0 identically 1 in the ball of radius 2 and χ_1 identically 1 on the support of χ_0. Everything commutes in the above identity since they are all constant coefficient Fourier multipliers.

Since the characteristic set of G_ϕ is disjoint from the support of $1-\chi_0$, the operator $(G_\phi - G^c_\phi)$ is a ΨDO with symbol in $S^{-2}_1(\mathbb{R}^n)$.

The mapping properties of G^c_ϕ come from the mapping properties of G_ϕ and the fact that $\chi_1(hD)$ has a compactly supported symbol. □

The explicit representation of $\tilde{\Delta}_\phi$ in (4.2) shows that its characteristic set lies in the sphere $|\xi'| = 1$, and so in particular if G^c_ϕ is multiplied by a Fourier side cutoff function supported away from that sphere, the resulting operator is well behaved. The following lemma makes this somewhat more precise.

Lemma 4.4. *Let $\tilde\rho(\xi') \in S^{-\infty}(\mathbb{R}^{n-1})$ be a smooth symbol with support compactly contained in $|\xi'| < 1$. Then $\tilde\rho(hD')\tilde G^c_\phi = Op_h(S^{-\infty}(\mathbb{R}^n)) + hm(x', hD)\tilde G_\phi$ for some $m(x',\xi) \in S^{-\infty}(\mathbb{R}^n)$.*

Proof. By the construction in Proposition 4.3,

$$\tilde G^c_\phi = \gamma^* G^c_\phi := (\gamma^{-1})^* \circ G^c_\phi \circ \gamma^*,$$

with $G^c_\phi = \chi_0(hD)G_\phi$. We compute

$$\tilde\rho(hD')\gamma^*(G^c_\phi) = \tilde\rho(hD')\gamma^*(\chi_0(hD))\tilde G_\phi = (\tilde\rho(hD')\tilde\chi(x', hD) + hOp_h(S^{-\infty}(\mathbb{R}^n)))\tilde G_\phi,$$

where $\tilde\chi(x',\xi) = \chi_0(D\gamma(x')^T\xi)$ is the pull-back symbol. By the composition formula in Proposition 2.3,

$$\begin{aligned}\tilde\rho(hD')\gamma^*(G^c_\phi) &= (Op_h(\tilde\rho\tilde\chi) + hOp_h(S^{-\infty}(\mathbb{R}^n)))\tilde G_\phi\\ &= \gamma^*(Op_h(\chi_0\rho)G_\phi) + hOp_h(S^{-\infty}(\mathbb{R}^n))\tilde G_\phi\end{aligned}$$

where $\rho(x',\xi) = \tilde\rho(\xi' + \xi_n K(x'))$ is the push-forward symbol. Observe that since G_ϕ is a constant coefficient, $Op_h(\chi_0\rho)G_\phi = Op_h\left(\frac{\chi_0(\xi)\rho(x',\xi)}{|\xi'|^2+\xi_n^2-1+2i\xi_n}\right)$. Since $\tilde\rho(\xi')$ vanishes in an open neighbourhood of $|\xi'| = 1$, the symbol

$$\frac{\chi_0(\xi)\rho(x',\xi)}{|\xi'|^2 + \xi_n^2 - 1 + 2i\xi_n}$$

belongs to $S^{-\infty}(\mathbb{R}^n)$. □

4.1. **Modified factorization.** To add boundary determination to the Green's function, we want to take advantage of the fact that $h^2\tilde\Delta_\phi$ factors into two parts, one of which is elliptic and resembles the operator described in Section 3.

Indeed, the symbol of $\frac{1}{1+K^2}h^2\tilde\Delta_\phi$ which appears in (4.2) factors formally as

$$\begin{aligned}&\frac{1}{1+K^2}h^2\tilde\Delta_\phi\\ &\quad= \xi_n^2 - 2i\xi_n\left(\frac{1_h + \xi'\cdot K}{1+|K|^2}\right) - \frac{(1-|\xi'|^2)}{1+|K|^2}\\ &\quad= \left(\xi_n - i\left(\frac{(1_h + iK\cdot\xi') - \sqrt{(1_h + iK\cdot\xi')^2 - (1-|\xi'|^2)(1+|K|^2)}}{1+|K|^2}\right)\right)\\ &\quad\times\left(\xi_n - i\left(\frac{(1_h + iK\cdot\xi') + \sqrt{(1_h + iK\cdot\xi')^2 - (1-|\xi'|^2)(1+|K|^2)}}{1+|K|^2}\right)\right),\end{aligned}$$

where $1_h = \left(1 + \frac{h\Delta'_{x'}f}{2}\right)$. Note that the second factor here is elliptic. The problem is that the square root is not smooth at its branch cut, so this does not give a proper factorization at the operator level. The obvious thing to do is to take a smooth approximation to the square root, but for our purposes we will require something more subtle.

We take the branch of the square root that has nonnegative real part, and seek to avoid the branch cut, which happens when the argument of the square root lies on the negative real axis. From examination of the square root, we see that this occurs when $K\cdot\xi' = 0$ and $|\xi'|^2 + h\frac{\Delta_{x'}f}{2} \le |K|^2(1+|K|^2)^{-1}$. By ensuring that ξ' avoids this set, we can guarantee that the argument of the square root stays away from the branch cut.

Thus let $0 < c < c' < 1$ be a constant such that $\frac{|K|^2}{1+|K|^2} < c$ for all x' and let $\tilde{\rho}_0(\xi')$ be a smooth function in ξ' such that $\tilde{\rho}_0 = 1$ for $|\xi'|^2 \leq c$ and $\mathrm{supp}(\tilde{\rho}_0) \Subset B_{c'}$. Introduce a second cutoff $\tilde{\rho}$ such that it is identically 1 on $|\xi'|^2 \leq c'$ but $\mathrm{supp}(\tilde{\rho}) \Subset B_{\sqrt{c'}}$. Observe that for $h > 0$ sufficiently small

$$\inf_{\xi \in \mathrm{supp}\tilde{\rho},\ x' \in \mathbb{R}^{n-1}} \left| \xi_n^2 - 2i\xi_n \frac{(1_h + i\xi' \cdot K)}{1+|K|^2} - \frac{(1-|\xi'|^2)}{1+|K|^2} \right| > 0. \tag{4.3}$$

Since the branch cut of the square root occurs when

$$|\xi'|^2 + h\frac{\Delta_{x'} f}{2} \leq |K|^2(1+|K|^2)^{-1},$$

it follows that for ξ' in the support of $1 - \tilde{\rho}_0$ and $h > 0$ sufficiently small, the function

$$(1_h + iK \cdot \xi')^2 - (1-|\xi'|^2)(1+K^2)$$

stays uniformly away from the branch cut of the square root. As such we may define

$$r := (1-\tilde{\rho}_0)\sqrt{(1_h + iK \cdot \xi')^2 - (1-|\xi'|^2)(1+|K|^2)} \tag{4.4}$$

and factor

$$\begin{aligned} \xi_n^2 - 2i\xi_n \frac{(1_h + i\xi' \cdot K)}{1+|K|^2} - \frac{(1-|\xi'|^2)}{1+|K|^2} &= (\xi_n - \tilde{a}_- + hm_0)(\xi_n - \tilde{a}_+ - hm_0) \\ &\quad + \tilde{a}_0 + h\sum_{|\alpha|=1} \partial_{\xi'}^\alpha \tilde{a}_- \partial_{x'}^\alpha \tilde{a}_+ - hm_0\tilde{a}_- + h^2 m_0^2 \end{aligned} \tag{4.5}$$

with $m_0(x', \xi') := \tilde{a}_+^{-1} \sum_{|\alpha|=1} \partial_{\xi'}^\alpha \tilde{a}_- \partial_{x'}^\alpha \tilde{a}_+$. Here the $\tilde{a}_\pm$ and $\tilde{a}_0$ are defined by

$$\begin{aligned} \tilde{a}_\pm &= i\left(\frac{(1_h + iK \cdot \xi') \pm r}{1+|K|^2}\right) \\ \text{and} \quad \tilde{a}_0 &= \frac{(1_h + iK \cdot \xi')^2 - (1-|\xi'|^2)(1+|K|^2) - r^2}{(1+|K|^2)^2}. \end{aligned} \tag{4.6}$$

Observe that the support of $\tilde{a}_0$ is compactly contained in the interior of the set where $\tilde{\rho} = 1$.

We now quantize (4.5) to see that

$$\frac{1}{1+K^2} h^2 \tilde{\Delta}_\phi = QJ + \tilde{a}_0(x', hD') - h\tilde{e}_1(x', hD') + h^2\tilde{e}_0(x', hD'), \tag{4.7}$$

where $\tilde{e}_1 = m_0\tilde{a}_- \in S_1^1(\mathbb{R}^{n-1})$, $\tilde{e}_0 \in S_1^0(\mathbb{R}^{n-1})$, and Q and J are the operators with symbols $\xi_n - \tilde{a}_- + hm_0$ and $\xi_n - \tilde{a}_+ - hm_0$, respectively. Observe that the $O(h)$ term in the composition formula for QJ is killed by one of the $O(h)$ terms in (4.5).

Although this decomposition still gives us an $O(h)$ error, the symbol $\tilde{e}_1$ vanishes when $|\xi'| = 1$. In particular it vanishes on the characteristic set of $h^2\tilde{\Delta}_\phi$ which is $\{\xi_n = 0, |\xi'| = 1\}$ by (4.2). We use this observation to show that $h\tilde{e}_1(x', hD')\tilde{G}_\phi$ behaves one order of h better than expected.

Lemma 4.5. *Let $\tilde{E}_1$ denote $\tilde{e}_1(x', hD')$. The operator $\tilde{E}_1\tilde{G}_\phi$ is of the form*

$$\tilde{E}_1\tilde{G}_\phi = (\tilde{E}_1\tilde{G}_\phi)^c + Op_h(S_1^1 S_1^{-2}) + h\tilde{e}_1'(x', hD')\tilde{G}_\phi + hOp_h(S^{-\infty}(\mathbb{R}^n))\tilde{G}_\phi$$

with $\tilde{e}_1' \in S_1^0(\mathbb{R}^{n-1})$ and

$$(\tilde{E}_1\tilde{G}_\phi)^c : L^2 \to_{h^0} H^k, \quad (\tilde{E}_1\tilde{G}_\phi)^c : L^{p'} \to_{h^{-1}} H^k \ \forall k \in \mathbb{N}.$$

Here the notation $T : X \to_{h^m} Y$ indicates that the norm of the operator T from X to Y is bounded by $O(h^m)$.

Proof. We use the fact that $\tilde{e}_1$ takes value zero on the characteristic set of $\tilde{G}_\phi$. First write

$$\tilde{E}_1 = \tilde{e}_1(x', hD') = Op_h(\tilde{a}_+^{-1} m_0 \tilde{a}_- \tilde{a}_+) = Op_h(\tilde{a}_+^{-1} m_0) Op_h(\tilde{a}_- \tilde{a}_+) + hOp_h \tilde{e}_1'(x', hD')$$

for some $\tilde{e}_1' \in S_1^0(\mathbb{R}^{n-1})$. Note that

$$Op_h(\tilde{a}_- \tilde{a}_+)\tilde{G}_\phi = Op_h(\tilde{a}_- \tilde{a}_+)\gamma^*(\chi G_\phi) + Op_h(\tilde{a}_- \tilde{a}_+)\gamma^*((1-\chi)G_\phi)$$

for some compactly supported smooth function $\chi(\xi)$ which is identically 1 on the ball of radius 2. This means that

$$\tilde{E}_1 \tilde{G}_\phi = Op_h(\tilde{a}_+^{-1} m_0) Op_h(\tilde{a}_- \tilde{a}_+)\gamma^*(\chi G_\phi) + Op_h(S_1^1 S_1^{-2}) + h\tilde{E}_1' \tilde{G}_\phi. \tag{4.8}$$

From Proposition 4.3 $\tilde{G}_\phi = \gamma^* G_\phi$ where G_ϕ is the Fourier multiplier $\frac{1}{\xi_n^2 + i2\xi_n + (1-|\xi'|^2)}$. We compute the $Op_h(\tilde{a}_- \tilde{a}_+)\gamma^*(\chi G_\phi)$ portion of this operator

$$\begin{aligned} Op_h(\tilde{a}_- \tilde{a}_+)\gamma^*(\chi(hD)G_\phi) &= Op_h(\tilde{a}_- \tilde{a}_+)\gamma^*(\chi(hD))\gamma^*(G_\phi) \\ &= Op_h(\tilde{a}_- \tilde{a}_+)(\tilde{\chi}(x, hD) + hOp_h(S^{-\infty}))\gamma^*(G_\phi), \end{aligned}$$

where $\tilde{\chi}(x,\xi) \in S^{-\infty}$ is the pulled-back symbol of $\chi(\xi)$. Compose $Op_h(\tilde{a}_- \tilde{a}_+)\tilde{\chi}(x, hD)$ using symbol calculus to yield

$$Op_h(\tilde{a}_- \tilde{a}_+)\gamma^*(\chi G_\phi) = \gamma^*(Op_h(a_- a_+)(\chi(hD)G_\phi)) + hOp_h(S^{-\infty})\gamma^*(G_\phi),$$

where $a_\pm(x,\xi) := (\gamma^* \tilde{a}_\pm)(x,\xi) = \tilde{a}_\pm(x', D\gamma^T \xi)$.

We claim that $Op_h(a_- a_+)(\chi(hD)G_\phi)$ can be written as the sum of a ΨDO with symbol in $S^{-\infty}(\mathbb{R}^n)$ and a part containing the characteristic set

$$(Op_h(a_- a_+)(\chi(hD)G_\phi))^c : L^2 \to_{h^0} H^k, \quad (Op_h(a_- a_+)(\chi(hD)G_\phi))^c : L^{p'} \to_{h^{-1}} H^k. \tag{4.9}$$

Inserting this into (4.8) would give us the lemma.

We verify our claim. Observe that

$$a_+ a_- = \frac{\rho_0(2-\rho_0)(1_h + iK \cdot (\xi' + \xi_n K))^2}{(1+|K|^2)^2} + \frac{(1-\rho_0)^2(1 - |\xi' + \xi_n K|^2)}{1+|K|^2}, \tag{4.10}$$

where $\rho_0(x',\xi) = \tilde{\rho}_0(\xi' + \xi_n K)$. Now $\tilde{\rho}_0(x',\xi') = 0$ if $|\xi'| \geq c'$ for some $c' < 1$ and $K(x')$ is uniformly bounded. Therefore $\rho_0(x',\xi) = 0$ if

$$\frac{1+c'}{2} \leq |\xi'| \leq 2 - c' \quad \text{and} \quad |\xi_n| \leq \frac{1-c'}{2(\sup_{x'} |K(x')| + 1)}.$$

Since the characteristic set of the Fourier multiplier $\frac{1}{\xi_n^2 + i2\xi_n + (1-|\xi'|^2)}$ is compactly contained in this set, let $\chi_2(\xi)$ be a cutoff which is supported in this set and let 1 be in a neighbourhood of the characteristic set and define

$$(Op_h(a_- a_+)(\chi(hD)G_\phi))^c := Op_h(a_- a_+)(\chi(hD)\chi_2(hD)G_\phi).$$

We now write $(Op_h(a_- a_+)(\chi(hD)G_\phi))$ as a sum of two operators

$$(Op_h(a_- a_+)(\chi(hD)G_\phi))^c + (Op_h(a_- a_+)(\chi(hD)(1-\chi_2(hD))G_\phi).$$

The second expression is ΨDO of order $-\infty$ since $(\chi(hD)(1-\chi_2(hD))G_\phi)$ vanishes identically near the characteristic set of G_ϕ and is therefore a compactly supported smooth multiplier.

It remains to establish (4.9) for the part containing the characteristic set given by

$$(Op_h(a_-a_+)(\chi(hD)G_\phi))^c = Op_h(a_-a_+)(\chi(hD)\chi_2(hD)G_\phi).$$

Since ρ_0 vanishes identically on the support of χ_2 and χ_2 is a constant coefficient Fourier multiplier, it follows from (4.10) that

$$Op_h(a_+a_-)\chi(hD)\chi_2(hD)G_\phi = Op_h\left(\frac{(1-|\xi'+\xi_n K|^2)}{(1+|K|^2)^2}\right)\chi(hD)\chi_2(hD)G_\phi.$$

Note since $Op_h(\frac{(1-|\xi'+\xi_n K|^2)}{(1+|K|^2)^2})$ is a differential operator, proving (4.9) amounts to proving estimates for the operators

$$Op_h(\frac{(1-|\xi'|^2)\chi(\xi)}{\xi_n^2+i2\xi_n+(1-|\xi'|^2)}) \quad \text{and} \quad Op_h(\frac{\xi_n\chi(\xi)\langle\xi'\rangle}{\xi_n^2+i2\xi_n+(1-|\xi'|^2)}).$$

Crucially, these are both bounded Fourier multipliers with compact support and therefore map $L^2\to H^k$ for all $k\in\mathbb{N}$ with norm $O(1)$. Therefore

$$Op_h(a_+a_-)\chi_2(hD)\chi(hD) : L^2\to H^k$$

with norm $O(1)$.

Moving on to the $L^{p'}\to H^k$ estimate we write $\chi(hD)G_\phi = \chi(hD)G_\phi\chi_{100}(hD)$ where $\chi_{100}(\xi)$ is identically 1 on the support of χ. The estimate is then a result of the L^2 estimate and the fact that $\chi_{100}(hD) : L^{p'}\to_{h^0} W^{1,p'}\hookrightarrow_{h^{-1}} L^2$ by Sobolev embedding. Therefore

$$Op_h(a_+a_-)\chi_2(hD)\chi(hD) : L^{p'}\to_{h^{-1}} H^k.$$

□

5. Parametrices on the half space

In this section we construct parametrices for $h^2\tilde{\Delta}_\phi$ on the upper half space which give a vanishing trace on the boundary. By a change of variables, we will later use these to build the Green's function of Theorem 1.3. Because the factoring in (4.7) contains a large error term A_0 at small frequencies, we will perform two separate constructions—one for the large frequency case (on $\mathrm{supp}(1-\tilde\rho)$) and one for the small frequency case (on $\mathrm{supp}(\tilde\rho)$). We split the two frequency cases by using the cutoff function $\tilde\rho:\mathbb{R}^{n-1}\to\mathbb{R}$ defined above equation (4.3).

5.1. **Parametrix for $h^2\tilde{\Delta}_\phi$ at large frequency.** Let $\tilde{G}_\phi$ be the Green's function from Proposition 4.3, and $J^+ := J^{-1}\mathbf{1}_{\mathbb{R}^n_+}$ where J^{-1} is defined as in Proposition 3.3. Let $\tilde\Omega\subset\mathbb{R}^n_+$ be a smooth bounded open subset of the upper half space (with possibly a portion of the boundary intersecting $x_n=0$). We show that the operator

$$P_l := (1-\tilde\rho(hD'))J^+J\tilde{G}_\phi \tag{5.1}$$

is a suitable parametrix for the operator $h^2\tilde{\Delta}_\phi$ in $\tilde\Omega$ at large frequencies.

We begin by showing that P_l has mapping properties like those of $\tilde{G}_\phi$.

Proposition 5.1. *The map P_l satisfies, for $\delta>0$,*

$$P_l : L^2_\delta(\mathbb{R}^n)\to_{h^{-1}} H^1_{\delta-1}(\mathbb{R}^n),\quad P_l : L^{p'}(\mathbb{R}^n)\to_{h^{-2}} L^p(\mathbb{R}^n).$$

Furthermore, $P_lv\in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $P_lv\,|_{x_n=0}=0$ for all $v\in L^{p'}(\mathbb{R}^n)$.

Proof. The weighted L^2 Sobolev norms come as a direct consequence of the mapping properties of $\tilde{G}_\phi$ and the fact that J, J^{-1} has symbols in $S^k_0(\mathbb{R}^n)$.

For the mapping property from $L^{p'}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, we split $\tilde{G}_\phi = \tilde{G}^c_\phi + (\tilde{G}_\phi - \tilde{G}^c_\phi)$ following Propostion 4.3 and observe

$$J^+ J\tilde{G}^c_\phi : L^{p'} \xrightarrow[h^{-2}]{\tilde{G}^c_\phi} W^{k,p} \xrightarrow{J} W^{k-1,p} \xrightarrow{J^+} W^{1,p} \quad \text{and}$$

$$J^+ J(\tilde{G}_\phi - \tilde{G}^c_\phi) : L^{p'} \xrightarrow{\tilde{G}_\phi - \tilde{G}^c_\phi} W^{2,p'} \xrightarrow{J} W^{1,p'} \hookrightarrow_{h^{-1}} L^2 \xrightarrow{J^+} H^1 \hookrightarrow_{h^{-1}} L^p.$$

The above diagrams also show that $P_l v \in H^1_{\rm loc}$ for all $v \in L^{p'}$ by omitting the last Sobolev embedding. The trace property then comes from the definition of P_l and Proposition 3.3. □

In the following statement we denote $\mathbf{1}_{\tilde{\Omega}}$ to be the indicator function of $\tilde{\Omega}$. If $v \in L^r(\tilde{\Omega})$ we use the notation $\mathbf{1}_{\tilde{\Omega}} v$ to denote its trivial extension to a function in $L^r(\mathbb{R}^n)$.

Proposition 5.2. *Let $\tilde{\Omega} \subset \mathbb{R}^n_+$ be a bounded domain with $\partial\tilde{\Omega} \cap \{x_n = 0\} \neq \varnothing$. Denote by $\mathbf{1}_{\tilde{\Omega}}$ the indicator function of $\tilde{\Omega}$. Then P_l is a parametrix at large frequencies with vanishing trace on the boundary of the upper half space, in the sense that for all $v \in L^{p'}(\tilde{\Omega})$,*

$$\mathbf{1}_{\tilde{\Omega}} h^2 \tilde{\Delta}_\phi P_l \mathbf{1}_{\tilde{\Omega}} v = (1 - \tilde{\rho}(hD') + R_l + hR'_l)v,$$

with

$$P_l \mathbf{1}_{\tilde{\Omega}} v \in H^1_{\rm loc}(\mathbb{R}^n), \quad P_l \mathbf{1}_{\tilde{\Omega}} v\,|_{\partial\tilde{\Omega}\cap\{x_n=0\}} = 0,$$

where $R_l = \mathbf{1}_{\tilde{\Omega}} R_l \mathbf{1}_{\tilde{\Omega}}$ and $R'_l = \mathbf{1}_{\tilde{\Omega}} R'_l \mathbf{1}_{\tilde{\Omega}}$ have the estimates

$$R_l : L^2 \to_h L^2, \quad R_l : L^{p'} \to_{h^0} L^2, \quad R'_l : L^r \to_{h^0} L^r, \quad 1 < r < \infty.$$

Proof. We compute in the sense of distributions on $\mathbb{R}^n_+$ acting on $C^\infty_0(\mathbb{R}^n_+)$. Using (4.7) we have

$$\begin{aligned} h^2\tilde{\Delta}_\phi P_l &= (1-\tilde{\rho})h^2\tilde{\Delta}_\phi J^+ J\tilde{G}_\phi + [h^2\tilde{\Delta}_\phi, \tilde{\rho}]J^+ J\tilde{G}_\phi \\ &= (1-\tilde{\rho})(1+K^2)(QJ + \tilde{A}_0 + h\tilde{E}_1 + h^2\tilde{E}_0)J^+ J\tilde{G}_\phi + [h^2\tilde{\Delta}_\phi, \tilde{\rho}]J^+ J\tilde{G}_\phi \\ &= (1-\tilde{\rho})(1+K^2)(Q\mathbf{1}_{\mathbb{R}^n_+} J\tilde{G}_\phi) + (1-\tilde{\rho})(1+K^2)\tilde{A}_0 J^+ J\tilde{G}_\phi \\ &\quad + h(1-\tilde{\rho})(1+K^2)\tilde{E}_1 J^+ J\tilde{G}_\phi + h^2(1-\tilde{\rho})(1+K^2)\tilde{E}_0 J^+ J\tilde{G}_\phi \\ &\quad + [h^2\tilde{\Delta}_\phi, \tilde{\rho}]J^+ J\tilde{G}_\phi. \end{aligned}$$

The first term requires some care. Applying this operator to functions $v \in C^\infty_0(\mathbb{R}^n)$ and testing it against $u \in C^\infty_0(\mathbb{R}^n_+)$ yields

$$\langle Q^*(1+K^2)(1-\tilde{\rho})^* u, (\mathbf{1}_{\mathbb{R}^n_+} J\tilde{G}_\phi)v\rangle_{L^2(\mathbb{R}^n)}.$$

The operator Q^* is a ΨDO in the ξ' direction but it is only a differential operator in the ξ_n direction. Therefore the support does not spread in the x_n direction. The operator $\tilde{\rho}(hD')$ is an operator only in the ξ' direction and therefore does not spread support in the x_n direction. As such $Q^*(1+K^2)(1-\tilde{\rho})^* u$ vanishes in an open neighbourhood containing the closure of the lower half space and therefore for all $u \in C^\infty_0(\mathbb{R}^n_+)$ and $v \in C^\infty_0(\mathbb{R}^n)$,

$$\langle Q^*(1+K^2)(1-\tilde{\rho})^* u, (\mathbf{1}_{\mathbb{R}^n_+} J\tilde{G}_\phi)v\rangle_{L^2(\mathbb{R}^n)} = \langle Q^*(1+K^2)(1-\tilde{\rho})^* u, (J\tilde{G}_\phi)v\rangle_{L^2(\mathbb{R}^n)}.$$

Therefore we may continue our computation:

$$\begin{aligned} h^2\tilde{\Delta}_\phi P_l &= (1-\tilde{\rho})(1+K^2)(QJ\tilde{G}_\phi) + (1-\tilde{\rho})(1+K^2)\tilde{A}_0 J^+ J\tilde{G}_\phi \\ &\quad + h(1-\tilde{\rho})(1+K^2)\tilde{E}_1 J^+ J\tilde{G}_\phi + h^2(1-\tilde{\rho})(1+K^2)\tilde{E}_0 J^+ J\tilde{G}_\phi \\ &\quad + [h^2\tilde{\Delta}_\phi, \tilde{\rho}]J^+ J\tilde{G}_\phi. \end{aligned}$$

At this juncture we invoke the factorization (4.7) again and plug the relation

$$h^2\tilde{\Delta}_\phi - (1+K^2)(\tilde{A}_0 - h\tilde{E}_1 + h^2\tilde{E}_0) = (1+K^2)QJ$$

into the first term. Since $h^2\tilde{\Delta}_\phi G_\phi = I$, we get for all $v \in C_0^\infty(\mathbb{R}^n)$,

$$h^2\tilde{\Delta}_\phi P_l v = (1-\tilde{\rho})v + R_1 v + R_2 v + [h^2\tilde{\Delta}_\phi, \tilde{\rho}]J^+ J\tilde{G}_\phi v \tag{5.2}$$

as a distribution on $\mathbb{R}^n_+$ (i.e., integrating against functions in $C_0^\infty(\mathbb{R}^n_+)$) where

$$R_1 = h(1-\tilde{\rho})(1+K^2)\tilde{E}_1(1-J^+J)\tilde{G}_\phi, \tag{5.3}$$

$$R_2 = (1-\tilde{\rho})(1+K^2)(\tilde{A}_0 - \tilde{A}_0 J^+ J + h^2\tilde{E}_0 - h^2\tilde{E}_0 J^+ J)\tilde{G}_\phi. \tag{5.4}$$

Lemma 5.3. *The last term of* (5.2) *can be estimated by*

$$[h^2\tilde{\Delta}_\phi, \tilde{\rho}]J^+ J\tilde{G}_\phi = h^2 R_0'' + hR_0',$$

where $R_0' : L^r \to L^r$, $R_0'' : L^2_\delta \to_{h^{-1}} L^2_{\delta-1}$, *and* $R_0'' : L^{p'} \to_{h^{-2}} L^p$.

Lemma 5.4. *The operator* R_1 *from* (5.3) *can be written as* $R_1 = R_1' + R_1''$ *where*

$$\|R_1'\|_{L^2\to L^2} + h\|R_1'\|_{L^{p'}\to L^2} \le Ch, \quad \|R_1''\|_{L^2_\delta \to L^2_{\delta-1}} + h\|R_1''\|_{L^{p'}\to L^p} \le Ch.$$

Lemma 5.5. *The operator* R_2 *from equation* (5.4) *maps* $L^2_\delta \to L^2_{\delta-1}$ *with norm* $O(h)$ *while* $R_2 : L^{p'} \to L^p$ *with norm* $O(1)$.

We leave the proofs of these lemmas until the end of the section.

The remainder terms in (5.2) can now be estimated using Lemmas 5.3, 5.4, and 5.5 to show that, when tested against $u \in C_0^\infty(\tilde{\Omega})$,

$$\mathbf{1}_{\tilde{\Omega}} h^2\tilde{\Delta}_\phi P_l \mathbf{1}_{\tilde{\Omega}} v = (1 - \tilde{\rho}(hD') + R_l + hR_l')v,$$

where $R_l = \mathbf{1}_{\tilde{\Omega}} R_l \mathbf{1}_{\tilde{\Omega}}$ and $R_l' = \mathbf{1}_{\tilde{\Omega}} R_l' \mathbf{1}_{\tilde{\Omega}}$ have the estimates

$$R_l : L^2 \to_h L^2, \quad R_l : L^{p'} \to_{h^0} L^2, \quad R_l' : L^r \to_{h^0} L^r, \quad 1 < r < \infty.$$

The trace property of the operator $P_l\mathbf{1}_{\tilde{\Omega}}$ on $\partial\tilde{\Omega} \cap \{x_n = 0\}$ is a result of Proposition 5.1. Note that the L^2 bounds in Proposition 5.2 are unweighted because of the conjugation with indicator functions of $\tilde{\Omega}$. □

Proof of Lemma 5.3. We have

$$[h^2\tilde{\Delta}_\phi, \tilde{\rho}]J^+ J\tilde{G}_\phi = [K^2, \tilde{\rho}]h^2 D_n^2 J^+ J\tilde{G}_\phi - 2[K \cdot hD_{x'}, \tilde{\rho}]hD_n J^+ J\tilde{G}_\phi.$$

Some care will be needed in treating the term involving $h^2D_n^2$ hitting $J^+ = J^{-1}\mathbf{1}_{\mathbb{R}^n_+}$. We are only considering the expressions as maps to distributions on $\mathbb{R}^n_+$, so for all $u \in C_0^\infty(\mathbb{R}^n_+)$ and $v \in C_0^\infty(\mathbb{R}^n)$,

$$\langle hD_n u, hD_n J^{-1}\mathbf{1}_{\mathbb{R}^n_+} v\rangle = \langle hD_n u, (1 - FJ^+)v\rangle = \langle u, hD_n v - Fv - F^2 J^+ v\rangle.$$

Here we used the fact that $J = hD_n + F(x', hD')$ for some $F(x', \xi') \in S^1_1(\mathbb{R}^{n-1})$ and the tangential operator $F(x', hD')$ commutes with the indicator function of the upper half space.

Combining the two expressions we obtain

$$(5.5)\qquad [h^2\tilde{\Delta}_\phi,\tilde{\rho}]J^+J\tilde{G}_\phi = [K^2,\tilde{\rho}](hD_n - F - F^2J^+)J\tilde{G}_\phi - 2[K\cdot hD_{x'},\tilde{\rho}](1-FJ^+)J\tilde{G}_\phi.$$

We decompose $\tilde{G}_\phi$ in (5.5) into its ΨDO part and its characteristic part as stated in Proposition 4.3. The ΨDO part of (5.5) is a bounded map from $L^r \to L^r$ with a gain in h obtained from the commutator. Therefore, the part containing the ΨDO belongs to the hR_0' bin.

For the part containing the characteristic set, we expand $[h^2\tilde{\Delta}_\phi,\tilde{\rho}]J^+J\tilde{G}_\phi^c$ as

$$\begin{aligned}&[h^2\tilde{\Delta}_\phi,\tilde{\rho}]J^+J\tilde{G}_\phi^c\\ &\quad= [K^2,\tilde{\rho}]\tilde{\rho}_1(hD_n - F - F^2J^+)J\tilde{G}_\phi^c - 2[K\cdot hD_{x'},\tilde{\rho}]\tilde{\rho}_1(1-FJ^+)J\tilde{G}_\phi^c\\ &\quad+ [K^2,\tilde{\rho}](1-\tilde{\rho}_1)(hD_n - F - F^2J^+)J\tilde{G}_\phi^c\\ &\quad- 2[K\cdot hD_{x'},\tilde{\rho}](1-\tilde{\rho}_1)(1-FJ^+)J\tilde{G}_\phi^c,\end{aligned}$$

where $\tilde{\rho}_1(\xi')$ is chosen to be identically 1 in a neighbourhood compactly containing the support of $\tilde{\rho}$ but $\mathrm{supp}(\tilde{\rho}_1) \Subset \{|\xi'| < 1\}$. By disjoint support, $[K^2,\tilde{\rho}](1-\tilde{\rho}_1)$ and $[K\cdot hD_{x'},\tilde{\rho}](1-\tilde{\rho}_1)$ both belong to $h^\infty S^{-\infty}(\mathbb{R}^{n-1})$. Since $\tilde{G}_\phi^c : L^2_\delta \to_{h^{-1}} H^k_\delta$ and $L^{p'} \to_{h^{-2}} W^{k,p}$, the last two terms in the above expression for $[h^2\tilde{\Delta}_\phi,\tilde{\rho}]J^+J\tilde{G}_\phi^c$ can be sorted into the h^2R_0'' bin.

The only thing remaining is to treat the terms on the support of $\tilde{\rho}_1$. We will treat the first term and the second term is dealt with in the same manner. We claim that modulo errors in the bin h^2R_0'' we can commute $\tilde{\rho}_1(hD')$ so that it appears next to $\tilde{G}_\phi^c$:

$$(5.6)\quad [K^2,\tilde{\rho}]\tilde{\rho}_1(hD_n - F - F^2J^+)J\tilde{G}_\phi^c = [K^2,\tilde{\rho}](hD_n - F - F^2J^+)J\tilde{\rho}_1\tilde{G}_\phi^c + h^2R_0''.$$

Since $\tilde{\rho}_1(\xi')$ vanishes identically near $|\xi'| = 1$, Lemma 4.4 asserts that,

$$\tilde{\rho}_1\tilde{G}_\phi^c = Op_h(S^{-\infty}(\mathbb{R}^n)) + hm(x',hD)\tilde{G}_\phi^c$$

for some $m(x,\xi) \in S^{-\infty}(\mathbb{R}^n)$ and therefore

$$[K^2,\tilde{\rho}]\tilde{\rho}_1(hD_n - F - F^2J^+)J\tilde{G}_\phi^c = hR_0' + h^2R_0''.$$

This proves the lemma up to verifying (5.6).

It only remains to verify (5.6) by checking that all the commutator terms with $\tilde{\rho}_1$ can be sorted into the h^2R_0'' bin by using Proposition 3.3, Lemma 3.2, and Proposition 2.3 in conjunction with the mapping properties of $\tilde{G}_\phi^c$ given by Proposition 4.3. We only write out explicitly the argument for commuting with J^+ as it is slightly more challenging than the others. First, observe that by Proposition 3.3

$$J^{-1} = j^{-1}(x',hD)(1 + hm_1(x',hD) + h^2m_2(x',hD))^{-1},$$

where $m_1(x',hD)$ and $m_2(x',hD)$ take $L^r \to L^r$ and $H^k_\delta \to H^k_\delta$ with the inverse given by Neumann series. Therefore

$$[K^2,\tilde{\rho}]\tilde{\rho}_1F^2J^+J\tilde{G}_\phi^c = [K^2,\tilde{\rho}]\tilde{\rho}_1F^2j^{-1}(x',hD)\mathbf{1}_{\mathbb{R}^n_+}J\tilde{G}_\phi^c + h^2R_0''.$$

Standard calculus for commutators then allows us to commute $\tilde{\rho}_1$ with F^2 and $j^{-1}(x', hD)$ to obtain

$$[K^2, \tilde{\rho}]\tilde{\rho}_1 F^2 J^+ J\tilde{G}^c_\phi = [K^2, \tilde{\rho}]F^2 j^{-1}(x', hD)\tilde{\rho}_1 \mathbf{1}_{\mathbb{R}^n_+} J\tilde{G}^c_\phi + h^2 R''_0.$$

We can commute $\tilde{\rho}_1$ with $\mathbf{1}_{\mathbb{R}^n_+}$ with no commutator since $\tilde{\rho}_1$ is an operator in the x' direction only. Commuting with J using the standard commutator calculus then gives us (5.6). □

Proof of Lemma 5.4. We begin with the $h\tilde{E}_1\tilde{G}_\phi$ term in (5.3). By Lemma 4.5,

$$h\tilde{E}_1\tilde{G}_\phi = h(\tilde{E}_1\tilde{G}_\phi)^c + hOp_h(S^1_1 S^{-2}_1) + h^2\tilde{e}'_1(x', hD')\tilde{G}_\phi + h^2 Op_h(S^{-\infty}(\mathbb{R}^n))\tilde{G}_\phi \tag{5.7}$$

with $\tilde{e}'_1 \in S^0_1(\mathbb{R}^{n-1})$ and

$$(\tilde{E}_1\tilde{G}_\phi)^c : L^2 \to_{h^0} H^k, \quad (\tilde{E}_1\tilde{G}_\phi)^c : L^{p'} \to_{h^{-1}} H^k \quad \forall k \in \mathbb{N}.$$

By Proposition 4.3, the third term of (5.7) can be written as

$$h^2\tilde{e}'_1(x', hD')\tilde{G}_\phi = h^2\tilde{e}'_1(x', hD')\tilde{G}^c_\phi + h^2\tilde{e}'_1(x', hD')(\tilde{G}_\phi - \tilde{G}^c_\phi),$$

where $(\tilde{G}_\phi - \tilde{G}^c_\phi)$ is a ΨDO with symbol in $S^{-2}_1(\mathbb{R}^n)$. Therefore (5.7) becomes

$$\begin{aligned} h\tilde{E}_1\tilde{G}_\phi = h(\tilde{E}_1\tilde{G}_\phi)^c + hOp_h(S^1_1 S^{-2}_1 + S^0_1 S^{-2}_1) \\ + h^2\tilde{e}'_1(x', hD')\tilde{G}^c_\phi + h^2 Op_h(S^{-\infty}(\mathbb{R}^n))\tilde{G}_\phi \end{aligned} \tag{5.8}$$

with

$$(\tilde{E}_1\tilde{G}_\phi)^c : L^2 \to_{h^0} H^k, \quad (\tilde{E}_1\tilde{G}_\phi)^c : L^{p'} \to_{h^{-1}} H^k \quad \forall k \in \mathbb{N}$$

and

$$\tilde{e}'_1(x', hD')\tilde{G}^c_\phi : L^2_\delta \to_{h^{-1}} H^k_{\delta-1}, \quad \tilde{e}'_1(x', hD')\tilde{G}^c_\phi : L^{p'} \to_{h^{-2}} W^{p,k} \quad \forall k \in \mathbb{N}.$$

We see then that the first and second terms belong to the R'_1 bin while the third and fourth terms belong to the R''_1 bin.

We proceed next with the $h\tilde{E}_1 J^+ J\tilde{G}_\phi$ term of (5.3):

$$h\tilde{E}_1 J^+ J\tilde{G}_\phi = hJ^+ J\tilde{E}_1\tilde{G}_\phi + h[\tilde{E}_1, J^{-1}]\mathbf{1}_{\mathbb{R}^n_+} J\tilde{G}_\phi + hJ^+[J, \tilde{E}_1]\tilde{G}_\phi. \tag{5.9}$$

In the above calculation we commuted $\tilde{E}_1$ and $\mathbf{1}_{\mathbb{R}^n_+}$ since $\tilde{E}_1$ only acts in the x' direction.

The first term above can be handled using (5.8)—note that there is enough regularity so that applying $\mathbf{1}_{\mathbb{R}^n_+} J$ presents no difficulty. For the first commutator term of (5.9), Lemma 3.2 and Proposition 3.3 show that $[\tilde{E}_1, J^{-1}] = hm(x, hD)$ for some

$$m(x, hD) : L^r \to L^r, \quad m(x, hD) : L^2_\delta \to L^2_\delta.$$

Therefore, splitting $\tilde{G}_\phi$ into its characteristic part $\tilde{G}^c_\phi$ and its ΨDO part $\tilde{G}_\phi - \tilde{G}^c_\phi$ as in Proposition 4.3 we have

$$L^{p'} \xrightarrow{\tilde{G}_\phi - \tilde{G}^c_\phi} W^{2,p'} \xrightarrow{J} W^{1,p'} \hookrightarrow_{h^{-1}} L^2 \xrightarrow{\mathbf{1}_{R^n_+}} L^2 \xrightarrow[h]{[J^{-1}, \tilde{E}_1]} L^2$$

and

$$L^2 \xrightarrow{\tilde{G}_\phi - \tilde{G}^c_\phi} H^2 \xrightarrow{J} H^1 \xrightarrow[h]{[\tilde{E}_1, J^{-1}]\mathbf{1}_{\mathbb{R}^n_+}} L^2$$

and so $h[\tilde{E}_1, J^{-1}]\mathbf{1}_{\mathbb{R}^n_+} J(\tilde{G}_\phi - \tilde{G}^c_\phi)$ belongs to the R'_1 bin. For the characteristic part

$$L^{p'} \xrightarrow[h^{-2}]{\tilde{G}^c_\phi} W^{k,p} \xrightarrow{J} W^{k-1,p} \xrightarrow{\mathbf{1}_{R^n_+}} L^p \xrightarrow[h]{[J^{-1},\tilde{E}_1]} L^p$$

and

$$L^2_\delta \xrightarrow{\tilde{G}^c_\phi} H^k_{\delta-1} \xrightarrow{J} H^{k-1}_{\delta-1} \xrightarrow[h]{[\tilde{E}_1, J^{-1}]\mathbf{1}_{\mathbb{R}^n_+}} L^2_{\delta-1}$$

and therefore $h[\tilde{E}_1, J^{-1}]\mathbf{1}_{\mathbb{R}^n_+} J\tilde{G}^c_\phi$ belongs to the R''_1 bin.

For the $J^+[J, \tilde{E}_1]\tilde{G}_\phi$ term, splitting $\tilde{G}_\phi$ into its characteristic part $\tilde{G}^c_\phi$ and its ΨDO part $\tilde{G}_\phi - \tilde{G}^c_\phi$ we have

$$L^{p'} \xrightarrow{\tilde{G}_\phi - \tilde{G}^c_\phi} W^{2,p'} \xrightarrow[h]{[J,\tilde{E}_1]} W^{1,p'} \hookrightarrow_{h^{-1}} L^2 \xrightarrow{J^+} H^1,$$

$$L^2 \xrightarrow{\tilde{G}_\phi - \tilde{G}^c_\phi} H^2 \xrightarrow[h]{[J,\tilde{E}_1]} H^1 \xrightarrow{J^+} H^1.$$

Therefore $hJ^+[J, \tilde{E}_1](\tilde{G}_\phi - \tilde{G}^c)$ belongs to the R'_1 bin. For the characteristic part, $J^+[J, \tilde{E}_1]\tilde{G}^c$ behaves like

$$L^{p'} \xrightarrow[h^{-2}]{\tilde{G}^c_\phi} W^{k,p} \xrightarrow[h]{[J,\tilde{E}_1]} W^{k-1,p} \xrightarrow{J^+} W^{1,p}, \quad L^2_\delta \xrightarrow[h^{-1}]{\tilde{G}^c_\phi} H^k_{\delta-1} \xrightarrow[h]{[J,\tilde{E}_1]} H^{k-1}_{\delta-1} \xrightarrow{J^+} H^1_{\delta-1}$$

and therefore $hJ^+[J, \tilde{E}_1]\tilde{G}^c$ belongs to R''_1 bin. □

Proof of Lemma 5.5. The terms involving $h^2\tilde{E}_0$ can be estimated directly using the estimates for $\tilde{G}_\phi$ and P_l in Propositions 4.3 and 5.1. The terms involving $\tilde{A}_0$ can be estimated by observing that since $\tilde{\rho}(\xi')$ is chosen to be identically 1 in a neighbourhood of the support of $\tilde{a}_0(x', \xi')$, the operator

$$(1 - \tilde{\rho}(hD'))(1 + K^2)\tilde{A}_0 \in h^\infty Op_h(S^\infty(\mathbb{R}^{n-1})).$$

□

5.2. **Parametrix for $h^2\tilde{\Delta}_\phi$ at small frequency.** Here we want to look for a parametrix for $h^2\tilde{\Delta}_\varphi$ at low frequencies. We begin by defining $p(x', \xi)$ to be the symbol of $h^2\tilde{\Delta}_\varphi$:

$$p(x', \xi) := (1 + K^2)\xi_n^2 - 2i\xi_n(1_h + i\xi' \cdot K) - (1 - |\xi'|^2).$$

Thanks to the fact that $\tilde{\rho}$ is chosen to be disjoint from the characteristic set of $p(x', \xi)$ we may define

$$P_s := \frac{\tilde{\rho}}{p}(x', hD).$$

The following proposition says that P_s inverts $h^2\tilde{\Delta}_\phi$ at small frequencies, up to an $O(h)$ error.

Proposition 5.6. *P_s is a bounded operator $P_s : L^r \to W^{2,r}$ for all $r \in (1, \infty)$. Moreover, for all $r \in (1, \infty)$.*

$$h^2\tilde{\Delta}_\phi P_s = \tilde{\rho} + hR_s$$

for some $R_s : L^r \to L^r$ bounded uniformly in h.

Proof. We want to use the symbol calculus developed in Section 2. However, we have the complication that $1/p(x',\xi)$ is not a proper symbol, because of the zeros of $p(x',\xi)$. Therefore it is not immediately evident that $\tilde{\rho}/p(x',\xi)$ lies in the symbol class $S^{-\infty}S_1^{-2}$, as we would want.

We can remedy this by writing

$$\tilde{\rho}(\xi')/p(x',\xi) = (1-\chi_{100}(\xi))\tilde{\rho}(\xi')/p(x',\xi) + \chi_{100}(\xi)\tilde{\rho}(\xi')/p(x',\xi),$$

where $\chi_{100}(\xi) \in S^{-\infty}(\mathbb{R}^n)$ is a smooth cutoff function supported only for $|\xi| < 100$, and identically one in the ball $|\xi| \leq 50$.

Now note that by (4.3), $p(x',\xi)$ is properly elliptic on the support of $\tilde{\rho}(\xi')$, and therefore $\chi_{100}(\xi)\tilde{\rho}(\xi')/p(x',\xi) \in S^{-\infty}(\mathbb{R}^n)$. Moreover, since the characteristic set of $p(x',\xi)$ lies well inside the set where $\chi_{100} \equiv 1$, we have that $(1-\chi_{100}(\xi))/p(x',\xi) \in S_1^{-2}(\mathbb{R}^n)$.

Therefore P_s can be understood as the sum of two operators, one of which is in the symbol class $S^{-\infty}(\mathbb{R}^n)$ and the other of which is in the symbol class $S^{-\infty}S_1^{-2}$. Then Proposition 2.2 asserts that $P_s : L^r \to W^{2,r}$ is a bounded operator and Proposition 2.3 asserts that

$$h^2\tilde{\Delta}_\varphi \mathrm{Op}_h\left(\frac{\tilde{\rho}}{p}\right) = \mathrm{Op}_h((1-\chi_{100})\tilde{\rho}) + \mathrm{Op}_h(\chi_{100}\tilde{\rho}) + hR_s = \mathrm{Op}_h(\tilde{\rho}) + hR_s$$

as we wanted. □

It turns out that our small frequency parametrix preserves support in the x_n direction.

Proposition 5.7. *Suppose $v \in L^r(\mathbb{R}^n)$, with $1 < r < \infty$, and* $\mathrm{supp}(v)$ *is contained in the closure of $\mathbb{R}^n_+$. Then both* $\mathrm{supp}(P_s v)$ *and* $\mathrm{supp}(R_s v)$ *are contained in $\bar{\mathbb{R}}^n_+$, where R_s is the operator from Proposition* 5.6. *In particular, $P_s v \mid_{x_n=0} = 0$ if* $\mathrm{supp}(v) \subset \bar{\mathbb{R}}^n_+$.

Proof. Let $v \in C_c^\infty(\mathbb{R}^n)$. Then

$$\mathrm{Op}_h\left(\frac{\tilde{\rho}}{p}\right) v(x) = h^{-n}\int_{\mathbb{R}^n} \frac{\tilde{\rho}(\xi')}{p(x',\xi)}\hat{v}(\xi)e^{i\xi\cdot x/h}\,d\xi. \tag{5.10}$$

We split the integral on the right into x' and x_n variables and get

$$h^{-n}\int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'/h}\int_{-\infty}^{\infty}\frac{\tilde{\rho}(\xi')}{p(x',\xi)}\hat{v}(\xi)e^{i\xi_n x_n/h}\,d\xi_n\,d\xi'.$$

Consider the inner integral

$$\int_{-\infty}^{\infty}\frac{\tilde{\rho}(\xi')}{p(x',\xi)}\hat{v}(\xi)e^{i\xi_n x_n/h}\,d\xi_n.$$

For fixed ξ' and x', we can write the Fourier transform of v in the ξ_n variable explicitly to get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\tilde{\rho}(\xi')\mathcal{F}_{x'}v(\xi',s)e^{i\xi_n(x_n-s)/h}}{p(x',\xi)}\,d\xi_n\,ds. \tag{5.11}$$

Suppose now that $\mathrm{supp}(v)$ is contained in $\mathbb{R}^n_+$ so that the integral over s in (5.11) is only taken over $s \geq \delta > 0$. We want to show that (5.11) vanishes when $x_n \leq 0$. This is done by showing that the inner $d\xi_n$ integral of (5.11) vanishes if $x_n < 0$ and $s > 0$. We do this by using residue calculus.

To evaluate the $d\xi_n$ integral of (5.11) when $x_n < 0$ and $s > 0$ we should take a contour on the lower half plane. The integral vanishes if we can verify that the zeros of $p(x',\xi)$ as a polynomial in ξ_n for values of ξ' on $\mathrm{supp}(\tilde\rho(\xi')) \Subset \{|\xi'| < 1\}$ all belong to the upper half plane.

Factoring $p(x',\xi)$ as a quadratic function in ξ_n, we have

$$p(x',\xi) = -(1+|K|^2)(\xi_n - a_+)(\xi_n - a_-), \tag{5.12}$$

where

$$a_\pm = i\frac{1_h + iK\cdot\xi' \pm \sqrt{(1_h + iK\cdot\xi')^2 - (1-|\xi'|^2)(1+|K|^2)}}{1+|K|^2}.$$

and the square root is defined by choosing angles between $(-\pi,\pi]$. With this choice we see that a_+ has a positive imaginary part when $h > 0$ is sufficiently small.

We will now argue that the same holds for $a_-(x',\xi')$ for ξ' on the support of $\tilde\rho$. Note that $K(x')$ is compactly supported so this clearly holds for $h > 0$ small and x' outside the support of K. Define$D : \mathrm{supp}(K(x')) \times \mathrm{supp}(\tilde\rho(\xi')) \to \mathbb{C}$ by

$$D(x',\xi') = (1_h + iK\cdot\xi')^2 - (1-|\xi'|^2)(1+|K|^2)$$

and one easily sees that $D(x',\xi') \in \overline{\mathbb{R}}_-$ if and only if

$$\xi' \in \mathcal{N} := \{\xi'\cdot K = 0, |\xi'|^2 \le \frac{(1+|K|^2) - 1_h^2}{1+|K|^2}\}.$$

Let $\hat{\mathcal{N}}$ to be a small neighbourhood containing $\mathcal{N}$. On the connected set $\mathrm{supp}(\tilde\rho)\setminus\hat{\mathcal{N}}$, $\sqrt{D(x',\xi')}$ is a continuous function if $h > 0$ is small enough. If the imaginary part of a_- vanishes on $\mathrm{supp}(\tilde\rho)\setminus\hat{\mathcal{N}}$, then by an appropriate choice of $\xi_n \in \mathbb{R}$ the factor $(\xi_n - a_-)$ in (5.12) can be made to vanish. But $p(x',\xi)$ is elliptic on support of $\tilde\rho(\xi')$ by (4.3) so the imaginary part of a_- cannot vanish on the support of $\tilde\rho(\xi')$. On the other hand, by choosing $\hat{\mathcal{N}}$ small enough we will have that the imaginary part of a_- takes on positive value somewhere on $\mathrm{supp}(\tilde\rho)\setminus\hat{\mathcal{N}}$. By connectedness the imaginary part of a_- must be positive everywhere on $\mathrm{supp}(\tilde\rho)\setminus\hat{\mathcal{N}}$.

Meanwhile on $\hat{\mathcal{N}}$ the function $D(x',\xi')$ takes on value sufficiently close to $\overline{\mathbb{R}}_-$. Therefore by our chosen branch of the square root, $\sqrt{D(x',\xi'}$ has small real part. So $a_-(x',\xi')$ has positive imaginary part on here as well.

We are now able to conclude, at least in the case when $v \in C_c^\infty(\mathbb{R}^n)$ is supported in $\mathbb{R}^n_+$, that

$$P_s v(x',x_n) = 0 \text{ for } x_n \le 0. \tag{5.13}$$

Now from Proposition 5.6 we have

$$\|P_s v\|_{W^{2,r}(\mathbb{R}^n)} \le \|v\|_{L^r(\mathbb{R}^n)},$$

and it follows from the trace theorem that for any fixed $x_n \le 0$,

$$h\|P_s v(\cdot,x_n)\|_{W^{1,r}(\mathbb{R}^{n-1})} \le \|v\|_{L^r(\mathbb{R}^n)}. \tag{5.14}$$

If $v \in L^r(\mathbb{R}^n)$ is supported in the closure of $\mathbb{R}^n_+$, we can approximate it with $C_0^\infty(\mathbb{R}^n)$ functions supported in $\mathbb{R}^n_+$. The trace property (5.14) then allows us to conclude

$$\mathrm{Op}_h\left(\frac{\rho}{p}\right) v(x',x_n) = 0$$

for $x_n \leq 0$. This shows that P_s has the desired support property. The support property for R_s then follows from writing

$$h^2\tilde{\Delta}_\phi P_s - \tilde{\rho}(hD') = hR_s$$

and noting that every operator on the left hand side of this equation has the desired support property. □

6. Dirichlet Green's function and Carleman estimates

6.1. **Green's function for single graph domains.** By combining Sections 5.1 and 5.2 we see that $\mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}$ is a parametrix for the operator $h^2\tilde{\Delta}_\phi$ in the domain $\tilde{\Omega}$. As one expects, this parametrix can be modified into a Green's function.

In this section we consider domains with a component of the boundary which coincides with the graph of a function. In particular, let Ω be a bounded domain in $\mathbb{R}^n$, and suppose $f \in C_0^\infty(\mathbb{R}^{n-1})$ such that Ω lies in the set $\{x_n > f(x')\}$ with a portion of the boundary $\Gamma \subset \partial\Omega$ lying on the graph $\{x_n = f(x')\}$. Denote by γ the change of variable $(x', x_n) \mapsto (x', x_n - f(x'))$. Set $\tilde{\Omega}$ and $\tilde{\Gamma}$ to be the image of Ω and Γ under this change of variables.

Proposition 6.1. *There exists a Green's function G_Γ which satisfies the relation*

$$\langle h^2\Delta_\phi^* u, G_\Gamma v\rangle = \langle u, v\rangle$$

for all $u \in C_0^\infty(\Omega)$ and is of the form

$$\gamma^* G_\Gamma := (\gamma^{-1})^* \circ G_\Gamma \circ \gamma^* = \mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}(I + R)$$

with R obeying the estimates

$$R : L^{p'}(\tilde{\Omega}) \to_{h^0} L^2(\tilde{\Omega}), \quad R : L^2(\tilde{\Omega}) \to_h L^2(\tilde{\Omega}).$$

The Green's function G_Γ satisfies the estimates

$$G_\Gamma : L^2(\Omega) \to_{h^{-1}} L^2(\Omega),\ G_\Gamma : L^{p'}(\Omega) \to_{h^{-2}} L^p(\Omega).$$

Furthermore, $G_\Gamma v \in H^1(\Omega)$ for all $v \in L^{p'}$ and $G_\Gamma v\mid_\Gamma = 0$.

Proof. Change coordinates $(x', x_n) \mapsto (x', x_n - f(x'))$ so that $\tilde{\Gamma} \subset \{x_n = 0\}$ and let $\tilde{\Delta}_\phi$ be the pulled-back conjugated Laplacian described in (4.2). All equalities below are in the sense of distributions in $\tilde{\Omega}$. By Propositions 5.2 and 5.6, for any $v \in L^{p'}(\tilde{\Omega})$,

$$\langle h^2\tilde{\Delta}_\phi^* u, \mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}} v\rangle = \langle u, v + (hR_s + hR_l' + R_l)v\rangle \quad \forall u \in C_0^\infty(\tilde{\Omega})$$

with R_s and R_l' mapping $L^r \to L^r$ with no loss in h while

$$R_l : L^2 \to_h L^2,\ \ R_l : L^{p'} \to_{h^0} L^2.$$

Let $S : L^r \to L^r$ denote the inverse of $(1 + hR_l' + hR_s)$ by Neumann series. Then in $\tilde{\Omega}$ we have

$$h^2\tilde{\Delta}_\phi \mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}} S = I + R_l S$$

with $R_l S : L^2 \to_h L^2$ while $R_l S : L^{p'} \to_{h^0} L^2$. Therefore, for all $v \in L^{p'}(\tilde{\Omega})$ the Neumann series

$$(1 + R_l S)^{-1} v := v - \sum_{k=0}^{\infty} (-R_l S)^k (R_l S) v \in L^{p'}$$

is well-defined and the series converge in $L^2(\tilde{\Omega})$. Then we have the operator $\mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}S(1 + R_lS)^{-1}$ is a right inverse of $h^2\tilde{\Delta}_\phi$ in $\tilde{\Omega}$. Define G_Γ by

$$G_\Gamma = \gamma^* \circ \mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}S(1 + R_lS)^{-1} \circ (\gamma^{-1})^*.$$

Direct computation verifies that this is a Green's function in the original coordinates.

For verifying the estimates of $G_\Gamma v$ and its trace along Γ it is more convenient to work with the operator $\mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}S(1 + R_lS)^{-1}$ and deduce the analogous properties for G_Γ. We first check that $\mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}S(1 + R_lS)^{-1}v \in H^1(\tilde{\Omega})$ for all $v \in L^{p'}$ and that the trace vanishes on $\tilde{\Gamma} \subset \{x_n = 0\}$.

By Proposition 5.1 the operator P_l maps $L^{p'}$ into H^1_{loc} has vanishing trace on $\{x_n = 0\}$. By Proposition 5.6 $P_s w$ is an element of $W^{2,p'}(\mathbb{R}^n) \hookrightarrow H^1(\mathbb{R}^n)$ which vanishes in $\{x_n \leq 0\}$ if $w \in L^{p'}(\mathbb{R}^n)$ is supported only on the closure of $\mathbb{R}^n_+$. Therefore we conclude that $\mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}S(1 + R_lS)^{-1}v \in H^1(\tilde{\Omega})$ has trace zero on $\tilde{\Gamma}$ for all $v \in L^{p'}(\tilde{\Omega})$ and thus G_Γ has vanishing trace on Γ.

To verify the mapping properties of $\mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}S(1 + R_lS)^{-1}$ write

$$\begin{aligned} &\mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}S(1 + R_lS)^{-1} \\ &\quad = \mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}S(I - \sum_{k=0}^{\infty}(R_lS)^k(R_lS)) \\ &\quad = \mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}S - \mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}S\sum_{k=0}^{\infty}(R_lS)^k(R_lS). \end{aligned} \tag{6.1}$$

Since $S : L^r \to L^r$, inserting an $L^2(\tilde{\Omega})$ function would yield, by Propositions 5.1 and 5.6, an H^1 function with a loss of h^{-1} in the first term and no loss in the second. For mappings from $L^{p'}$ we only need to concern ourselves with the first term since the Neumann sum maps $L^{p'} \to L^2$ with no loss in h and we can refer to the L^2 estimate for $\mathbf{1}_{\tilde{\Omega}}(P_s + P_l)\mathbf{1}_{\tilde{\Omega}}S$.

To analyze the mapping properties of the first term of (6.1) observe that due to Propositions 5.1 and 5.6,

$$\mathbf{1}_{\tilde{\Omega}}P_l\mathbf{1}_{\tilde{\Omega}}S : L^{p'} \xrightarrow{S} L^{p'} \xrightarrow{\mathbf{1}_{\tilde{\Omega}}} L^{p'} \xrightarrow[h^{-2}]{P_l} L^p \xrightarrow{\mathbf{1}_{\tilde{\Omega}}} L^p$$

and

$$\mathbf{1}_{\tilde{\Omega}}P_s\mathbf{1}_{\tilde{\Omega}}S : L^{p'} \xrightarrow{S} L^{p'} \xrightarrow{\mathbf{1}_{\tilde{\Omega}}} L^{p'} \xrightarrow{P_s} W^{2,p'} \hookrightarrow_{h^{-2}} L^p \xrightarrow{\mathbf{1}_{\tilde{\Omega}}} L^p.$$

□

This finishes the proof of Theorem 1.3 in the case when Γ lies in a single graph. In the next section we move on to the general case where Γ is a disjoint union of graphs.

6.2. **Proof of Theorem 1.3-Dirichlet Green's function.** To prove Theorem 1.3, we first develop the necessary tools for gluing together Green's functions. Let Ω be a bounded domain and let Γ be a subset of $\partial\Omega$ which coincides with the graph $\{x_n = f(x')\}$ of a smooth compactly supported function f. Without loss of generality we may assume that there is an open neighbourhood $\Omega_\Gamma \subset \mathbb{R}^n$ of Γ for which $\Omega_\Gamma \cap \Omega$ lies in the set $\{x_n > f(x')\}$, and that

$$\Omega_\Gamma \cap \partial\Omega \cap \{x_n = f(x')\} = \bar{\Gamma}.$$

Then $\Gamma' := \Omega_\Gamma \cap \partial\Omega$ is an open subset of the boundary such that $\Gamma \Subset \Gamma'$ and compact subsets of $\Gamma'\backslash\bar{\Gamma}$ are strictly above the graph $x_n = f(x')$.

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be supported inside Ω_Γ with $\chi = 1$ near Γ. We can arrange that $\operatorname{supp}(\chi) \cap \partial\Omega \subset \Gamma'$. We can also arrange for the derivatives of χ to have the following support property:

$$\exists\epsilon > 0 \mid \operatorname{supp}(\mathbf{1}_\Omega D\chi) \subset \{(x', x_n) \mid x_n \geq f(x') + \epsilon\}. \tag{6.2}$$

In this setting choose an open subset $\mathcal{O} \subset \Omega \cap \{(x', x_n) \mid x_n > f(x')\}$ which contains Γ' as a part of its boundary and whose closure contains the support of $\chi\mathbf{1}_\Omega$. Set G_Γ to be the Green's function constructed in Proposition 6.1 for the domain $\mathcal{O}$ with vanishing trace on Γ. We may then define

$$\Pi_\Gamma : L^{p'}(\Omega) \to_{h^{-2}} L^p(\Omega),\ \ \Pi_\Gamma : L^2(\Omega) \to_{h^{-1}} H^1(\Omega) \tag{6.3}$$

by

$$\Pi_\Gamma := \chi\mathbf{1}_\Omega(G_\phi - G_\Gamma)\mathbf{1}_\mathcal{O}.$$

Note that G_Γ is not defined on the portion of Ω that lies below the graph of f, but this point is rendered moot when we multiply by χ. Observe that by Proposition 6.1 one has the trace identity

$$\Pi_\Gamma v \in H^1(\Omega),\ \ (\Pi_\Gamma v)\mid_\Gamma = (G_\phi v)\mid_\Gamma,\quad \forall v \in L^{p'}(\Omega). \tag{6.4}$$

Lemma 6.2. *One has the estimates*

$$h^2\Delta_\phi\mathbf{1}_\Omega\Pi_\Gamma\mathbf{1}_\Omega : L^{p'}(\Omega) \to_{h^0} L^2(\Omega),\ \ h^2\Delta_\phi\mathbf{1}_\Omega\Pi_\Gamma\mathbf{1}_\Omega : L^2(\Omega) \to_{h^1} L^2(\Omega).$$

With this lemma we are in a position to construct a general Green's function for the $h^2\Delta_\phi$ on a general domain Ω. Let $\omega \in \mathbb{R}^n$ be a unit vector and let $\Gamma \subset \partial\Omega$ be compactly contained in $\{x \in \partial\Omega \mid \omega \cdot \nu(x) > 0\}$. Without loss of generality we may assume as before that $\omega = (0', 1)$. Assume in addition that Γ as a union of its connected components Γ_j each of which lies in the graph of $x_n = f_j(x')$ for some smooth compactly supported function f_j. For each Γ_j construct χ_j and Π_{Γ_j} as earlier. One then, by (6.4), has that

$$\Big(G_\phi v - \sum_{j=1}^k \Pi_{\Gamma_j} v\Big)\mid_\Gamma = 0 \quad \forall v \in L^{p'}(\Omega).$$

Furthermore by Lemma 6.2, $h^2\Delta_\phi\mathbf{1}_\Omega\big(G_\phi - \sum_{j=1}^k \Pi_{\Gamma_j}\big)\mathbf{1}_\Omega = I + R'$ with

$$R' : L^2(\Omega) \to_h L^2(\Omega),\ \ R' : L^{p'}(\Omega) \to_{h^0} L^2(\Omega).$$

Note that as before we can invert by Neumann series since $L^{p'}$ gets mapped by R' to L^2 with no loss and the Neumann series converge in L^2. Theorem 1.3 is now complete by the estimates of (6.3), Lemma 4.1, and Lemma 4.2. All that remains is to give a proof of Lemma 6.2.

Proof of Lemma 6.2. By Proposition 6.1, G_Γ is by construction a right inverse for $h^2\Delta_\phi$ in Ω, and $\chi\mathbf{1}_\Omega$ is supported only on Ω, so $\chi h^2\Delta_\phi\mathbf{1}_\Omega G_\Gamma v(x) = \chi v(x)$ as distributions on Ω. Meanwhile G_ϕ is an honest right inverse for $h^2\Delta_\phi$ on $\mathbb{R}^n$, so $h^2\Delta_\phi\mathbf{1}_\Omega G_\phi = I$ as distributions on Ω. Therefore as distributions on Ω,

$$h^2\Delta_\phi\Pi_\Gamma v(x) = [h^2\Delta_\phi, \chi_j]\mathbf{1}_\Omega(G_\phi - G_{\Gamma_j})\mathbf{1}_\mathcal{O} v(x). \tag{6.5}$$

To analyze this term we will change coordinates by $(x', x_n) \mapsto (x', x_n - f(x'))$ and mark the pushed-forward domains, functions, and operators with a tilde. Then by the push-forward expression for the operator G_Γ stated in Proposition 6.1, the right side of (6.5) is

$$[h^2\tilde{\Delta}_\phi, \tilde{\chi}]\mathbf{1}_{\tilde{\Omega}}(\tilde{G}_\phi - (P_s + P_l)\mathbf{1}_{\tilde{\Omega}}(I + R))\mathbf{1}_{\tilde{\mathcal{O}}},$$

where

$$R : L^{p'}(\tilde{\Omega}) \to_{h^0} L^2(\tilde{\Omega}), \quad R : L^2(\tilde{\Omega}) \to_h L^2(\tilde{\Omega}).$$

Computing the commutator $[h^2\tilde{\Delta}_\phi, \tilde{\chi}]$ explicitly in conjunction with the operator estimates in Propositions 5.6 and 5.1 we have that

(6.6)

$$[h^2\tilde{\Delta}_\phi, \tilde{\chi}]\mathbf{1}_{\tilde{\Omega}}(\tilde{G}_\phi - (P_s+P_l)\mathbf{1}_{\tilde{\Omega}}S(1+R_lS)^{-1})\mathbf{1}_{\tilde{\mathcal{O}}} = [h^2\tilde{\Delta}_\phi, \tilde{\chi}]\mathbf{1}_{\tilde{\Omega}}(\tilde{G}_\phi - (P_s+P_l))\mathbf{1}_{\tilde{\Omega}} + E,$$

where

$$E : L^{p'}(\tilde{\Omega}) \to_{h^0} L^2(\tilde{\Omega}), \quad E : L^2(\tilde{\Omega}) \to_{h^1} L^2(\tilde{\Omega}).$$

Returning to (6.6), we see that E has the desired estimates, so it remains only to analyze the first term of (6.6)

$$[h^2\tilde{\Delta}_\phi, \tilde{\chi}]\mathbf{1}_{\tilde{\Omega}}(\tilde{G}_\phi - (P_s + P_l))\mathbf{1}_{\tilde{\Omega}}.$$

Since we are only doing the computation as distributions on $\tilde{\Omega}$, the first-order differential operator $[h^2\tilde{\Delta}_\phi, \tilde{\chi}]$ commutes with the indicator function $\mathbf{1}_{\tilde{\Omega}}$, and we have

$$[h^2\tilde{\Delta}_\phi, \tilde{\chi}]\mathbf{1}_{\tilde{\Omega}}(\tilde{G}_\phi - (P_s + P_l))\mathbf{1}_{\tilde{\Omega}} = \mathbf{1}_{\tilde{\Omega}}[h^2\tilde{\Delta}_\phi, \tilde{\chi}](\tilde{G}_\phi - (P_s + P_l))\mathbf{1}_{\tilde{\Omega}}.$$

Now P_s maps L^2 to L^2 with no loss of h's, and $L^{p'}$ to $W^{2,p'} \hookrightarrow_{h^{-1}} H^1$. Meanwhile the commutator $[h^2\tilde{\Delta}_\phi, \tilde{\chi}]$ maps H^1 to L^2 with the gain of h, so the term involving P_s has the desired behaviour. Therefore the only term of difficulty is

(6.7)

$$[h^2\tilde{\Delta}_\phi, \tilde{\chi}]\mathbf{1}_{\tilde{\Omega}}(\tilde{G}_\phi - P_l)\mathbf{1}_{\tilde{\Omega}} = [h^2\tilde{\Delta}_\phi, \tilde{\chi}]\mathbf{1}_{\tilde{\Omega}}(I - J^+J)\tilde{G}_\phi\mathbf{1}_{\tilde{\Omega}} = \mathbf{1}_{\tilde{\Omega}}[h^2\tilde{\Delta}_\phi, \tilde{\chi}]J^{-1}\mathbf{1}_{\mathbb{R}^n_-}J\tilde{G}_\phi\mathbf{1}_{\tilde{\Omega}}.$$

By (6.2) the term $\mathbf{1}_{\tilde{\Omega}}[h^2\tilde{\Delta}_\phi, \tilde{\chi}]$ is a first-order differential operator whose coefficients are supported in $\{x_n \geq \epsilon > 0\}$. This allows us to apply Lemma 3.4 to obtain the estimate

$$[h^2\tilde{\Delta}_\phi, \tilde{\chi}]J^{-1}\mathbf{1}_{\mathbb{R}^n_-} : L^2_\delta \to_{h^2} L^2_\delta.$$

Inserting this estimate back into (6.7) and splitting $\tilde{G}_\phi$ by using Proposition 4.3 we see that

$$[h^2\tilde{\Delta}_\phi, \tilde{\chi}]\mathbf{1}_{\tilde{\Omega}}(\tilde{G}_\phi - P_l)\mathbf{1}_{\tilde{\Omega}} : L^{p'}(\tilde{\Omega}) \to_{h^0} L^2(\tilde{\Omega}),$$
$$[h^2\tilde{\Delta}_\phi, \tilde{\chi}]\mathbf{1}_{\tilde{\Omega}}(\tilde{G}_\phi - P_l)\mathbf{1}_{\tilde{\Omega}} : L^2(\tilde{\Omega}) \to_h L^2(\tilde{\Omega}).$$

Therefore we see that every term in (6.6) has the desired form. ☐

6.3. **Carleman estimates.** The Carleman estimates in Theorem 1.4 now follow from the existence of the Green's function G_Γ.

Proof of Theorem 1.4. Let $u \in C^2(\bar{\Omega})$ be a function which vanishes along $\partial\Omega$ and $\partial_\nu u\,|_{\Gamma^c} = 0$, and let $v \in C_0^\infty(\Omega)$. Integrating by parts, we have

$$\langle h^2\Delta_\phi^* u, G_\Gamma v\rangle_\Omega = \langle u, v\rangle_\Omega \tag{6.8}$$

with the boundary terms vanishing because of the boundary conditions on u and the boundary behaviour of $G_\Gamma v$. Equation (6.8) implies that

$$\|h^2\Delta_\phi u\|_{H^{-1}_\Gamma(\Omega)}\|G_\Gamma v\|_{H^1_\Gamma(\Omega)} \geq |\langle u, v\rangle_\Omega|$$

and

$$\|h^2\Delta_\phi u\|_{L^{p'}(\Omega)}\|G_\Gamma v\|_{L^p(\Omega)} \geq |\langle u, v\rangle_\Omega|.$$

Applying the boundedness results for G_Γ and taking the supremum over $v \in C_0^\infty(\Omega)$ completes the proof. $\square$

7. Complex geometrical optics and the inverse problem

Let $\Omega \subset \mathbb{R}^n$, $\omega \in \mathbf{S}^{n-1}$ and $\Gamma \subset \partial\Omega$ be an open subset of the boundary compactly contained in $\{x \in \partial\Omega \mid \nu(x) \cdot \omega > 0\}$ where ν denotes the normal vector. Assume in addition that in coordinates given by $(x', x_n) \in \omega^\perp \oplus \mathbb{R}\omega$ that Γ is the disjoint union of open subsets Γ_j such that Γ_j is the graph of $x_n = f_j(x')$. By Theorem 1.3 there exists a Green's function G_Γ for $h^2\Delta_\phi$ with vanishing trace on Γ and

$$G_\Gamma : L^2(\Omega) \to_{h^{-1}} L^2(\Omega), \quad G_\Gamma : L^{p'}(\Omega) \to_{h^{-2}} L^p(\Omega).$$

7.1. **Semiclassical solvability.** Let ω be a unit vector and let $\Gamma \subset \partial\Omega$ be as before. We have the following solvability result, resembling the one in [24] (see the explanation of this method in [13]), but with an additional term.

Proposition 7.1. *Let $L \in L^2(\Omega)$ with $\|L\|_{L^2} \le Ch^2$, and let $q \in L^{n/2}(\Omega)$. For all $a = a_h \in L^\infty$ with $\|a_h\|_{L^\infty} \le C$, there exists a solution of*

$$h^2(\Delta_\phi + q)r = h^2 qa + L \quad r\mid_\Gamma = 0 \tag{7.1}$$

with estimates $\|r\|_{L^2} \le o(1)$ and $\|r\|_{L^p} \le O(1)$.

Proof. We try solutions of the form $r = G_\Gamma(\sqrt{|q|}v + L)$ for $v \in L^2$ with $\|v\|_{L^2} \le Ch^2$. Supposing this can be accomplished, then using $\|L\|_{L^2} \le Ch^2$,

$$\begin{aligned}
\|r\|_{L^2} &\le \|G_\Gamma(\sqrt{|q|}v)\|_{L^2} + \|G_\Gamma(L)\|_{L^2} \\
&\le \|G_\Gamma(\sqrt{|q|}^\flat v)\|_{L^2} + \|G_\Gamma(\sqrt{|q|}^\sharp v)\|_{L^p} + \|G_\Gamma(L)\|_{L^2} \\
&\le \frac{C_\epsilon}{h}\|v\|_{L^2} + \frac{C}{h^2}\|\sqrt{|q|}^\sharp v\|_{L^{p'}} + Ch,
\end{aligned}$$

where for any $\epsilon > 0$ we decompose $\sqrt{|q|} = \sqrt{|q|}^\sharp + \sqrt{|q|}^\flat$ with $\sqrt{|q|}^\flat \in L^\infty$ and $\|\sqrt{|q|}^\sharp\|_{L^n} \le \epsilon$. Therefore,

$$\|r\|_{L^2} \le \left(\frac{C_\epsilon}{h} + \frac{C\epsilon}{h^2}\right)\|v\|_{L^2} + Ch = o(1)$$

by taking $h \to 0$ and using that $\|v\|_2 \le Ch^2$.

For the L^p norm, observe that

$$\|L\|_{L^{p'}} \le \|L\|_{L^2} \le Ch^2 \quad \text{and} \quad \|\sqrt{|q|}v\|_{p'} \le \|q\|_{L^{n/2}}\|v\|_{L^2} \le Ch^2.$$

The mapping property of G_Γ from $L^{p'} \to_{h^{-2}} L^p$ then gives the result.

We now show that we can indeed construct such a v. Inserting the ansatz into (7.1) and writing $q = e^{i\theta}|q|$ for some $\theta(\cdot) : \Omega \to \mathbb{R}$ we see that it suffices to construct $v \in L^2$ solving the integral equation

$$(1 + h^2 e^{i\theta}\sqrt{|q|}G_\Gamma\sqrt{|q|})v = h^2(e^{i\theta}\sqrt{|q|}a - e^{i\theta}\sqrt{|q|}G_\Gamma(L))$$

with $\|v\|_{L^2} \le Ch^2$. Observe that the right side is $O(h^2)$ in L^2 norm due to the fact that $\|L\|_{L^2} \le Ch^2$ so it suffices to show that $h^2 e^{i\theta}\sqrt{|q|}G_\Gamma\sqrt{|q|} : L^2 \to L^2$ is bounded by $o(1)$ as $h \to 0$ and invert by Neumann series. Indeed, writing $\sqrt{|q|} = \sqrt{|q|}^\sharp + \sqrt{|q|}^\flat$ we have

$$\sqrt{|q|}G_\Gamma\sqrt{|q|} = \sqrt{|q|}^\flat G_\Gamma\sqrt{|q|}^\flat + \sqrt{|q|}^\sharp G_\Gamma\sqrt{|q|}^\flat + \sqrt{|q|}^\flat G_\Gamma\sqrt{|q|}^\sharp.$$

Each of the three pieces have the following mapping properties:

$$\sqrt{|q|}^\flat G_\Gamma \sqrt{|q|}^\flat : L^2 \xrightarrow{\sqrt{|q|}^\flat} L^2 \xrightarrow[h^{-1}]{G_\Gamma} L^2 \xrightarrow{\sqrt{|q|}^\flat} L^2$$

$$\sqrt{|q|}^\sharp G_\Gamma \sqrt{|q|}^\flat : L^2 \xrightarrow{\sqrt{|q|}^\flat} L^2 \hookrightarrow L^{p'} \xrightarrow[h^{-2}]{G_\Gamma} L^p \xrightarrow[o(1)]{\sqrt{|q|}^\sharp} L^2$$

$$\sqrt{|q|}^\flat G_\Gamma \sqrt{|q|}^\sharp : L^2 \xrightarrow[o(1)]{\sqrt{|q|}^\sharp} L^{p'} \xrightarrow[h^{-2}]{G_\Gamma} L^p \xrightarrow{\sqrt{|q|}^\flat} L^p \hookrightarrow L^2.$$

Therefore we have that $h^2 e^{i\theta}\sqrt{|q|}G_\Gamma\sqrt{|q|} : L^2 \to_{o(1)} L^2$ as $h \to 0$. □

7.2. **Ansatz for the Schrödinger equation.** We briefly summarize the ansatz construction procedure given in [21]; see also the explanation in [5]. Let $\phi(x)$ and $\psi(x)$ be linear functions satisfying $D(\phi+i\psi)\cdot D(\phi+i\psi) = 0$. If $\Gamma \subset \partial\Omega$ is an open subset of the boundary satisfying $D\phi\cdot\nu(x) \geq \epsilon_0 > 0$ for all $x \in \bar{\Gamma}$, we first look to construct a solution to

$$h^2\Delta_\phi(e^{i\psi/h} + a_h) = L, \quad (e^{i\psi/h} + a_h)\mid_\Gamma = 0$$

with $\|L\|_{L^2} \leq Ch^2$ and $a_h \in L^\infty$. By the fact that $\nabla\phi\cdot\nu(x) \geq \epsilon_0 > 0$ for all $x \in \Gamma$, we can apply Borel's lemma to construct $\ell \in C^\infty$ such that

$$D\ell\cdot D\ell(x) = d(x,\Gamma)^\infty \quad \ell\mid_\Gamma = (\phi+i\psi)\mid_\Gamma \quad \partial_\nu\ell\mid_\Gamma = -\partial_\nu(\phi+i\psi)\mid_\Gamma .$$

Since we are working with linear weights we will need a slightly more general h-dependent phase function than $\phi+i\psi$. Let $\xi \in \mathbb{R}^n$ be a fixed vector which is orthogonal to both $D\phi$ and $D\psi$, and let $\psi_h(x)$ be a linear function defined by $\psi_h(x) = (\xi - \omega_h)\cdot x$ where

$$\omega_h = \frac{1-\sqrt{1-h^2|\xi|^2}}{h}D\psi \tag{7.2}$$

is a vector of length $O(h)$. Observe that in this setting the linear function $\phi + i(\psi + h\psi_h)$ still solves the eikonal equation

$$D(\phi+i(\psi+h\psi_h))\cdot D(\phi+i(\psi+h\psi_h)) = 0.$$

We now construct $b \in \mathbb{C}^\infty(\Omega)$ supported close to Γ such that

$$e^{-\ell/h}h^2\Delta(e^{\ell/h}e^{i\psi_h}b) = d(x,\Gamma)^\infty + O_{L^\infty}(h^2), \quad b\mid_\Gamma = -1. \tag{7.3}$$

Using the fact that $D\ell\cdot D\ell = d(x,\Gamma)^\infty$ and $D\psi_h = \xi - \omega_h$ with $|\omega_h| \leq Ch$ we see that this amounts to solving the transport equation

$$bD\ell\cdot\xi + b\Delta\ell + 2D\ell\cdot Db = d(x,\Gamma)^\infty, \quad b\mid_\Gamma = -1.$$

Taking advantage of the fact that $-\partial_\nu\mathrm{Re}(\ell)\mid_\Gamma = \partial_\nu\phi\mid_\Gamma \geq \epsilon_0 > 0$ we can again solve the iterative equation and use Borel's lemma to construct $b \in C^\infty(\Omega)$ supported in an arbitrarily small neighbourhood of Γ satisfying this approximate equation. We have therefore constructed $b \in C^\infty$ solving (7.3).

By the fact that $\nabla\phi\cdot\nu(x) \geq \epsilon_0 > 0$ we have, by choosing the support of b sufficiently small, that $\mathrm{Re}(\phi(x) - \ell(x)) \sim d(x,\Gamma)$ on $\mathrm{supp}(b)$. By analyzing separately the case when $d(x,\Gamma) \leq \sqrt{h}$ and $d(x,\Gamma) \geq \sqrt{h}$ we have that (7.3) becomes

$$h^2\Delta_\phi(e^{\frac{\ell-\phi}{h}}e^{i\psi_h}b) = O_{L^\infty}(h^2), \quad b\mid_\Gamma = -1.$$

By the fact that $h^2\Delta e^{\frac{\phi+i\psi+hi\psi_h}{h}}=0$ and $\ell\mid_\Gamma=(\phi+i\psi)\mid_\Gamma$ we have

$$(7.4)\; h^2\Delta_\phi(e^{\frac{i\psi+hi\psi_h}{h}}+e^{\frac{i\psi+hi\psi_h}{h}}a_h)=L,\quad \|L\|_{L^\infty}\le Ch^2,\quad (1+a_h)\mid_\Gamma=0,$$

where $a_h:=e^{\frac{\ell-\phi-i\psi}{h}}b$ with $\|a_h\|_{L^\infty}\le C$ and $a_h(x)\to 0$ for all $x\in\Omega$ as $h\to 0$.

This discussion allows us to construct the suitable CGO for solving our inverse problem. Indeed, let ω and ω' be two unit vectors which are mutually orthogonal. Define $\phi(x)=\omega\cdot x$ and $\psi(x)=\omega'\cdot x$. Let $\xi\in\mathbb{R}^n$ be another vector satisfying $\omega\cdot\xi=\omega'\cdot\xi=0$ and define $\psi_h(x):=(\xi-\omega_h)\cdot x$ where ω_h is as in (7.2). Construct $\ell,b\in C^\infty(\Omega)$ so that (7.4) is satisfied. Applying Proposition 7.1 to (7.4) proves the following.

Proposition 7.2. *Let ω and ω' be two unit vectors which are mutually orthogonal. Let $\Gamma\subset\partial\Omega$ be an open subset compactly contained in $\{x\in\partial\Omega\mid\omega\cdot\nu(x)>0\}$. For all $q\in L^{n/2}$ there exists solutions to*

$$(\Delta+q)u=0,\quad u\in H^1(\Omega),\quad u\mid_\Gamma=0$$

of the form

$$u=e^{\frac{\omega\cdot x+i\omega'\cdot x+hi\psi_h}{h}}(1+a_h+r)$$

with $\|a_h\|_{L^\infty}\le C$, $a_h\to 0$ pointwise in Ω as $h\to 0$. The remainder $r\in L^p$ satisfies the estimates $\|r\|_{L^2}=o(1)$ and $\|r\|_p\le C$ as $h\to 0$.

7.3. **Recovering the coefficients.** In this section we prove Theorem 1.1. Let ω be a unit vector sufficiently close to ω_0 such that there exists an open set Γ_+ such that

$$\partial\Omega\backslash\mathbf{B}\subset\subset\Gamma_+\Subset\{x\in\partial\Omega\mid\omega\cdot\nu(x)>0\},\quad \partial\Omega\backslash\mathbf{F}\subset\subset\Gamma_-\Subset\{x\in\partial\Omega\mid\omega\cdot\nu(x)<0\}.$$

Let $\xi\in\mathbb{R}^n$ be any vector orthogonal to ω and choose a third vector ω' of unit length which is perpendicular to both ξ and ω.

By Theorem 7.2 there exists solutions $u_\pm\in H^1(\Omega)$ solving

$$(\Delta+q_1)u_+=0,\quad u_+\mid_{\Gamma_+}=0,\quad (\Delta+q_2)u_-=0,\quad u_-\mid_{\Gamma_-}=0$$

of the form

$$u_\pm=e^{\frac{\pm\omega+i\omega'+hi\psi_h^\pm}{h}}(1+a_h^\pm+r_\pm),\quad \|r_\pm\|_{L^2}=o(1),\quad \|r_\pm\|_{L^p}=O(1),$$

where $\psi_h^\pm(x):=(\pm\xi-\omega')\cdot x$.

Since $u_\pm$ are solutions belonging to $H^1(\Omega)$ and vanishing on $\partial\Omega\backslash\mathbf{B}$ and $\partial\Omega\backslash\mathbf{F}$, respectively, we have the following boundary integral identity (see Lemma A.1 of [13]):

$$\int_\Omega \bar{u}_-(q_1-q_2)u_+=0.$$

Inserting the expressions for $u_\pm$ gives

$$0=\int_\Omega e^{2i\xi\cdot x}q(1+a_h^-a_h^++a_h^-+a_h^++a_h^-r_++a_h^+r_-+r_-+r_++r_+r_-),$$

where $q=q_1-q_2$. The function $q\in L^{n/2}\subset L^1$ and

$$\|a_h^\pm\|_{L^\infty}\le C,\quad \lim_{h\to 0}a_h^\pm(x)=0\ \ \forall x\in\Omega$$

by (7.4). Therefore, terms $\lim_{h\to 0}\int_\Omega e^{2i\xi\cdot x}q(a_h^- a_h^+ + a_h^- + a_h^+) = 0$. For the terms involving $\int_\Omega e^{2i\xi}qa_h^\pm r_\mp$, we note that for all $\epsilon > 0$ we may split $q = q^\sharp + q^\flat$ where $q^\flat \in L^\infty$ while $\|q^\sharp\|_{L^{n/2}} \le \epsilon$. Then, using the fact that $\|a_h^\pm\|_{L^\infty} \le C$,

$$\left|\int_\Omega e^{2i\xi\cdot x}qa_h^\pm r_\mp\right| \le C(\|q^\flat\|_{L^\infty}\|r_\mp\|_{L^2} + \|q^\sharp\|_{L^{n/2}}\|r_\mp\|_{L^p}),$$

where $p = \frac{2n}{n-2}$. By the estimates on $r_\mp$ given in Proposition 7.2 we have that $\lim_{h\to 0}\|r_\mp\|_{L^2} = 0$ and $\|r_\mp\|_{L^p} \le C$. Therefore, the limit

$$\lim_{h\to 0}\left|\int_\Omega e^{2i\xi\cdot x}qa_h^\pm r_\mp\right| \le C\epsilon$$

for all $\epsilon > 0$ and therefore the limit vanishes. The terms $\int_\Omega e^{2i\xi}q(r_- + r_+)$ can be estimated the same way. For the last term, we again decompose, for all $\epsilon > 0$, $q = q^\flat + q^\sharp$. The integral $|\int_\Omega e^{2i\xi\cdot x}qr_-r_+|$ is then estimated by

$$\int_\Omega |q^\flat r_- r_+| + \int |q^\sharp r_- r_+| \le \|q^\flat\|_{L^\infty}\|r_+\|_{L^2}\|r_-\|_{L^2} + \|q^\sharp\|_{L^{n/2}}\|r_-\|_{L^p}\|r_+\|_{L^p}.$$

The L^p norms of $r_\pm$ stay uniformly bounded while the L^2 norms vanish when $h \to 0$. Therefore the limit

$$\lim_{h\to 0}\left|\int_\Omega e^{2i\xi\cdot x}qr_-r_+\right| \le C\|q^\sharp\|_{L^{n/2}} \le C\epsilon$$

for all $\epsilon > 0$ and therefore vanishes.

This means that $\mathcal{F}(q)(\xi) = 0$ for all ξ which are orthogonal to ω. Note that varying ω in a small neighbourhood does not change the fact that Γ lies in the set $\{x \in \partial\Omega | \omega\cdot\nu(x) > 0\}$, and so the construction in Proposition 7.2 still applies. Then varying ω in a small neighbourhood and using the analyticity of the Fourier transform for q compactly supported we have that $q = q_1 - q_2 = 0$. ☐

8. Appendix

Here we will provide proofs for Propositions 2.2 and 2.3 from Section 2.

Proposition 8.1. *Let $a(x',\xi)$ be in $S_1^0(\mathbb{R}^n)$ or $S_0^{-k(n)}(\mathbb{R}^n)$ for some $k(n)$ large depending only on the dimension. If $b(x',\xi') \in S_1^0(\mathbb{R}^{n-1})$, then*

$$ab(x', hD) : L^r \to L^r$$

is uniformly bounded in $h > 0$.

Proof. If $a \in S_0^{-k(n)}$, then $ab \in S_0^{-k(n)}$ so we may directly appeal to (2.8). So we only need to treat the case when $a \in S_1^0$.

It suffices to show that for $a(x',\xi) \in S_1^{-k}(\mathbb{R}^n)$ with $k \ge 0$ and $b(x',\xi') \in S_1^0(\mathbb{R}^{n-1})$ one can write

(8.1)
$$(ab)(x',hD)u = a(x',hD)b(x',hD')u - \sum_{1\le|\alpha|\le k(n)} h^{|\alpha|}Op_h(\partial_x^\alpha b\partial_\xi^\alpha a) - Op_h(m)$$

with $m \in S_0^{-k(n)}$. The last term $Op_h(m)$ takes $L^r \to L^r$ by (2.8). The first term is a composition of an operator taking $L^r(\mathbb{R}^{n-1}) \to L^r(\mathbb{R}^{n-1})$ (leaving the x_n direction untouched) and an operator from $L^r(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ by (2.8). The middle term involves sums of derivatives $\partial_x^\alpha b \partial_\xi^\alpha a \in S_1^0 S_1^{-k-|\alpha|}$ with $|\alpha| \geq 1$. This means we can inductively apply (8.1) until we land in $S_1^0 S_1^{-k(n)}(\mathbb{R}^n)$ and apply (2.8).

To derive (8.1) simply use the standard methods as in [35]. First we have that

$$Op_h(a)Op_h(b) = Op_h(c),$$

where

$$c(x', \xi) = \frac{1}{(2\pi h)^n} \int\int e^{-iy\cdot(\xi-\eta)} a(x', \eta) b(y' + x', \xi') d\eta dy.$$

Standard computation then yields that

$$c(x', \xi) = a(x', \xi) b(x', \xi') + \sum_{1 \leq |\alpha| \leq k(n)} h^{|\alpha|} \partial_x^\alpha b(x', \xi') \partial_\xi^\alpha a(x', \xi) + h^{k(n)+1} m(x', \xi),$$

where $m(x', \xi)$ has the explicit representation

$$m(x', \xi) = \sum_{|\alpha| = k(n)+1} \int e^{iy\cdot\eta} \frac{(1 + \Delta_\eta)^{n+1} \partial_\eta^\alpha a(x', \eta + \xi)(1 + \Delta_y)^{n+k(n)+1} \int_0^1 \partial_x^\alpha b(x' + thy', \xi') dt}{(1 + |\eta|^2)^{k(n)+n+1}(1 + |y|^2)^{n+1}} d\eta dy.$$

To verify that $m(x', \xi) \in S_0^{-k(n)-1}(\mathbb{R}^n)$, note that if $|\alpha| = k(n) + 1$, then

$$|(1 + \Delta_\eta)^{n+1} \partial_\eta^\alpha a(x', \eta + \xi)| \leq C\langle \xi + \eta \rangle^{-k(n)-1}.$$

Combined with a $\langle \eta \rangle^{2k(n)+2+2n}$ in the denominator and using Peetre's inequality we have

$$\langle \eta \rangle^{-2k(n)-2-2n} |(1 + \Delta_\eta)^{n+1} \partial_\eta^\alpha a(x', \eta + \xi)| \leq C\langle \xi \rangle^{-k(n)-1} \langle \eta \rangle^{-n-1}.$$

Differentiating $m(x', \xi)$ and passing the derivative under the integral using Lebesgue theory shows that $m(x', \xi) \in S_0^{-k(n)-1}(\mathbb{R}^n)$. □

Composition of two ΨDO operators in this class can be described by the composition calculus

$$b(x, hD)a(x, hD) = ab(x, hD) + h \sum_{|\alpha|=1} (\partial_\xi^\alpha b \partial_x^\alpha a)(x, hD) + h^2 m(x, hD)$$

with the remainder explicitly computed as

$$m(x, \xi) = \sum_{|\alpha|=2} \int_{\mathbb{R}^{2n}} \frac{e^{i\eta\cdot y}}{\langle \eta \rangle^{2N} \langle y \rangle^{2N}} (I + \Delta_\eta)^N \partial_\eta^\alpha b(x, \eta + \xi)(I + \Delta_y)^N \int_0^1 \partial_x^\alpha a(x + \theta hy, \xi) d\theta dy d\eta \quad (8.2)$$

for all $N \in \mathbb{N}$. This leads to the following statement about the remainder term of the composition.

Lemma 8.2. *Let* $a \in S_1^{k_1} S_1^{\ell_1} \cup S_1^{k_1} S_0^{-k(n)+\ell_1}$ *and* $b \in S_1^{-k_1} S_1^{-\ell_1} \cup S_1^{-k_1} S_0^{-\ell_1-k(n)}$; *then one has*

$$b(x', hD)a(x', hD) = ab(x', hD) + h \sum_{|\alpha|=1} (\partial_\xi^\alpha b \partial_x^\alpha a)(x', hD) + h^2 m(x', hD)$$

with

$$m(x', hD) : L^r \to L^r$$

norm independent of $h > 0$.

Proof. We have that

$$b(x'hD)a(x',hD) = ab(x',hD) + h\sum_{|\alpha|=1}\partial_\xi^\alpha b\partial_{x'}^\alpha a(x',hD) + h^2 m(x',hD),$$

where $m(x,\xi)$ is given by (8.2). By taking N large enough in (8.2) we see that

$$m(x',hD)u = \sum_{|\alpha|=2}\int_{\mathbb{R}^{2n}} \frac{e^{iy\cdot\eta}}{\langle\eta\rangle^N\langle y\rangle^N}\int_0^1 m^\alpha_{\theta,y,h,\eta}(x',hD)ud\theta dyd\eta, \tag{8.3}$$

where for each (α,θ,y,h,η), $m^\alpha_{\theta,y,h,\eta}(x',\xi)$ is a symbol of the form

$$m^\alpha_{\theta,y,h,\eta}(x',\xi) = \langle\eta\rangle^{-N}\langle y\rangle^{-N}(I+\Delta_\eta)^N\partial_\eta^\alpha b(x',\eta+\xi)(1+\Delta_y)^N\partial_x^\alpha a(x+h\theta y,\xi).$$

Since $a\in S_1^{k_1}S_1^{\ell_1}\cup S_1^{k_1}S_0^{-k(n)+\ell_1}$ and $b\in S_1^{-k_1}S_1^{-\ell_1}\cup S_1^{-k_1}S_0^{-k(n)-\ell_1}$ we may write $a=a^ta^v$ and $b=b^tb^v$ where

$$a^t(x',\xi')\in S_1^{k_1}, b^t(x',\xi')\in S_1^{-k_1}, a^v(x',\xi)\in S_1^{\ell_1}\cup S_0^{-k(n)+\ell_1}, b^v(x',\xi)\in S_1^{-\ell_1}\cup S_0^{-k(n)-\ell_1}.$$

We see then that for each (α,θ,y,h,η) the symbol $m^\alpha_{\theta,y,h,\eta}(x',\xi)$ is a sum of finitely many (depending on the choice of N) terms of the form

$$\langle\eta\rangle^{-N}\langle y\rangle^{-N}\partial_\xi^{\beta_1}b^t_\eta(x',\xi')(\theta h)^{|\beta_2|}\partial_{x'}^{\beta_2}a^t_{\theta,hy'}(x',\xi')\partial_\xi^{\beta_3}b^v_\eta(x',\xi)(\theta h)^{|\beta_4|}\partial_{x'}^{\beta_4}a^v_{\theta,hy'}(x',\xi). \tag{8.4}$$

Here

$$b_\eta(x',\xi) := b(x',\xi+\eta),\quad a_{\theta,hy'}(x',\xi) := a(x'+\theta hy',\xi).$$

If $N\ge k(n)$ is chosen to be sufficiently large, one has by Peetre's inequality

$$\begin{aligned}&\langle\eta\rangle^{-N}\langle y\rangle^{-N}\partial_\xi^{\beta_1}b^t_\eta(x',\xi')\partial_\xi^{\beta_2}a^t_{\theta,hy'}(x',\xi')\partial_\xi^{\beta_3}b^v_\eta(x',\xi)\partial_\xi^{\beta_4}a^v_{\theta,hy'}(x',\xi)\\ &\quad\le C\langle\xi'\rangle^{-|\beta_1|-|\beta_2|}\langle\xi\rangle^{-|\beta_3|-|\beta_4|}\end{aligned}$$

if

$$a^t(x',\xi')\in S_1^{k_1}, b^t(x',\xi')\in S_1^{-k_1}, a^v(x',\xi)\in S_1^{\ell_1}, b^v(x',\xi)\in S_1^{-\ell_1}.$$

The constant is independent of η, y', θ, and h. The analogous conclusion can be made if $a^v(x',\xi)\in S_0^{-k(n)+\ell_1}$ or $b^v(x',\xi)\in S_0^{-k(n)-\ell_1}$. Therefore we conclude that, as a function of (x',ξ), (8.4) is a symbol in $S_1^0S_1^0\cup S_1^0S^{-k(n)}$ whose seminorms are uniformly bounded in η, y', θ, and h. Since $m^\alpha_{\theta,y,h,\eta}(x',\xi)$ is a finite sum of these objects, we may apply Proposition 8.1 to obtain

$$\sup_{(\alpha,\theta,y,h,\eta)}\|m^\alpha_{\theta,y,h,\eta}(x',hD)\|_{L^r\to L^r}\le C$$

if $N\ge k(n)$ is chosen large enough. Choosing $N\ge n+2$ in (8.3) we get that

$$\|m(x',hD)\|_{L^r\to L^r}\le C\sup_{(\alpha,\theta,y,h,\eta)}\|m^\alpha_{\theta,y,h,\eta}(x',hD)\|_{L^r\to L^r}\int_{\mathbb{R}^{2n}}\langle y\rangle^{-N}\langle\eta\rangle^{-N}d\eta dy.$$

□

The composition formula given by Lemma 8.2 in conjunction with the mapping property asserted in Proposition 8.1 also allows us to deduce Proposition 2.2 by composition with suitable powers of $\langle hD'\rangle\langle hD\rangle$.

Proposition 8.3 (Proposition 2.2)**.** *If $b(x',\xi')\in S_1^k$ and $a(x',\xi)\in S_1^\ell\cup S_0^{-k(n)+\ell}$, then*

$$ba(x',hD): W^{m,l,r}\to W^{m-k,l-\ell,r}$$

with norm uniformly bounded in $h>0$.

Proof. Since pre-composition by $\langle hD'\rangle^{-k}\langle hD\rangle^{-\ell}$ amounts to multiplication of symbols without remainders, it suffices to show that symbols $a(x',\xi)\in S_1^k S_1^\ell \cup S_1^k S_0^{-k(n)+\ell}$ take $L^r \to W^{-k,-\ell,r}$. Indeed, by Lemma 8.2 we have that

$$\langle hD'\rangle^{-k}\langle hD\rangle^{-\ell}a(x',hD) = c(x',hD) + h^2 m(x',hD),$$

where $c(x',hD) = \langle \xi'\rangle^{-k}\langle\xi\rangle^{-\ell}a(x',\xi) + h\sum_{|\alpha|=1}\partial_\xi^\alpha(\langle\xi'\rangle^{-k}\langle\xi\rangle^{-\ell})\partial_{x'}^\alpha a(x',\xi)$ and $m(x',hD) : L^r \to L^r$. ☐

Now we turn to the proof of Proposition 2.3.

Proposition 8.4 (Proposition 2.3). *If* $a \in S_1^{k_1}S_1^{\ell_1}\cup S_1^{k_1}S_0^{-k(n)+\ell_1}$ *and* $b\in S_1^{k_2}S_1^{\ell_2}\cup S_1^{k_2}S_0^{-k(n)+\ell_2}$*, then*

$$b(x'hD)a(x',hD) = ab(x',hD) + h\sum_{|\alpha|=1}(\partial_\xi^\alpha b\partial_{x'}^\alpha a)(x',hD) + h^2 m(x',hD),$$

where $m(x',hD) : W^{k,\ell,r}\to W^{k-k_1-k_2,\ell-\ell_1-\ell_2,r}$.

Proof. The proof goes along the same idea as Lemma 8.2 except that to show the boundedness of the remainder in the mixed Sobolev norms one uses Proposition 2.2.

We have that

$$b(x'hD)a(x',hD) = ab(x',hD) + h\sum_{|\alpha|=1}\partial_\xi^\alpha b\partial_{x'}^\alpha a(x',hD) + h^2 m(x',hD),$$

where $m(x,\xi)$ is given by (8.2). By taking N large enough in (8.2) we see that

$$m(x',hD)u = \sum_{|\alpha|=2}\int_{\mathbb{R}^{2n}}\frac{e^{iy\cdot\eta}}{\langle\eta\rangle^N\langle y\rangle^N}\int_0^1 m^\alpha_{\theta,y,h,\eta}(x',hD)u\,d\theta dy d\eta, \tag{8.5}$$

where for each (α,θ,y,h,η), $m^\alpha_{\theta,y,h,\eta}(x',\xi)$ is a symbol of the form

$$m^\alpha_{\theta,y,h,\eta}(x',\xi) = \langle\eta\rangle^{-N}\langle y\rangle^{-N}(I+\Delta_\eta)^N\partial_\eta^\alpha b(x',\eta+\xi)(1+\Delta_y)^N\partial_x^\alpha a(x+h\theta y,\xi).$$

We may then proceed as in the proof of Lemma 8.2 to conclude that for $N\geq k(n)$ sufficiently large, $m^\alpha_{\theta,y,h,\eta}(x',\xi)$ is a finite sum of symbols in $S_1^{k_1+k_2}S_1^{\ell_1+\ell_2}\cup S_1^{k_1+k_2}S_0^{-k(n)+\ell_1+\ell_2}$ whose seminorms are uniformly bounded in (θ,y,h,η). We can now use Proposition 2.2 to conclude that

$$m^\alpha_{\theta,y,h,\eta}(x',hD) : W^{k,\ell,r}\to W^{k-k_1-k_2,\ell-\ell_1-\ell_2,r}$$

with norm uniformly bounded independent of (θ,y,h,η). Choosing $N\geq n+2$ in (8.5) we get that

$$m(x',hD) : W^{k,\ell,r}\to W^{k-k_1-k_2,\ell-\ell_1-\ell_2,r}$$

and the proof is complete. ☐

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