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LOCAL $C^{1,\beta}$-REGULARITY AT THE BOUNDARY OF TWO DIMENSIONAL SLIDING ALMOST MINIMAL SETS IN $\mathbb{R}^3$

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Abstract. In this paper, we will give a $C^{1,\beta}$-regularity result on the boundary for two dimensional sliding almost minimal sets in $\mathbb{R}^3$. This effect may apply to the regularity of the soap films at the boundary, and may also lead to the existence of a solution to the Plateau problem with sliding boundary conditions proposed by Guy David in the case that the boundary is a 2-dimensional smooth submanifold.

1. Introduction

Jean Taylor, in [13], proved a celebrated regularity result of Almgren almost minimal sets, that gives a complete classification of the local structure of 2-dimensional (almost) minimal sets, that is, every 2-dimensional almost minimal set E, in an open set $U \subseteq \mathbb{R}^3$ with gauge function $h(t) \leq Ct^\alpha$, is local $C^{1,\beta}$ equivalent to a 2-dimensional minimal cone. This result may apply to many actual surfaces, soap films are considered as typical examples. In [5], Guy David gave a new proof of this result and generalized it to any codimension. Even with this very nice regularity property, we still do not know the behavior of almost minimal sets E at the boundary $\overline{E} \cap \partial U$, since it could be more and more complicated when points tend to the boundary, that is, the behaves of soap films at the boundary is not clear.

In [7], Guy David proposed to consider the Plateau Problem with sliding boundary conditions, since it is very natural to the soap films, here we mean that the soap films can be consider as sliding almost minimal sets. We see that, away from the boundary, sliding almost minimal sets are almost minimal, Jean Taylor's regularity also applies, so that we already know the behavior of sliding almost minimal sets except at the boundary. Indeed, the feature that allow surfaces moving along the boundary could make the local structure more simple. Motivated by these, the regularity at the boundary would be well worth our considering. In fact, we are looking for a result similar to Jean Talyor's, for which together with Jean Taylor's theorem will imply the local Lipschitz retract property of sliding (almost) minimal sets, and the existence of minimizers for the sliding Plateau Problem will easily

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follows. Certainly we will get the whole story about the regularity of the soap films.

In [13], Jean Taylor gave a full list of a two dimensional minimal cones in $\mathbb{R}^3$, that is, planes, cones of type $\mathbb{Y}$, and cones of type $\mathbb{T}$. One of the advantages for the sliding boundary conditions is that we perceived the chance to determine the possibility of minimal cones in the upper half space Ω_0 of $\mathbb{R}^3$, where minimal cone is a cone which is minimal under the sliding deformations. Indeed, there are seven kinds of cones which are minimal, they are $\partial\Omega_0$, cones of type $\mathbb{V}$, cones of type $\mathbb{P}_+$, cones of type $\mathbb{Y}_+$, cones of type $\mathbb{T}_+$ and cones $\partial\Omega_0 \cup Z$ where Z are cones of type $\mathbb{P}_+$ or $\mathbb{Y}_+$, see Section 3 in [9] for the precise definition of cones of type $\mathbb{P}_+$, $\mathbb{Y}_+$, $\mathbb{T}_+$ and $\mathbb{V}$. Let us refer to Remark 3.11 in [9] for the claim there are at most seven, Theorem 3.10 in [9] proved some cones are minimal, and the rest is proved by Cavallotto [2]. We ascertain that there are only three kinds of cones which are minimal and contains the boundary $\partial\Omega_0$, they are $\partial\Omega_0$ and $\partial\Omega_0 \cup Z$ where Z is cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$, see Theorem 3.10 in [9] for the statement.

Another advantages of the sliding almost minimal sets is that they are not far from usual almost minimal sets, away from the boundary, they are almost minimal, we have also the monotonicity of density property, and at the boundary we can establish a similar monotonicity of density property without too much effort, see Theorem 2.3 for precise statement. But in fact, the monotonicity of density property is not enough, we have estimated the decay of the almost density, and that is also possible with sliding on the boundary, see Corollary 3.16.

In [9], we proved a Hölder regularity of two dimensional sliding almost minimal set at the boundary. That is, suppose that $\Omega \subseteq \mathbb{R}^3$ is a closed domain with boundary $\partial\Omega$ a C^1 manifold of dimension 2, $E \subseteq \Omega$ is a 2 dimensional sliding almost minimal set with sliding boundary $\partial\Omega$, and that $\partial\Omega \subseteq E$. Then E, at the boundary, is locally biHölder equivalent to a sliding minimal cone in the upper half space Ω_0. In this paper, we will generalized the biHölder equivalence to a $C^{1,\beta}$ equivalence when the gauge function h satisfies that $h(t) \le Ct^{\alpha_1}$ and $\partial\Omega$ is a 2 dimensional $C^{1,\alpha}$ manifold. Let us refer to Theorem 1.2 for details. Where the sliding minimal cones always contain the boundary $\partial\Omega_0$, namely only there kinds of cones can appear: $\partial\Omega_0$ and $\partial\Omega_0 \cup Z$, where Z are cones of type $\mathbb{P}_+$ or $\mathbb{Y}_+$.

Let us introduce some notation and definitions before state our main theorem. A gauge function is a nondecreasing function $h : [0,\infty) \to [0,\infty]$ with $\lim_{t\to 0} h(t) = 0$. Let Ω be a closed domain of $\mathbb{R}^3$, L be a closed subset in $\mathbb{R}^3$, $E \subseteq \Omega$ be a given set. Let $U \subseteq \mathbb{R}^3$ be an open set. A family of mappings $\{\varphi_t\}_{0\le t\le 1}$, from E into Ω, is called a sliding deformation of E in U, while $\varphi_1(E)$ is called a competitor of E in U, if following properties hold:

- $\varphi_t(x) = x$ for $x \in E \setminus U$, $\varphi_t(x) \subseteq U$ for $x \in E \cap U$, $0 \le t \le 1$,
- $\varphi_t(x) \in L$ for $x \in E \cap L$, $0 \le t \le 1$,
- the mapping $[0,1] \times E \to \Omega, (t,x) \mapsto \varphi_t(x)$ is continuous,
- φ_1 is Lipschitz and $\varphi_0 = \mathrm{id}_E$.

Definition 1.1. Let $L, \Omega \subseteq \mathbb{R}^3$ be two closed sets, $L \subseteq \Omega$. We say that an nonempty set $E \subseteq \Omega$ is locally sliding almost minimal at $x \in E$ with sliding boundary L and with gauge function h, called (Ω, L, h) locally sliding almost at $x \in E$ for short, if $\mathcal{H}^2 \llcorner E$ is locally finite, and for any sliding deformation $\{\varphi_t\}_{0\le t\le 1}$ of E in $B(x,r)$, we have that

$$\mathcal{H}^2(E \cap B(x,r)) \le \mathcal{H}^2(\varphi_1(E) \cap B(x,r)) + h(r)r^2.$$

We say that E is sliding almost minimal with sliding boundary L and gauge function h, denote by $SAM(\Omega, L, h)$ the collection of all such sets, if E is locally sliding almost minimal at all points $x \in E$.

For any $x \in \mathbb{R}^3$, we let $\boldsymbol{\tau}_x : \mathbb{R}^3 \to \mathbb{R}^3$ be the translation defined by $\boldsymbol{\tau}_x(y) = y+x$, and let $\boldsymbol{\mu}_r : \mathbb{R}^3 \to \mathbb{R}^3$ be the mapping defined by $\boldsymbol{\mu}_r(y) = ry$ for any $r > 0$. For any $S \subseteq \mathbb{R}^3$ and $x \in S$, a blow-up limit of S at x is any closed set in $\mathbb{R}^3$ that can be obtained as the Hausdorff limit of a sequence $\boldsymbol{\mu}_{1/r_k} \circ \boldsymbol{\tau}_{-x}(S)$ with $\lim_{k\to\infty} r_k = 0$. A set X in $\mathbb{R}^3$ is called a cone centered at the origin 0 if for any $\boldsymbol{\mu}_t(X) = X$ for any $t \geq 0$; in general, we call a cone X centered at x if $\boldsymbol{\tau}_{-x}(X)$ is a cone centered at 0. We denote by $\mathrm{Tan}(S, x)$ the tangent cone of S at x, see Section 2.1 in [1]. We see that if there is unique blow-up limit of S at x, then it coincide with the tangent cone $\mathrm{Tan}(S, x)$. Our main theorem is the following.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^3$ be a closed set such that the boundary $\partial\Omega$ is a 2-dimensional manifold of class $C^{1,\alpha}$ for some $\alpha > 0$ and $\mathrm{Tan}(\Omega, z)$ is a half space for any $z \in \partial\Omega$. Let $E \subseteq \Omega$ be a closed set such that $E \supseteq \partial\Omega$ and E is a sliding almost minimal set with sliding boundary $\partial\Omega$ and with gauge function h satisfying that*

$$h(t) \leq C_h t^{\alpha_1},\ 0 < t \leq t_0,\ \textit{for some } C_h > 0, \alpha_1 > 0 \textit{ and } t_0 > 0.$$

Then for any $x_0 \in \partial\Omega$, there is unique blow-up limit of E at x_0; moreover, there exist a radius $r > 0$, a sliding minimal cone Z in Ω_0 with sliding boundary $\partial\Omega_0$, and a mapping $\Phi : \Omega_0 \cap B(0, 2r) \to \Omega$ of class $C^{1,\beta}$, which is a diffeomorphism between its domain and image, such that $\Phi(0) = x_0$, $\Phi(\partial\Omega_0 \cap B(0, 2r)) \subseteq \partial\Omega$, $|\Phi(x) - x_0 - x| \leq 10^{-2}r$ for $x \in B(0, 2r)$, and

$$E \cap B(x_0, r) = \Phi(Z) \cap B(x_0, r).$$

The theorem above, together with the Jean Taylor's theorem, will imply that any sliding almost minimal set E as in the theorem is local Lipschitz neighborhood retract. This effect may gives the existence of a solution to the Plateau problem with sliding boundary conditions in a special case, see Theorem 8.1.

2. Lower bound of the decay for the density

In this section, we will consider a simple case that Ω is a half space and L is its boundary; without loss of generality, we assume that Ω is the upper half space, and change the notation to be Ω_0 for convenience, i.e.

$$\Omega_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \geq 0\}, L_0 = \partial\Omega_0.$$

It is well known that for any 2-rectifiable set E, there exists an approximate tangent plane $\mathrm{Tan}(E, y)$ of E at y for $\mathcal{H}^2$-a.e. $y \in E$. We will denote by $\theta(y) \in [0, \pi/2]$ the angle between the segment $[0, y]$ and the plane $\mathrm{Tan}(E, y)$, by $\theta_x(y) \in [0, \pi/2]$ the angle between the segment $[x, y]$ and the plane $\mathrm{Tan}(E, y)$, for $x \in \mathbb{R}^3$.

For any gauge function h in this paper, we always assume that there is a number $r_h > 0$ such that

$$\int_0^{r_h} \frac{h(2t)}{t} dt < \infty, \tag{2.1}$$

and put

$$h_1(t) = \int_0^t \frac{h(2s)}{s} ds,\ \text{for } 0 \leq t \leq r_h.$$

For any mapping $f : E \subseteq \mathbb{R}^m \to \mathbb{R}^n$, we denote by $\operatorname{ap} J_k f(x) = \| \wedge_k \operatorname{ap} Df(x)\|$ the k dimensional approximate Jacobian of f at x, if f is approximate differentiable at x, see Section 3.2.1 in [10]

In this section, we will compare a set E to the cone over $E \cap \partial B(0, r)$, then establish a monotonicity of density formula for any 2-rectifiable set E which is locally sliding almost minimal at 0, see Theorem 2.3.

Lemma 2.1. *Let $E \subseteq \Omega_0$ be any 2-rectifiable set. Then, by putting $u(r) = \mathcal{H}^2(E \cap B(x, r))$, we have that u is differentiable almost every $r > 0$, and for such r,*

$$\mathcal{H}^1(E \cap \partial B(x, r)) \le u'(r).$$

Proof. Considering the function $\psi : \mathbb{R}^3 \to \mathbb{R}$ defined by $\psi(y) = |y - x|$, we have that, for any $y \neq x$ and $v \in \mathbb{R}^3$,

$$D\psi(y)v = \left\langle \frac{y - x}{|y - x|}, v \right\rangle,$$

thus

$$\operatorname{ap} J_1(\psi|_E)(y) = \sup\{|D\psi(y)v| : v \in \operatorname{Tan}(E, x), |v| = 1\} = \cos\theta_x(y). \tag{2.2}$$

Employing Theorem 3.2.22 in [10], we have that, for any $0 < r < R < \infty$,

$$\int_r^R \mathcal{H}^1(E \cap \partial B(x, t))dt = \int_{E\cap B(x,R)\setminus B(x,r)} \cos_x(y) d\mathcal{H}^2(y) \le u(R) - u(r),$$

we get so that, for almost every $r \in (0, \infty)$,

$$\mathcal{H}^1(E \cap \partial B(x, t)) \le u'(r).$$

□

Lemma 2.2. *Let E be a 2-rectifiable (Ω_0, L_0, h) locally sliding almost minimal at $x \in E$.*

- *If $x \in E \cap L_0$, then for $\mathcal{H}^1$-a.e. $r \in (0, \infty)$,*

$$\mathcal{H}^2(E \cap B(x, r)) \le \frac{r}{2}\mathcal{H}^1(E \cap \partial B(x, r)) + h(2r)(2r)^2. \tag{2.3}$$

- *If $x \in E \setminus L_0$, then inequality (2.3) holds for $\mathcal{H}^1$-a.e. $r \in (0, \operatorname{dist}(x, L_0))$.*

Proof. If $\mathcal{H}^2(E \cap \partial B(x, r)) > 0$, then $\mathcal{H}^1(E \cap \partial B(x, r)) = \infty$, and nothing need to do. We assume so that $\mathcal{H}^2(E \cap \partial B(x, r)) = 0$.

Let $f : [0, \infty) \to [0, \infty)$ be any Lipschitz function, we let $\phi : \Omega_0 \to \Omega_0$ be defined by

$$\phi(y) = f(|y - x|)\frac{y - x}{|y - x|}.$$

Then, for any $y \neq x$ and any $v \in \mathbb{R}^3$, by putting $\tilde{y} = y - x$, we have that

$$D\phi(y)v = \frac{f(|\tilde{y}|)}{|\tilde{y}|}v + \frac{|\tilde{y}|f'(|\tilde{y}|) - f(|\tilde{y}|)}{|\tilde{y}|^2}\left\langle \frac{\tilde{y}}{|\tilde{y}|}, v \right\rangle \tilde{y}$$

If the tangent plane $\operatorname{Tan}^2(E, y)$ of E at y exists, we take $v_1, v_2 \in \operatorname{Tan}^2(E, y)$ such that $|v_1| = |v_2| = 1$, v_1 is perpendicular to $y = x$, and that v_2 is perpendicular to v_1, let v_3 be a vector in $\mathbb{R}^3$ which is perpendicular to $\operatorname{Tan}^2(E, y)$ and $|v_3| = 1$, then

$$\tilde{y} = \langle \tilde{y}, v_2\rangle v_2 + \langle \tilde{y}, v_3\rangle v_3 = |\tilde{y}|\cos\theta_x(y)v_2 + |\tilde{y}|\sin\theta_x(y)v_3,$$

and

$$D\phi(y)v_1 \wedge D\phi(y)v_2 = \frac{f(|\tilde{y}|)^2}{|\tilde{y}|^2} v_1 \wedge v_2 + \frac{|\tilde{y}| f'(|\tilde{y}|) f(|\tilde{y}|) - f(|\tilde{y}|)^2}{|\tilde{y}|^3} \cos\theta_x(y) v_1 \wedge \tilde{y},$$

thus

$$\begin{aligned} \operatorname{ap} J_2(\phi|_E)(y) &= \|D\phi(y)v_1 \wedge D\phi(y)v_2\| \\ &= \frac{f(|\tilde{y}|)}{|\tilde{y}|}\left(f'(|\tilde{y}|)^2 \cos^2\theta_x(y) + \frac{f(|\tilde{y}|)^2}{|\tilde{y}|^2} \sin^2\theta_x(y) \right)^{1/2}. \end{aligned}$$

We consider the function $\psi : \mathbb{R}^3 \to \mathbb{R}$ defined by $\psi(y) = |y - x|$. Then, by (2.2), we have that

$$\operatorname{ap} J_1(\psi|_E)(y) = \cos\theta_x(y).$$

For any $\xi \in (0, r/2)$, we consider the function f defined by

$$f(t) = \begin{cases} 0, & 0 \leq t \leq r - \xi \\ \frac{r}{\xi}(t - r + \xi), & r - \xi < t \leq r \\ t, & t > r. \end{cases}$$

Then we have that

$$\operatorname{ap} J_2(\phi|_E)(y) \leq \frac{f(|\tilde{y}|) f'(|\tilde{y}|)}{|\tilde{y}|} \cos\theta_x(y) + \frac{f(|\tilde{y}|)^2}{|\tilde{y}|^2} \sin\theta_x(y).$$

Applying Theorem 3.2.22 in [10], by putting $A_\xi = E \cap B(0, r) \setminus B(0, r - \xi)$, we get that

$$\begin{aligned} \mathcal{H}^2(\phi(E \cap B(0, r))) &\leq \int_{A_\xi} \frac{r^2}{\xi^2} \cdot \frac{|\tilde{y}| - r + \xi}{|\tilde{y}|} \cos\theta_x(y) d\mathcal{H}^2(y) + \frac{r^2}{(r - \xi)^2} \mathcal{H}^2(A_\xi) \\ &= \int_{r-\xi}^{r} \frac{r^2(t - r + \xi)}{\xi^2 t} \mathcal{H}^1(E \cap \partial B(x, t)) dt + 4\mathcal{H}^2(A_\xi), \end{aligned}$$

thus

$$\mathcal{H}^2(E \cap B(0, r)) \leq (2r)^2 h(2r) + \lim_{\xi \to 0+} r^2 \int_{r-\xi}^{r} \frac{t - r + \xi}{t\xi^2} \mathcal{H}^1(E \cap \partial B(x, t)) dt.$$

Since the function $g(t) = \mathcal{H}^1(E \cap B(x, t))/t$ is a measurable function, we have that, for almost every r,

$$\lim_{\xi \to 0+} \int_0^{\xi} \frac{tg(t - r + \xi)}{\xi^2} dt = \frac{1}{2} g(r),$$

thus for such r,

$$\mathcal{H}^2(E \cap B(x, r)) \leq (2r)^2 h(2r) + \frac{r}{2} \mathcal{H}^1(E \cap \partial B(x, r)).$$

□

For any set $E \subseteq \mathbb{R}^3$, we set

$$\Theta_E(x, r) = r^{-2} \mathcal{H}^2(E \cap B(x, r)), \text{ for any } r > 0,$$

and denote by $\Theta_E(x) = \lim_{r \to 0+} \Theta_E(x, r)$ if the limit exist, we may drop the script E if there is no danger of confusion.

Theorem 2.3. *Let E be a 2-rectifiable (Ω_0, L_0, h) locally sliding almost minimal at $x \in E$.*

- *If $x \in L_0$, then $\Theta(x, r) + 8h_1(r)$ is nondecreasing as $r \in (0, r_h)$.*
- *If $x \notin L_0$, then $\Theta(x, r) + 8h_1(r)$ is nondecreasing as $r \in (0, \min\{r_h, \operatorname{dist}(x, L)\})$.*

Proof. From Lemma 2.2 and Lemma 2.1, by putting $u(r) = \mathcal{H}^2(E \cap B(x,r))$, we get that, if $x \in L$,

$$u(r) \leq \frac{r}{2}u'(r) + h(2r)(2r)^2, \tag{2.4}$$

for almost every $r \in (0,\infty)$; if $x \notin L$, then (2.4) holds for almost every $r \in (0, \min\{r_h, \operatorname{dist}(x,L)\})$.

We put $v(r) = r^{-2}u(r)$, then $v'(r) \geq -8r^{-1}h(2r)$, we get that $\Theta(x,r) + 8h_1(r)$ is nondecreasing. □

Remark 2.4. Let E be a 2-rectifiable (Ω_0, L_0, h) locally sliding almost minimal at some point $x \in E$. Then by Theorem 2.3, we get that $\Theta_E(x)$ exists.

3. Estimation of upper bound

In the previous section, we get a monotonicity of density formula, that is $\Omega_E(x,r) - \Theta_E(x) + 8h_1(r)$ is nondecreasing, thus we get the estimation $\Theta_E(x,r) - \Theta_E(x) \geq -8h_1(r)$ when r small. But in fact we need a good estimation for $|\Omega_E(x,r) - \Theta_E(x)|$, so we have to get some estimation for upper bound. The main purpose of this section is get the control of $\mathcal{H}^2(E \cap B(0,r))$ by a convex combination of $\mathcal{H}^2(Z \cap B(0,r))$ and $\Theta_E(0)r^2$, where Z is the cone over $E \cap \partial B(0,r)$, see Theorem 3.15 and Corollary 3.16.

Let $\mathscr{Z}$ be a collection of cones. We say that a set $E \subseteq \mathbb{R}^3$ is locally $C^{k,\alpha}$-equivalent (resp. C^k-equivalent) to a cone in $\mathscr{Z}$ at $x \in E$ for some nonnegative integer k and some number $\alpha \in (0,1]$, if there exist $\varrho_0 > 0$ and $\tau_0 > 0$ such that for any $\tau \in (0,\tau_0)$ there is $\varrho \in (0,\varrho_0)$, a cone $Z \in \mathscr{Z}$ and a mapping $\Phi : B(0,2\varrho) \to \mathbb{R}^3$, which is a homeomorphism of class $C^{k,\alpha}$ (resp. C^k) between $B(0,2\varrho)$ and its image $\Phi(B(0,2\varrho))$ with $\Phi(0) = x$, satisfying that

$$\|\Phi - \mathrm{id} - \Phi(0)\|_\infty \leq \varrho\tau \tag{3.1}$$

and

$$E \cap B(x,\varrho) \subseteq \Phi\left(Z \cap B\left(0, 2\varrho\right)\right) \subseteq E \cap B(x, 3\varrho). \tag{3.2}$$

Similarly, if $\Omega \subseteq \mathbb{R}^3$ is a closed set with the boundary $\partial\Omega$ is a 2-dimensional manifold, a set $E \subseteq \Omega$ is called locally $C^{k,\alpha}$-equivalent to a sliding minimal cone Z in Ω_0 at $x \in E \cap \partial\Omega$, if there exist $\varrho_0 > 0$ and $\tau_0 > 0$ such that for any $\tau \in (0,\tau_0)$ there is $\varrho \in (0,\varrho_0)$ and a mapping $\Phi : B(0,2\varrho) \cap \Omega_0 \to \Omega$, which is a diffeomorphism of class $C^{k,\alpha}$ between its domain and image with $\Phi(0) = x$ satisfying that $\Phi(L_0 \cap B(0,2\varrho)) \subseteq \partial\Omega$ and (3.1) and (3.2).

Suppose that $\Omega \subseteq \mathbb{R}^3$ is closed set with the boundary $\partial\Omega$ is a 2-dimensional C^1 manifold. Suppose that $E \subseteq \Omega$ is sliding almost minimal with sliding boundary $\partial\Omega$ and gauge function h. Then, by putting $U = \Omega \setminus \partial\Omega$, we see that $E \cap U$ is almost minimal in U, applying Jean Taylor's theorem, E is locally $C^{1,\beta}$-equivalent to a minimal cone at each point $x \in E \cap U$ for some $\beta > 0$ in case $h(r) \leq cr^\alpha$ for some $c > 0$, $\alpha > 0$, $r_0 > 0$ and $0 < r < r_0$. We see from [9, Theorem 6.1] that, at $x \in E \cap \partial\Omega$, E is locally $C^{0,\beta}$-equivalent to a sliding minimal cone in Ω_0 in case the gauge function h satisfying (2.1).

3.1. **Approximation of $E\cap\partial B(0,r)$ by rectifiable curves.** For any sets $X,Y\subseteq\mathbb{R}^3$, any $z\in\mathbb{R}^3$ and any $r>0$, we denote by $d_{z,r}$ the normalized local Hausdorff distance defined by

$$\begin{aligned}d_{z,r}(X,Y)=&\frac{1}{r}\sup\left\{\mathrm{dist}(x,Y):x\in X\cap\overline{B(z,r)}\right\}\\&+\frac{1}{r}\sup\left\{\mathrm{dist}(y,X):y\in Y\cap\overline{B(z,r)}\right\}.\end{aligned}$$

It is quite easy to see that for $r>0$,

- $d_{z,r}(X,Y)\leq d_{z,r}(X,Z)+d_{z,r}(Z,Y)$ if Z is a cone centered at z;
- $d_{z,r}(X,Y)=d_{z,1}(X,Y)$, if X and Y are cones centered at z;
- $d_{z,r}(X,Y)\leq d_{z,r}(X,E)+d_{z,r}(E,Y)$, if X and Y are cones centered at z, $E\cap\overline{B(0,r)}\neq\emptyset$, $d_{z,r}(X,E)\ll 1$ and $d_{z,r}(X,E)\ll 1$.

A cone in $\mathbb{R}^3$ is called of type $\mathbb{Y}$ if it is the union of three half planes with common boundary line and that make $120°$ angles along the boundary line. A cone $Z\subseteq\Omega_0$ is called of type $\mathbb{P}_+$ is if it is a half plane perpendicular to L_0; a cone $Z\subseteq\Omega_0$ is called of type $\mathbb{Y}_+$ is if $Z=\Omega_0\cap Y$, where Y is a cone of type $\mathbb{Y}$ perpendicular to L_0; for convenient, we will also use the notation $\mathbb{P}_+$, to denote the collection of all of cones of type $\mathbb{P}_+$, and $\mathbb{Y}_+$ to denote the collection of all of cones of type $\mathbb{Y}_+$.

For any set $E\subseteq\Omega_0$ with $0\in E$, and any $r>0$, we set

$$\begin{aligned}\varepsilon_P(r)&=\inf\{d_{0,r}(E,Z):Z\in\mathbb{P}_+\},\\\varepsilon_Y(r)&=\inf\{d_{0,r}(E,Z):Z\in\mathbb{Y}_+\}.\end{aligned}$$

If E is 2-rectifiable and $\mathcal{H}^2(E)<\infty$, then $E\cap\partial B(0,r)$ is 1-rectifiable and $\mathcal{H}^1(E\cap\partial B(0,r))<\infty$ for $\mathcal{H}^1$-a.e. $r\in(0,\infty)$, we denote by $\mathscr{R}_0$ the collection of such r; we now consider the function $u:(0,\infty)\to\mathbb{R}$ which is defined by $u(r)=\mathcal{H}^2(E\cap B(0,r))$, it is quite easy to see that u is nondecreasing, thus u is differentiable for $\mathcal{H}^1$-a.e.; we will denote by $\mathscr{R}$ the set $r\in(0,\infty)$ such that $\mathcal{H}^1(E\cap\partial B(0,r))<\infty$, u is differentiable at r, and for any continuous nonnegative function f

$$\lim_{\xi\to 0+}\frac{1}{\xi}\int_{t\in(r-\xi,r)}\int_{E\cap\partial B(0,t)}f(z)d\mathcal{H}^1(z)dt=\int_{E\cap\partial B(0,r)}f(z)d\mathcal{H}^1(z),\tag{3.3}$$

and

$$\sup_{\xi>0}\frac{1}{\xi}\int_{t\in(r-\xi,r)}\mathcal{H}^1(E\cap\partial B(0,t))dt<+\infty.\tag{3.4}$$

It is not hard to see that $\mathcal{H}^1((0,\infty)\setminus\mathscr{R})=0$, see for example Lemma 4.12 in [5].

Lemma 3.1. *Let $E\subseteq\mathbb{R}^3$ be a connected set. If $\mathcal{H}^1(E)<\infty$, then E is path connected.*

For a proof, see for example Lemma 3.12 in [8], so we omit it here.

Lemma 3.2. *Let $\mathbb{X}$ be a locally connected and simply connected compact metric space. Let A and B be two connected subsets of $\mathbb{X}$. If F is a closed subset of $\mathbb{X}$ such that A and B are contained in two different connected components of $\mathbb{X}\setminus F$, then there exists a connected closed set $F_0\subseteq F$ such that A and B still lie in two different connected components of $\mathbb{X}\setminus F_0$.*

Proof. See for example 52.III.1 on page 335 in [12], so we omit the proof here. □

For any $r>0$, we put $\mathfrak{z}_r=(0,0,r)\in\mathbb{R}^3$.

Lemma 3.3. *Let $E \subseteq \Omega_0$ be a 2-rectifiable set with $\mathcal{H}^2(E) < \infty$. Suppose that $0 \in E$, and that E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{P}_+$ at 0. Then for any $\tau \in (0, \tau_0)$ there exist $\mathfrak{r} = \mathfrak{r}(\tau) > 0$ such that, for any $r \in (0, \mathfrak{r}) \cap \mathscr{R}_0$ and $\varepsilon > \varepsilon_P(r)$, we can find $y_r \in E \cap \partial B(0, r) \setminus L_0$, $x_{r,1}, x_{r,2} \in E \cap L_0 \cap \partial B(0, r)$ and two simple curves $\gamma_{r,1}, \gamma_{r,2} \subseteq E \cap \partial B(0, r)$ satisfying that*

(1) $|y_r - \mathfrak{z}_r| \leq \varepsilon r$ and $|x_{r,1} - x_{r,2}| \geq (2 - 2\varepsilon)r$;
(2) $\gamma_{r,i}$ joins y_r and $x_{r,i}$, $i = 1, 2$;
(3) $\gamma_{r,1}$ and $\gamma_{r,2}$ are disjoint except for point y_r.

Proof. Since E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{P}_+$ at 0, for any $\tau \in (0, \tau_0)$, there exist $\varrho > 0$, sliding minimal cone Z of type $\mathbb{P}_+$, and a mapping $\Phi : \Omega_0 \cap B(0, 2\varrho) \to \Omega_0$ which is a homeomorphism between $\Omega_0 \cap B(0, 2\varrho)$ and $\Phi(\Omega_0 \cap B(0, 2\varrho))$ with $\Phi(0) = 0$ and $\Phi(\partial\Omega_0 \cap B(0, 2\varrho)) \subseteq \partial\Omega_0$ such that (3.1) and (3.2) hold. We new take $\mathfrak{r} = \varrho$. Then for any $r \in (0, \mathfrak{r})$,

$$\Phi^{-1}\left[E \cap \partial B(0, r)\right] \subseteq Z \cap B(0, 3\varrho).$$

Without loss of generality, we assume that $Z = \{(x_1, 0, x_3) \mid x_1 \in \mathbb{R}, x_3 \geq 0\}$. Applying Lemma 3.2 with $\mathbb{X} = Z \cap \overline{B(0, 3\varrho)}$, $F = \Phi^{-1}\left[E \cap \partial B(0, r)\right]$, $A = \{0\}$ and $B = Z \cap \partial B(0, 3\varrho)$, we get that there is a connected closed set $F_0 \subseteq F$ such that A and B lie in two different connected components of $\mathbb{X} \setminus F_0$, thus $\Phi(F_0) \subseteq E \cap \partial B(0, r)$ is connected. We put $a_1 = \{(x_1, 0, 0) \mid x_1 < 0\}$ and $a_2 = \{(x_1, 0, 0) \mid x_1 > 0\}$. Then $F_0 \cap a_i \neq \emptyset$, $i = 1, 2$; otherwise A and B are contained in a same connected component of $\mathbb{X} \setminus F_0$. We take $z_{r,i} \in F_0 \cap a_i$, and let $x_{r,i} = \Phi(z_{r,i}) \in E \cap \partial B(0, r)$. Then $|x_{r,1} - x_{r,2}| \geq (2 - 2\varepsilon)r$.

Since $\Phi(F_0)$ is connected and $\mathcal{H}^1(\Phi(F_0)) \leq \mathcal{H}^1(E \cap \partial B(0, r)) < \infty$, by Lemma 3.1, $\Phi(F_0)$ is path connected. But Φ is a homeomorphism, we get that $F_0 = \Phi^{-1}(\Phi(F_0))$ is path connected. Let γ be a simple curve which joins $z_{r,1}$ and $z_{r,2}$. We see that $B(\mathfrak{z}_r, \varepsilon r) \cap \gamma \neq \emptyset$, because $\varepsilon_P(r) < \varepsilon$ and $\mathfrak{z}_r \in Z$ for sliding minimal cone Z of type $\mathbb{P}_+$. We take $y_r \in B(\mathfrak{z}_r, \varepsilon r) \cap \gamma$. □

Lemma 3.4. *Let $E \subseteq \Omega_0$ be a 2-rectifiable set with $\mathcal{H}^2(E) < \infty$. Suppose that $0 \in E$, and that E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{Y}_+$ at 0. Then for any $\tau \in (0, \tau_0)$ there exist $\mathfrak{r} = \mathfrak{r}(\tau) > 0$ such that, for any $r \in (0, \mathfrak{r}) \cap \mathscr{R}_0$ and $\varepsilon > \varepsilon_Y(r)$, we can find $y_r \in E \cap \partial B(0, r) \setminus L_0$, $x_{r,1}, x_{r,2}, x_{r,3} \in E \cap L_0 \cap \partial B(0, r)$ and three simple curves $\gamma_{r,1}, \gamma_{r,2}, \gamma_{r,3} \subseteq E \cap \partial B(0, r)$ satisfying that*

(1) $|\mathfrak{z}_r - y_r| \leq \varepsilon r$, and there exists $Z \in \mathbb{Y}_+$ through 0 such that $\operatorname{dist}(x, Z) \leq \varepsilon r$ *for $x \in \gamma_{r,i}$;*
(2) $\gamma_{r,i}$ join y_r and $x_{r,i}$;
(3) $\gamma_{r,i}$ and $\gamma_{r,j}$ are disjoint except for point y_r.

Proof. Since E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{Y}_+$ at 0, for any $\tau \in (0, \tau_0)$, there exist $\tau > 0$, $\varrho > 0$, sliding minimal cone Z of type $\mathbb{Y}_+$, and a mapping $\Phi : \Omega_0 \cap B(0, 2\varrho) \to \Omega_0$ which is a homeomorphism between $\Omega_0 \cap B(0, 2\varrho)$ and $\Phi(\Omega_0 \cap B(0, 2\varrho))$ with $\Phi(0) = 0$ and $\Phi(\partial\Omega_0 \cap B(0, 2\varrho)) \subseteq \partial\Omega_0$ such that (3.1) and (3.2) hold. We now take $\mathfrak{r} = \varrho$. Then for any $r \in (0, \mathfrak{r})$,

$$\Phi^{-1}\left[E \cap \partial B(0, r)\right] \subseteq Z \cap B(0, 3\varrho).$$

Applying Lemma 3.2 with $\mathbb{X} = Z \cap \overline{B(0, 3\varrho)}$, $F = \Phi^{-1}\left[E \cap \partial B(0, r)\right]$, $A = \{0\}$ and $B = Z \cap \partial B(0, 3\varrho)$, we get that there is a connected closed set $F_0 \subseteq F$ such that A and B lie in two different connected components of $\mathbb{X} \setminus F_0$, thus $\Phi(F_0) \subseteq E \cap \partial B(0, r)$

is connected. We let a_i, $i = 1, 2, 3$, be the three component of $Z \cap L_0 \setminus A$. Then $F_0 \cap a_i \neq \emptyset$, $i = 1, 2, 3$; otherwise A and B are contained in a same connected component of $\mathbb{X} \setminus F_0$. We take $z_{r,i} \in F_0 \cap a_i$, and let $x_{r,i} = \Phi(z_{r,i}) \in E \cap \partial B(0, r)$. Then $|x_{r,1} - x_{r,2}| \geq (\sqrt{3} - 2\varepsilon)r$.

Using the same arguments as in the proof of Lemma 3.3, we get that F_0 is path connected. We see that Z is of type $\mathbb{Y}_+$, denote by $\ell(Z)$ the spine of Z, that is, the half line through 0 and perpendicular to $\partial\Omega_0$. We find a point $y_r \in F_0 \cap \ell(Z)$ and curves $\gamma_{r,i}$ satisfying the conditions. □

3.2. **Approximation of rectifiable curves in $\mathbb{S}^2$ by Lipschitz graph.** We denote by $\mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$. We say that a simple rectifiable curve $\gamma \subseteq \mathbb{S}^2$ is a Lipschitz graph with constant at most η, if it can be parametrized, after a rotation, by

$$z(t) = \left(\sqrt{1 - v(t)^2} \cos\theta(t), \sqrt{1 - v(t)^2} \sin\theta(t), v(t)\right),$$

where v is Lipschitz with $\mathrm{Lip}(v) \leq \eta$.

Lemma 3.5. *Let $T \in [\pi/3, 2\pi/3]$ be a number, and $\gamma : [0, T] \to \mathbb{S}^2$ a simple rectifiable curve given by*

$$\gamma(t) = \left(\sqrt{1 - v(t)^2} \cos\theta(t), \sqrt{1 - v(t)^2} \sin\theta(t), v(t)\right),$$

where v is a Lipschitz function with $v(0) = v(T) = 0$, θ is a continuous function with $\theta(0) = 0$ and $\theta(T) = T$. Then there is a small number $\tau_0 \in (0, 1)$ such that whenever $|v(t)| \leq \tau_0$, we have that

$$|v(t)| \leq 10\sqrt{\mathcal{H}^1(\gamma) - T}. \tag{3.5}$$

Moreover, there is an $\varepsilon_0 > 0$ such that (3.5) holds whenever $\mathcal{H}^1(\gamma) - T \leq \varepsilon_0$.

Proof. We let $A = \gamma(0) = (1, 0, 0)$, $B = \gamma(T) = (\cos T, \sin T, 0)$, and let $C = \gamma(t_0)$ be a point in γ such that

$$|v(t_0)| = \max\{|v(t)| : t \in [0, T]\}.$$

We let γ_i, $i = 1, 2$, be two curves such that $\gamma_1(0) = A$, $\gamma_1(1) = C$, $\gamma_2(0) = B$, $\gamma_2(1) = C$ and $\gamma_i \subseteq \gamma$, let $s = \inf\{s \in [0, 1] : \gamma_1(s) \in \gamma_2\}$, and put $D = \gamma_1(s)$. By setting $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ the arcs AD, BD and CD respectively, i.e., AD is the arc of the great circle on the unity sphere which joint the points A and D. Then we have that

$$\mathcal{H}^1(\gamma) \geq \mathcal{H}^1(\gamma_1 \cup \gamma_2) \geq \mathcal{H}^1(\mathfrak{C}_1) + \mathcal{H}^1(\mathfrak{C}_2) + \mathcal{H}^1(\mathfrak{C}_3).$$

We see that $\mathfrak{C}_1 \cup \mathfrak{C}_2$ is a simple Lipschitz curve joining A and B, and let $\gamma_3 : [0, \ell] \to \mathbb{S}^2$ giving by

$$\gamma_3(t) = \left(\sqrt{1 - w(t)^2} \cos\theta(t), \sqrt{1 - w(t)^2} \sin\theta(t), w(t)\right)$$

be its parametrization by length. We assume that $\gamma_3(t_1) = D$, then $w'(t) > 0$ on $(0, t_1)$, or $w'(t) < 0$ on $(0, t_1)$, thus $|w(t)| = \int_0^{t_1} |w'(t)| dt$.

We let the number $\tau_0 \in (0, 1)$ to be the small number τ_1 in Lemma 7.8 in [5]. If $\mathcal{H}^1(\gamma) - T \leq \tau_0$, then we have that

$$\int_0^\ell |w'(t)|^2 dt \leq 14(\ell - T),$$

thus

$$|w(t_1)| = \int_0^{t_1} |w'(t)|dt \le \left(t_1 \int_0^{t_1} |w'(t)|^2 dt\right)^{1/2} \le \sqrt{14\ell(\ell - T)}.$$

We get so that

$$\begin{aligned}|v(t_0)| &\le \mathcal{H}^1(\mathfrak{C}_3) + |w(t_1)| \le (\mathcal{H}^1(\gamma) - \ell) + \sqrt{14\ell(\ell - T)}\\ &\le \sqrt{14\mathcal{H}^1(\gamma)(\mathcal{H}^1(\gamma) - T)} \le 10\sqrt{\mathcal{H}^1(\gamma) - T}.\end{aligned}$$

If $\mathcal{H}^1(\gamma) - T > \tau_0$, then $v(t) \le \tau_0 \le 10\sqrt{\tau_0} \le 10\sqrt{\mathcal{H}^1(\gamma) - T}$. □

Lemma 3.6. *Let a and b be two points in $\Omega_0 \cap \partial B(0,1)$ satisfying*

$$\frac{\pi}{3} \le \mathrm{dist}_{\mathbb{S}^2}(a, b) \le \frac{2\pi}{3}.$$

Let γ be a simple rectifiable curve in $\Omega_0 \cap \partial B(0,1)$ which joins a and b, and satisfies

$$\mathrm{length}(\gamma) \le \mathrm{dist}_{\mathbb{S}^2}(a, b) + \tau_0,$$

where $\tau_0 > 0$ is as in Lemma 3.5. Then there is a constant $C > 0$ such that, for any $\eta > 0$, we can find a simple curve γ_* in $\Omega_0 \cap \partial B(0,1)$ which is a Lipschitz graph with constant at most η joining a and b, and satisfies that

$$\mathcal{H}^1(\gamma_* \setminus \gamma) \le \mathcal{H}^1(\gamma \setminus \gamma_*) \le C\eta^{-2}(\mathrm{length}(\gamma) - \mathrm{dist}_{\mathbb{S}^2}(a, b)).$$

Moreover, if we denote by $\Gamma_{a,b}$ the geodesic joining a and b, then we can assume that

$$\mathrm{dist}(x, \Gamma_{a,b}) \le \eta\, \mathrm{dist}(x, \{a, b\}), \forall x \in \gamma_*. \tag{3.6}$$

The proof will be the same as in [5, p.875-p.878], so we omit it.

3.3. **Comparison surfaces.** Let Γ be a Lipschitz curve in $\mathbb{S}^2$. We assume for simplicity that its extremities a and b lie in the horizontal plane. Let us assume that $a = (1, 0, 0)$ and $b = (\cos T, \sin T, 0)$ for some $T \in [\pi/3, 2\pi/3]$. We also assume that Γ is a Lipschitz graph with constant at most η, i.e. there is a Lipschitz function $s : [0, T] \to \mathbb{R}$ with $s(0) = s(T) = 0$ and $\mathrm{Lip}(s) \le \eta$, such that Γ is parametrized by

$$z(t) = (w(t)\cos t, w(t)\sin t, s(t)) \text{ for } t \in [0, T],$$

where $w(t) = (1 - |s(t)|^2)^{1/2}$.

We set

$$D_T = \{(r\cos t, r\sin t)| \mid 0 < r < 1, 0 < t < T\},$$

and consider the function $v : \overline{D}_T \to \mathbb{R}$ defined by

$$v(r\cos t, r\sin t) = \frac{rs(t)}{w(t)} \text{ for } 0 \le r \le 1 \text{ and } 0 \le t \le T.$$

For any function $f : \overline{D}_T \to \mathbb{R}$, we denote by Σ_f the graphs of f over $\overline{D}_T$.

Lemma 3.7. *There is a universal constant $\kappa > 0$ such that we can find a Lipschitz function u on $\overline{D}_T$ satisfying that*

$$\begin{gathered}\mathrm{Lip}(u) \le C\eta,\\ u(r, 0) = u(r\cos T, r\sin T) = 0, \text{ for } 0 \le r \le 1,\\ u(r\cos t, r\sin t) = v(r\cos t, r\sin t) \text{ for } 1 - 2\kappa \le r \le 1, 0 \le t \le T,\\ u(r\cos t, r\sin t) = 0, \text{ for } 0 \le r \le 2\kappa, 0 \le t \le T\end{gathered} \tag{3.7}$$

and

$$\mathcal{H}^2(\Sigma_v) - \mathcal{H}^2(\Sigma_u) \geq 10^{-4}(\mathcal{H}^1(\Gamma) - T). \tag{3.8}$$

The proof is the same as Lemma 8.8 in [5], we omit it here.

3.4. **Retractions.** In this subsection, we assume that $E \subseteq \Omega_0$ is a 2-rectifiable set satisfying that

(a) $\mathcal{H}^2(E) < \infty$, $0 \in E$,
(b) E is locally (Ω_0, L_0, h) sliding almost minimal at 0,
(c) E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$.

For any $r > 0$, we let $\varepsilon(r) = \varepsilon_P(r)$ if E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{P}_+$, and let $\varepsilon(r) = \varepsilon_Y(r)$, if E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{Y}_+$. Recall that $\mathscr{R}_0$ is denoted by the collection of radii $r \in (0, \infty)$ such that $\mathcal{H}^1(E \cap \partial B(0, r)) < \infty$. For any $r \in (0, \mathfrak{r}) \cap \mathscr{R}_0$, we will discuss two situations: first, if Z is a sliding minimal cone of type $\mathbb{P}_+$, we put $\mathcal{X}_r = \{x_{r,1}, x_{r,2}\}$, where $x_{r,1}$ and $x_{r,2}$ are considered as in Lemma 3.3. Second, if Z is a sliding minimal cone of type $\mathbb{Y}_+$, we put $\mathcal{X}_r = \{x_{r,1}, x_{r,2}, x_{r,3}\}$, where $x_{r,1}$, $x_{r,2}$ and $x_{r,3}$ are consider as in Lemma 3.4. We see that $\mathcal{X}_r \subseteq E \cap \partial B(0, r) \cap L_0$.

We take y_r as in Lemma 3.3 or Lemma 3.4. For any $x \in \mathcal{X}_r$, we let γ_x be the curve joining x and y_r which is considered as in Lemma 3.3 or Lemma 3.4, put $\Gamma_x = \boldsymbol{\mu}_{1/r}(\gamma_x)$. Then by Lemma 3.6, there is a curve $\Gamma_{x,*}$ on $\Omega_0 \cap \partial B(0, 1)$ joining $\boldsymbol{\mu}_{1/r}(x)$ and $\boldsymbol{\mu}_{1/r}(y_r)$ which is a Lipschitz graph with constant at most $\eta \leq 10^{-6}$. Let $\mathfrak{C}_x$ be the arc on $\partial B(0, 1)$ joining $\boldsymbol{\mu}_{1/r}(x)$ and $\boldsymbol{\mu}_{1/r}(y_r)$, let D_x and $\mathcal{M}_x$ be the cone over $\mathfrak{C}_x$ and $\Gamma_{x,*}$ respectively. By Lemma 3.7, we can find Lipschitz graph Σ_x corresponding to $\mathcal{M}_x$ such that (3.7) and (3.8) hold, that is,

$$\begin{gathered}\Sigma_x \cap B(0, 2\kappa) = D_x \cap B(0, 2\kappa), \\ \Sigma_x \cap \overline{B(0,1)} \setminus B(0, 1 - 2\kappa) = \mathcal{M}_x \cap \overline{B(0,1)} \setminus B(0, 1 - 2\kappa), \\ \mathcal{H}^2(\mathcal{M}_x \cap B(0, 1)) - \mathcal{H}^2(\Sigma_x) \geq 10^{-4}(\mathcal{H}^1(\Gamma_{x,*}) - \mathcal{H}^1(\mathfrak{C}_x)).\end{gathered} \tag{3.9}$$

We put

$$X = \bigcup_{x \in \mathcal{X}_r} D_x,\ \Gamma = \bigcup_{x \in \mathcal{X}_r} \Gamma_x,\ \Gamma_* = \bigcup_{x \in \mathcal{X}_r} \Gamma_{x,*},\ \mathfrak{C} = \bigcup_{x \in \mathcal{X}_r} \mathfrak{C}_x,\ \mathcal{M} = \bigcup_{x \in \mathcal{X}_r} \mathcal{M}_x,\ \Sigma = \bigcup_{x \in \mathcal{X}_r} \Sigma_x. \tag{3.10}$$

From (3.9), we see that

$$\mathcal{H}^2(\mathcal{M} \cap B(0, 1)) - \mathcal{H}^2(\Sigma) \geq 10^{-4} \left(\mathcal{H}^1(\Gamma_*) - \mathcal{H}^1(\mathfrak{C})\right). \tag{3.11}$$

By Lemma 3.5 and Lemma 3.6, we have that

$$d_H(\mathfrak{C}_x, \Gamma_{x,*}) \leq 10 \left(\mathcal{H}^1(\Gamma_{x,*}) - \mathcal{H}^1(\mathfrak{C}_x)\right)^{1/2} \leq 10 \left(\mathcal{H}^1(\Gamma_*) - \mathcal{H}^1(\mathfrak{C})\right)^{1/2}$$

and

$$d_{0,1}(X, \mathcal{M}) \leq d_H(\mathfrak{C}, \Gamma_*) \leq \max_{x \in \mathcal{X}_r} d_H(\mathfrak{C}_x, \Gamma_{x,*}) \leq 10 \left(\mathcal{H}^1(\Gamma_*) - \mathcal{H}^1(\mathfrak{C})\right)^{1/2}. \tag{3.12}$$

For any $r \in (0, \mathfrak{r}) \cap \mathscr{R}_0$, we put $j(r) = r^{-1}\mathcal{H}^1(E \cap \partial B(0, r)) - \mathcal{H}^1(\mathfrak{C})$, and denote by $\mathscr{R}_1$ the set $\{r \in (0, \mathfrak{r}) \cap \mathscr{R} : j(r) \leq \tau_0\}$, where τ_0 is the small number considered as in Lemma 3.5, $\mathscr{R} \subseteq \mathscr{R}_0$ is defined in (3.3) and (3.4). Then (3.12) implies that

$$d_{0,r}(X, M) \leq 10 j(r)^{1/2}. \tag{3.13}$$

Lemma 3.8. *If $\varepsilon(r) < 1/2$, then for any $\varepsilon \in (\varepsilon(r), 1/2)$, there is a sliding minimal cone $Z = Z_r$ such that*

$$d_{0,1}(X, Z) \leq 4\varepsilon.$$

Moreover, we have that

$$d_{0,r}(E, X) \leq 5\varepsilon(r).$$

Proof. There exists sliding minimal cone Z such that $d_{0,r}(E, Z) \leq \varepsilon$, thus for any $x \in \mathcal{X}_r$, there is $x_z \in Z \cap (L_0 \cap \partial B_r)$ satisfying that $|x - x_z| \leq 2\varepsilon r$. We get so that

$$d_H([x, y_r], [x_z, \mathfrak{z}_r]) \leq 2\varepsilon r.$$

Since $\operatorname{dist}(0, [x, y_r]) > r/2$ for any $x \in \mathcal{X}_r$, we have that

$$d_H(X \cap B(0, r/2), Z \cap B(0, r/2)) \leq 2\varepsilon r.$$

Thus

$$d_{0,1}(X, Z) = d_{0,r/2}(X, Z) \leq 4\varepsilon,$$

and

$$d_{0,r}(E, X) \leq d_{0,r}(E, Z) + d_{0,r}(Z, X) \leq 5\varepsilon.$$

□

Lemma 3.9. *Let $0 < \delta, \varepsilon < 1/2$ be positive numbers. Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be three unit vectors.*

- *If $|\langle v_2, v_i\rangle| \leq \delta$ for $i = 1, 3$, then for any $v \in \mathbb{R}^3$ with $\langle v, v_2\rangle = 0$ and $\operatorname{dist}(v, \operatorname{span}\{v_1, v_2\}) \leq \varepsilon|v|$, we have that*

  $$|\langle v, v_3\rangle - \langle v_1, v_3\rangle\langle v, v_1\rangle| \leq (\varepsilon + \delta)|v|, \text{ and } |\langle v, v_1\rangle| \geq (1 - \varepsilon - \delta)|v|.$$

- *If $\langle v_1, v_3\rangle < 1$ and $0 < 64\delta < 1 - \langle v_1, v_3\rangle$, then for any $w_1, w_3 \in \mathbb{R}^3$ with $\langle v_i, w_i\rangle \geq (1 - \delta)|w_i|$, $i = 1, 3$, we have that*

$$|w_1| + |w_3| \leq 2 \cdot (1 - \langle v_1, v_3\rangle)^{-1/2} |w_1 - w_3|. \tag{3.14}$$

Proof. We write $v = v^\perp + \lambda_1 v_1 + \lambda_2 v_2$, $\lambda_i \in \mathbb{R}$, $\langle v^\perp, v_i\rangle = 0$. Since $\langle v, v_2\rangle = 0$, we have that $\lambda_2 = -\lambda_1\langle v_1, v_2\rangle$, thus

$$\lambda_1 = \frac{\langle v, v_1\rangle}{1 - \langle v_1, v_2\rangle^2}, \ \lambda_2 = -\frac{\langle v, v_1\rangle\langle v_1, v_2\rangle}{1 - \langle v_1, v_2\rangle^2},$$

we get so that

$$v = v^\perp + \frac{\langle v, v_1\rangle v_1 - \langle v, v_1\rangle\langle v_1, v_2\rangle v_2}{1 - \langle v_1, v_2\rangle^2}, \tag{3.15}$$

and then

$$\langle v, v_3\rangle = \langle v^\perp, v_3\rangle + \frac{\langle v_1, v_3\rangle - \langle v_2, v_3\rangle\langle v_1, v_2\rangle}{1 - \langle v_1 . v_2\rangle^2}\langle v, v_1\rangle,$$

thus

$$|\langle v, v_3\rangle - \langle v_1, v_3\rangle\langle v, v_1\rangle| \leq \varepsilon|v| + \frac{\delta^2 + \delta}{1 - \delta^2}|v| \leq (\varepsilon + 2\delta)|v|.$$

We get also, from (3.15), that

$$|v| \leq |v^\perp| + \frac{1 + |\langle v_1 . v_2\rangle|}{1 - \langle v_1, v_2\rangle^2}|\langle v, v_1\rangle| \leq \varepsilon|v| + \frac{1}{1 - \delta}|\langle v, v_1\rangle|,$$

thus

$$|\langle v, v_1\rangle| \geq (1 - \varepsilon)(1 - \delta)|v| \geq (1 - \varepsilon - \delta)|v|.$$

We can certainly assume $w_i \neq 0$, otherwise the inequality (3.14) will be trivial true. Since $\langle v_i, w_i\rangle \geq (1-\delta)|w_i|$, we have that $\langle v_i, w_i/|w_i|\rangle \geq 1-\delta$, and

$$\left|v_i - w_i/|w_i|\right|^2 = 2 - 2\langle v_i, w_i/|w_i|\rangle \leq 2\delta,$$

we have so that

$$\begin{aligned}\left|\frac{w_1}{|w_1|} - \frac{w_3}{|w_3|}\right|^2 &= \left|\left(\frac{w_1}{|w_1|} - v_1\right) - \left(\frac{w_3}{|w_3|} - v_3\right) + (v_1 - v_3)\right|^2 \\ &\geq |v_1 - v_3|^2 - 2|v_1 - v_3|\left(\left|\frac{w_1}{|w_1|} - v_1\right| + \left|\frac{w_3}{|w_3|} - v_3\right|\right) \\ &\geq |v_1 - v_3|^2 - 4\sqrt{2\delta}|v_1 - v_3|,\end{aligned}$$

and

$$\langle w_1, w_3\rangle = |w_1||w_3|\left\langle \frac{w_1}{|w_1|}, \frac{w_3}{|w_3|}\right\rangle \leq |w_1||w_3|\left(\langle v_1, v_3\rangle + 2\sqrt{2\delta}|v_1 - v_3|\right).$$

Thus

$$|w_1 - w_3|^2 \geq |w_1|^2 + |w_3|^2 - 2|w_1||w_3|\left(\langle v_1, v_3\rangle + 2\sqrt{2\delta}|v_1 - v_3|\right) \geq (1-s)(|w_1| + |w_3|)^2,$$

where $s = \frac{1}{2}(1 + \langle v_1, v_3\rangle + 2\sqrt{2\delta}|v_1 - v_3|) \leq \frac{1}{2}(1 + \langle v_1, v_3\rangle) + \frac{1}{4}(1 - \langle v_1, v_3\rangle)$. Hence

$$|w_1| + |w_3| \leq (1-s)^{-1/2}|w_1 - w_3| \leq 2(1 - \langle v_1, v_3\rangle)^{-1/2}|w_1 - w_3|.$$

□

Lemma 3.10. *For any $r \in (0, \mathfrak{r}) \cap \mathscr{R}_1$, we let Σ be as in* (3.10)*. Then there is a universal constant $C > 0$ and a Lipschitz mapping $p : \Omega_0 \to \Sigma$ with* $\mathrm{Lip}(p) \leq C$*, such that $p(z) \in L$ for $z \in L$, and that $p(z) = z$ for $z \in \Sigma$.*

Proof. We see from (3.9) that

$$\Sigma \cap \overline{B(0,1)} \setminus B(0, 1-2\kappa) = \mathcal{M} \cap \overline{B(0,1)} \setminus B(0, 1-2\kappa)),$$

and

$$\Sigma \cap B(0, 2\kappa) = X \cap B(0, 2\kappa).$$

For any $z \in \Omega_0 \setminus \{0\}$, we denote by $\ell(z)$ the line which goes through 0 and z, and denote $\partial D_x = \ell(x) \cup \ell(y_r)$. Let $\sigma \in (0, 10^{-3})$ be fixed. We put

$$\begin{aligned} R^x &= \{z \in \Omega_0 \mid \mathrm{dist}(z, D_x) \leq \sigma\, \mathrm{dist}(z, \partial D_x)\}, \\ R_1^x &= \{z \in \Omega_0 \mid \mathrm{dist}(z, D_x) \leq \sigma\, \mathrm{dist}(z, \ell(y_r))\}, \end{aligned}$$

and

$$R = \bigcup_{x \in \mathcal{X}_r} R^x, R_1 = \bigcup_{x \in \mathcal{X}_r} R_1^x.$$

Then we see that $R^x \subseteq R_1^x$, and that both of them are cones,

$$R^{x_i} \cap R^{x_j} = R_1^{x_i} \cap R_1^{x_j} = \ell(y_r) \text{ for } x_i, x_j \in \mathcal{X}_r, x_i \neq x_j.$$

Since $\Gamma_{x,*}$ is a Lipschitz graph with constant at most η such that (3.6) hold, we have that

$$\mathcal{M}_x \subseteq R^x \text{ and } \Sigma_x \subseteq R^x,$$

when η small enough.

We will construct a Lipschitz retraction $p_0 : \Omega_0 \to R_1$ such that $p_0(z) = z$ for $z \in R_1$, $p_0(z) \in L_0$ for $z \in L_0$, and $\mathrm{Lip}(p_0) \leq 25$. We now distinguish two cases, depending on cardinality of $\mathcal{X}_r$.

Case 1. $\operatorname{card}(\mathcal{X}_r) = 2$. We assume that $\mathcal{X}_r = \{x_1, x_2\}$. Then $|y_r| = |x_1| = |x_2| = r$, and

$$0 \le \langle x_1, x_2\rangle + r^2 \le 2\varepsilon^2 r^2.$$

Since $|y_r - \mathfrak{z}_r| \le \varepsilon r$, we have that $|\langle y_r, x\rangle| \le \varepsilon r^2$ for any $x \in L \cap \partial B(0, r)$.

We now let e_1 and e_2 be two unit vectors in L_0 such that $\langle x_1, e_1\rangle = \langle x_2, e_1\rangle \ge 0$ and $e_2 = -e_1$. Then

$$0 \le \langle x_i, e_1\rangle \le \varepsilon r, \ \ i \in \{1, 2\}.$$

We let Ω_1' and Ω_2' be the two connected components of $\Omega_0 \setminus (\cup_i D_{x_i})$ such that $e_i \in \Omega_i'$. We put $\Omega_i = \Omega_i' \setminus R_1$. We claim that

$$|\langle z_1 - z_2, e_i\rangle| \le 10(\sigma + \varepsilon)|z_1 - z_2| \tag{3.16}$$

whenever $z_1, z_2 \in \partial\Omega_i$, $z_1 \ne z_2$, $i \in \{1, 2\}$.

Without loss of generality, we assume $z_1, z_2 \in \partial\Omega_1$, because for another case we will use the same treatment. We see that

$$\operatorname{dist}(z_i, D_{x_j}) = \sigma \operatorname{dist}(z_i, \ell(y_r)).$$

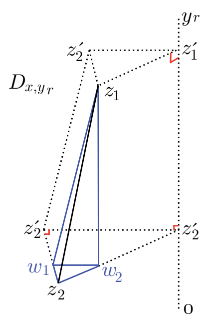

FIGURE 1. The angle between $z_1 - z_2$ and D_x is small.

(1) In case $z_1, z_2 \in \partial R_1^{x_i} \cap \overline{\Omega}_1$, without loss of generality, we assume that $z_1, z_2 \in \partial R_1^{x_1} \cap \overline{\Omega}_1$. We let $\widetilde{z}_i \in D_{x_1}$ and $z_i' \in \ell(y_r)$ be such that

$$|z_i - \widetilde{z}_i| = \operatorname{dist}(z_i, D_{x_1}), \ \ |z_i - z_i'| = \operatorname{dist}(z_i, \ell(y_r)), \ \ i \in \{1, 2\}.$$

We put

$$w_1 = z_1 - \widetilde{z}_1 + \widetilde{z}_2, \ \ w_2 = z_1 - z_1' + z_2',$$

then we get that $z_1 - z_2 = (z_1 - w_2) + (w_2 - z_2)$. Moreover, we have that $z_1 - w_2$ is perpendicular to $w_2 - z_2$ and parallel to y_r. Thus $|w_2 - z_2| \le |z_1 - z_2|$, $|z_1 - w_2| \le |z_1 - z_2|$ and

$$\operatorname{dist}(w_2 - z_2, \operatorname{span}\{x_1, y_r\}) = \sigma|w_2 - z_2|.$$

Applying Lemma 3.9, we get that

$$|\langle z_1 - w_2, e_1\rangle| \le \varepsilon|z_1 - w_2| \text{ and } |\langle w_2 - z_2, e_1\rangle| \le (\sigma + 3\varepsilon)\,|w_2 - z_2|,$$

thus

$$|\langle z_1 - z_2, e_1\rangle| \le |\langle z_1 - w_2, e_1\rangle| + |\langle w_2 - z_2, e_1\rangle| \le (\sigma + 4\varepsilon)\,|z_1 - z_2|.$$

(2) In case $z_1 \in \partial R^{x_1} \cap \overline{\Omega}_1$, $z_2 \in \partial R^{x_2} \cap \overline{\Omega}_1$. We let $\widetilde{z}_i \in D_{x_i}$ and $z_i' \in \ell(y_r)$ be such that

$$|z_i - \widetilde{z}_i| = \operatorname{dist}(z_i, D_{x_i}),\ |z_i - z_i'| = \operatorname{dist}(z_i, \ell(y_r)),\ i = 1, 2.$$

Then by Lemma 3.9, we have that

$$\left\langle z_i - z_i', \frac{x_i}{|x_i|} \right\rangle \geq (1 - \sigma - \varepsilon)|z_i - z_i'|,\ i = 1, 2.$$

Since $z_1 - z_2 = (z_1 - z_1') + (z_2' - z_2) + (z_1' - z_2')$, we have that

$$|\langle z_1' - z_2', e_1 \rangle| \leq \varepsilon |z_1' - z_2'| \leq \varepsilon |z_1 - z_2|$$

and

$$|\langle z_i - z_i', e_1 \rangle| \leq (\sigma + \varepsilon)\, |z_i - z_i'|,$$

we get so that

$$\begin{aligned} |\langle z_1 - z_2, e_1 \rangle| &\leq |\langle z_1 - z_1', e_1 \rangle| + |\langle z_2' - z_2, e_1 \rangle| + |\langle z_1' - z_2', e \rangle| \\ &\leq 2 \cdot (\sigma + \varepsilon)\, (|z_1 - z_1'| + |z_2 - z_2'|) + \varepsilon |z_1 - z_2|. \end{aligned} \tag{3.17}$$

Since $z_1' - z_2'$ is perpendicular to $z_1 - z_1'$ and $z_2 - z_2'$, $\langle x_1/|x_1|, x_2/|x_2| \rangle \leq -1 + 2\varepsilon^2$ and

$$\left\langle z_i - z_i', \frac{x_i}{|x_i|} \right\rangle \geq (1 - \sigma - \varepsilon)|z_i - z_i'|,\ i = 1, 2,$$

by (3.14) in Lemma 3.9, we get that

$$|z_1 - z_1'| + |z_2 - z_2'| \leq 2 \cdot \left(2 - 2\varepsilon^2\right)^{-1/2} |(z_1 - z_1') - (z_2 - z_2')| \leq 4|z_1 - z_2|.$$

Thus inequality (3.17) implies that

$$|\langle z_1 - z_2, e_1 \rangle| \leq (8\sigma + 9\varepsilon)|z_1 - z_2| \leq 10(\sigma + \varepsilon)|z_1 - z_2|,$$

and we finished the proof of the claim (3.16).

We now define $p_0 : \Omega_0 \to R_1$ as follows: for any $z \in \Omega_i$, we let $p_0(z)$ be the unique point in $\partial\Omega_i$ such that $p_0(z) - z$ parallels e; and for any $z \in R_1$, we let $p_0(z) = z$. Since $p_0(z) - z$ parallels e, we see that $p_0(L_0) \subseteq L_0$. We will check that

$$p_0 \text{ is Lipschitz with } \operatorname{Lip}(p_0) \leq \frac{2}{1 - 10(\sigma + \varepsilon)}.$$

Indeed, for any $z_1, z_2 \in \Omega_0$, we put

$$p_0(z_i) = z_i + t_i e,\ t_i \in \mathbb{R},$$

then

$$\begin{aligned} |t_1 - t_2| = |\langle (t_1 - t_2)e, e \rangle| &\leq |\langle p_0(z_1) - p_0(z_2), e \rangle| + |\langle z_1 - z_2, e \rangle| \\ &\leq 10(\sigma + \varepsilon)|p_0(z_1) - p_0(z_2)| + |z_1 - z_2|, \end{aligned}$$

and

$$|p_0(z_1) - p_0(z_2)| \leq |z_1 - z_2| + |t_1 - t_2| \leq 10(\sigma + \varepsilon)|p_0(z_1) - p_0(z_2)| + 2|z_1 - z_2|,$$

thus

$$|p_0(z_1) - p_0(z_2)| \leq \frac{2}{1 - 10(\sigma + \varepsilon)} |z_1 - z_2|.$$

Case 2. $\mathrm{card}(\mathcal{X}_r) = 3$. We assume that $\mathcal{X}_r = \{x_1, x_2, x_3\}$, then

$$|\langle x_i, y_r\rangle| \le \varepsilon r^2, \left(-\frac{1}{2} - \sqrt{3}\varepsilon\right) r^2 \le \langle x_i, x_j\rangle \le \left(-\frac{1}{2} + 2\varepsilon\right) r^2.$$

We put

$$e_1 = \frac{x_2 + x_3}{|x_2 + x_3|}, e_2 = \frac{x_1 + x_3}{|x_1 + x_3|}, e_3 = \frac{x_2 + x_1}{|x_2 + x_1|},$$

and let Ω_1', Ω_2' and Ω_3' be the three connected components of $\Omega_0 \setminus (\cup_i D_{x_i})$ such that $e_i \in \Omega_i'$. By putting $\Omega_i = \Omega_i' \setminus R_1$, we claim that

$$|\langle z_1 - z_2, e_i\rangle| \le \left(\frac{9}{10} + 5\sigma + 5\varepsilon\right) |z_1 - z_2| \tag{3.18}$$

whenever $z_1, z_2 \in \partial\Omega_i$, $z_1 \ne z_2$, $i \in \{1, 2, 3\}$.

Indeed, we only need to check the case $z_1, z_2 \in \partial\Omega_1$, and the other two cases will be the same. Since $(-1/2 - \sqrt{3}\varepsilon)r^2 \le \langle x_i, x_j\rangle \le (-1/2 + 2\varepsilon)r^2$, we have that $(1/2 - \varepsilon)r \le \langle x_i, e_1\rangle \le (1/2 + \varepsilon)r$ for $i = 2, 3$. In case $z_1, z_2 \in \partial R_1^{x_2} \cap \overline{\Omega}_1$ or $z_1, z_2 \in \partial R_1^{x_3} \cap \overline{\Omega}_1$. Let us assume that $z_1, z_2 \in \partial R_1^{x_2} \cap \overline{\Omega}_1$. Let $\widetilde{z}_i \in D_{x_2}$ and $z_i' \in \ell(y_r)$ be such that

$$|z_i - \widetilde{z}_i| = \mathrm{dist}(z_i, D_{x_2}),\ |z_i - z_i'| = \mathrm{dist}(z_i, \ell(y_r)),\ i = 1, 2.$$

We put

$$w_1 = z_1 - \widetilde{z}_1 + \widetilde{z}_2,\ w_2 = z_1 - z_1' + z_2',$$

then we get that $z_1 - w_2$ is perpendicular to $w_2 - z_2$ and parallel to y_r. Since $z_1 - z_2 = (z_1 - w_2) + (w_2 - z_2)$, we have that $|w_2 - z_2| \le |z_1 - z_2|$, $|z_1 - w_2| \le |z_1 - z_2|$ and

$$\mathrm{dist}(w_2 - z_2, \mathrm{span}\{x_1, y_r\}) = \sigma|w_2 - z_2|.$$

We apply Lemma 3.9 to get that $|\langle z_1 - w_2, e_1\rangle| \le \varepsilon|z_1 - w_2|$ and

$$|\langle w_2 - z_2, e_1\rangle| \le \left(\frac{1}{2} + \varepsilon + \sigma + \varepsilon\right) |w_2 - z_2|,$$

thus

$$|\langle z_1 - z_2, e_1\rangle| \le |\langle z_1 - w_2, e_1\rangle| + |\langle w_2 - z_2, e_1\rangle| \le \left(\frac{1}{2} + \sigma + 3\varepsilon\right) |z_1 - z_2|.$$

If $z_1 \in \partial R^{x_2} \cap \Omega_1$, $z_2 \in \partial R^{x_3} \cap \Omega_1$, we let $\widetilde{z}_i \in D_{x_i}$ and $z_i' \in \ell(y_r)$ be such that

$$|z_1 - \widetilde{z}_1| = \mathrm{dist}(z_1, D_{x_2}),\ |z_2 - \widetilde{z}_2| = \mathrm{dist}(z_2, D_{x_3})$$

and

$$|z_i - z_i'| = \mathrm{dist}(z_i, \ell(y_r)),\ i = 1, 2,$$

then $z_1' - z_2'$ is perpendicular to $z_1 - z_1'$ and $z_2 - z_2'$, and we get that $|(z_1 - z_1') - (z_2 - z_2')| \le |z_1 - z_2|$, since $z_1 - z_2 = (z_1 - z_1') - (z_2 - z_2') + (z_1' - z_2')$. We see that $|\langle z_1' - z_2', e_1\rangle| \le \varepsilon|z_1' - z_2'| \le \varepsilon|z_1 - z_2|$ and

$$|\langle z_i - z_i', e_1\rangle| \le \left(\frac{1}{2} + \varepsilon + \sigma + \varepsilon\right) |z_i - z_i'|,$$

thus

$$\begin{aligned} |\langle z_1 - z_2, e_1\rangle| &\le |\langle z_1 - z_1', e_1\rangle| + |\langle z_2 - z_2', e_1\rangle| + |\langle z_1' - z_2', e\rangle| \\ &\le \left(\frac{1}{2} + \sigma + 2\varepsilon\right) (|z_1 - z_1'| + |z_2 - z_2'|) + \varepsilon|z_1 - z_2|. \end{aligned} \tag{3.19}$$

By Lemma 3.9, we get that

$$\left\langle z_1 - z_1', \frac{x_2}{|x_2|} \right\rangle \geq (1-\sigma-\varepsilon)|z_1 - z_1'| \text{ and } \left\langle z_2 - z_2', \frac{x_3}{|x_3|} \right\rangle \geq (1-\sigma-\varepsilon)|z_2 - z_2'|,$$

and applying Lemma 3.9 again with $\langle x_2/|x_2|, x_3/|x_3|\rangle \leq -1/2 + 2\varepsilon$, we have that

$$|z_1 - z_1'| + |z_2 - z_2'| \leq 2\left(3/2 - 2\varepsilon\right)^{-1/2} |(z_1 - z_1') - (z_2 - z_2')| \leq \frac{9}{5}|z_1 - z_2|$$

We get, from (3.19), that

$$|\langle z_1 - z_2, e_1\rangle| \leq \left(\frac{9}{10} + 2\sigma + 5\varepsilon\right)|z_1 - z_2|,$$

and we proved our claim (3.18).

For any $z \in \Omega_i$, we now let $p_0(z)$ be the unique point in $\partial\Omega_i$ such that $p_0(z) - z$ parallels e_i; and for $z \in R_1$, we let $p_0(z) = z$. Then $p_0(L_0) \subseteq L_0$. We will check that

$$p_0 \text{ is Lipschitz with } \operatorname{Lip}(p_0) \leq 25.$$

Indeed, for any $z_1, z_2 \in \Omega_i$, we put

$$p_0(z_j) = z_j + t_j e_i,\ t_i \in \mathbb{R},\ j = 1, 2,$$

then

$$\begin{aligned} |t_1 - t_2| &= |\langle (t_1 - t_2)e_i, e_i\rangle| \leq |\langle p_0(z_1) - p_0(z_2), e_i\rangle| + |\langle z_1 - z_2, e_i\rangle| \\ &\leq \left(\frac{9}{10} + 5\sigma + 5\varepsilon\right)|p_0(z_1) - p_0(z_2)| + |z_1 - z_2|, \end{aligned}$$

and

$$|p_0(z_1) - p_0(z_2)| \leq |z_1 - z_2| + |t_1 - t_2| \leq \left(\frac{9}{10} + 5\sigma + 5\varepsilon\right)|p_0(z_1) - p_0(z_2)| + 2|z_1 - z_2|,$$

thus

$$|p_0(z_1) - p_0(z_2)| \leq \frac{2}{1/10 - 5(\sigma + \varepsilon)}|z_1 - z_2|.$$

By the definition of R^x and R_1^x, we have that

$$R^x = \{z \in R_1^x \mid \operatorname{dist}(z, D_x) \leq \sigma \operatorname{dist}(z, \ell(x))\}.$$

Similar as above, we will get that, for any $z_1, z_2 \in R_1^x \cap \partial R^x$ with $[z_1, z_2] \cap D_{x,y_r} = \emptyset$, if $\operatorname{card}(\mathcal{X}_r) = 2$ then

$$|\langle z_1 - z_2, e_i\rangle| \leq 10(\sigma + \varepsilon)|z_1 - z_2|;$$

if $\operatorname{card}(\mathcal{X}_r) = 3$ then

$$|\langle z_1 - z_2, e_i\rangle| \leq \left(\frac{9}{10} + 5\sigma + 5\varepsilon\right)|z_1 - z_2|,$$

where e_i is the vector in 2 with $z_1, z_2 \in \Omega_i$.

We now consider the mapping $p_1 : R_1 \to R$ defined by

$$p_1(z) = \begin{cases} z, & \text{for } z \in R, \\ z - te_i \in \partial R \cap \Omega_i, & \text{for } z \in \Omega_i. \end{cases}$$

By the same reason as above, we get that

$$\operatorname{Lip}(p_1) \leq \frac{2}{1/10 - 5\sigma - 5\varepsilon} \leq 25.$$

We define a mapping $p_2 : R\cap\overline{B(0,1)} \to \Sigma$ as follows: we see that Σ_x is the graph over D_x, thus for any $z \in R^x$, there is only one point in the intersection of Σ_x and the line which is perpendicular to D_x and through z, we let $p_2(z)$ to be the unique intersection point. That is, $p_2(z)$ is the unique point in Σ_x such that $p_2(z) - z$ is perpendicular to D_x. We will show that p_2 is Lipschitz and $\mathrm{Lip}(p_2) \leq 1 + 10^4 C\eta$. Indeed, we assume that Σ_x is the graph of founction u on D_x, then by Lemma 3.7 we have that $\mathrm{Lip}(u) \leq C\eta$. For any points $z_1, z_2 \in R^x$, we let $\widetilde{z}_i$, $i = 1, 2$, be the points in D_x such that $z_i - \widetilde{z}_i$ is perpendicular to D_x, then

$$|(p_2(z_1) - z_1) - (p_2(z_2) - z_2)| = |u(\widetilde{z}_1) - u(\widetilde{z}_2)| \leq \mathrm{Lip}(u)|\widetilde{z}_1 - \widetilde{z}_2| \leq \mathrm{Lip}(u)|z_1 - z_2|,$$

thus

$$|p_2(z_1) - p_2(z_2)| \leq (1 + \mathrm{Lip}(u))|z_1 - z_2| \leq (1 + 10^4 C\eta)|z_1 - z_2|.$$

Let $p_3 : \mathbb{R}^3 \to \mathbb{R}^3$ be the mapping defined by

$$p_3(x) = \begin{cases} x, & |x| \leq 1 \\ \frac{x}{|x|}, & |x| > 1. \end{cases}$$

Then $p = p_3 \circ p_2 \circ p_3 \circ p_1 \circ p_0$ is our desire mapping. □

Lemma 3.11. *For any $r \in (0, \mathfrak{r}) \cap \mathscr{R}_1$, we let Σ be as in* (3.10)*, and let Σ_r be given by $\boldsymbol{\mu}_r(\Sigma)$. Then we have that*

$$\mathcal{H}^2(E \cap B(0,r)) \leq \mathcal{H}^2(\Sigma_r) + C\int_{E\cap\partial B(0,r)} \mathrm{dist}(z, \Sigma_r) d\mathcal{H}^1(z) + (2r)^2 h(2r),$$

where $C > 0$ is a universal constant.

Proof. For any $\xi > 0$, we consider the function $\psi_\xi : [0, \infty) \to \mathbb{R}$ defined by

$$\psi_\xi(t) = \begin{cases} 1, & 0 \leq t \leq 1 - \xi \\ -\frac{t-1}{\xi}, & 1 - \xi < t \leq 1 \\ 0, & t > 1, \end{cases}$$

and the mapping $\phi_\xi : \Omega_0 \to \Omega_0$ defined by

$$\phi_\xi(z) = \psi_\xi(|z|)p(z) + (1 - \psi_\xi(|z|))z,$$

where p is the Lipschitz mapping considered in Lemma 3.10. We see that $\phi_\xi(L) \subseteq L$. For any $t \in [0, 1]$, we put

$$\varphi_t(z) = tr\phi_\xi(z/r) + (1 - t)z, \text{ for } z \in \Omega_0.$$

Then $\{\varphi_t\}_{0\leq t\leq 1}$ is a sliding deformation, and we get that

$$\mathcal{H}^2(E \cap \overline{B(0,r)}) \leq \mathcal{H}^2(\varphi_1(E) \cap \overline{B(0,r)}) + (2r)^2 h(2r).$$

Since $\psi_\xi(t) = 1$ for $t \in [0, 1 - \xi]$, we get that

$$\varphi_1(E \cap B(0, (1 - \xi)r)) = p(E \cap B(0, (1 - \xi)r)) \subseteq \Sigma_r.$$

We set $A_\xi = B(0, r) \setminus B(0, (1 - \xi)r)$. By Theorem 3.2.22 in [10], we get that

$$\mathcal{H}^2(\varphi_1(E \cap A_\xi)) \leq \int_{E\cap A_\xi} \mathrm{ap}\, J_2(\varphi_1|_E)(z) d\mathcal{H}^2(z). \tag{3.20}$$

For any $z \in A_\xi$ and $v \in \mathbb{R}^3$, by setting $z' = z/r$, we have that

$$D\varphi_1(z)v = \psi_\xi(|z'|)Dp(z')v + (1 - \psi_\xi(|z'|))v + \psi'_\xi(|z'|)\langle z/|z|, v\rangle(rp(z') - z).$$

For any $z\in A_\xi\cap E$, we let $v_1,v_2\in\operatorname{Tan}(E,x)$ be such that

$$|v_1|=|v_2|=1,\ v_1\perp z \text{ and } v_2\perp v_1,$$

then we have that $\langle z/|z|,v\rangle=\cos\theta(z)$, and that

$$|\psi_\xi(|z'|)Dp(z')v_i+(1-\psi_\xi(|z'|))v_i|\leq |Dp(z')v_i|\leq \operatorname{Lip}(p),$$

thus

$$\begin{aligned}\operatorname{ap}J_2(\varphi_1|_E)(z)&=|D\varphi_1(z)v_1\wedge D\varphi_1(z)v_2|\\ &\leq \operatorname{Lip}(p)^2+\frac{1}{\xi}\operatorname{Lip}(p)\cos\theta(z)|rp(z')-z|.\end{aligned}\tag{3.21}$$

Since $p(\widetilde{z})=\widetilde{z}$ for any $\widetilde{z}\in\Sigma$, we have that

$$|p(z')-z'|=|p(z')-p(\widetilde{z})+\widetilde{z}-z'|\leq(\operatorname{Lip}(p)+1)|\widetilde{z}-z'|,$$

then we get that

$$|p(z')-z'|\leq(\operatorname{Lip}(p)+1)\operatorname{dist}(z,\Sigma).$$

We now get, from (3.21), that

$$\operatorname{ap}J_2(\varphi_1|_E)(z)\leq \operatorname{Lip}(p)^2+\frac{1}{\xi}\operatorname{Lip}(p)(\operatorname{Lip}(p)+1)\operatorname{dist}(z,\Sigma_r)\cos\theta(z),$$

plug that into (3.20) to get that

$$\begin{aligned}\mathcal{H}^2(\varphi_1(E\cap A_\xi))&\leq C\mathcal{H}^2(E\cap A_\xi)+\frac{C}{\xi}\int_{E\cap A_\xi}\operatorname{dist}(z,\Sigma_r)\cos\theta(z)d\mathcal{H}^2(z)\\ &\leq C\mathcal{H}^2(E\cap A_\xi)+\frac{C}{\xi}\int_{(1-\xi)r}^{r}\int_{E\cap\partial B(0,t)}\operatorname{dist}(z,\Sigma_r)d\mathcal{H}^1(z)dt,\end{aligned}$$

we let $\xi\to0+$, then we get that, for such r,

$$\lim_{\xi\to0+}\mathcal{H}^2(\varphi_1(E\cap A_\xi))\leq Cr\int_{E\cap\partial B(0,r)}\operatorname{dist}(z,\Sigma_r)d\mathcal{H}^1(z),$$

thus

$$\mathcal{H}^2(E\cap B(0,r))\leq\mathcal{H}^2(\Sigma_r)+Cr\int_{E\cap\partial B(0,r)}\operatorname{dist}(z,\Sigma_r)d\mathcal{H}^1(z)+(2r)^2h(2r).$$

□

3.5. **The comparison statement.** For any $x,y\in\Omega_0\cap\partial B(0,1)$, if $|x-y|<2$, we denote by $g_{x,y}$ the unique geodesic on $\Omega_0\cap\partial B(0,1)$ which join x and y. We will denote by B_t the open ball $B(0,t)$ sometimes for short.

Lemma 3.12. *Let $\tau\in(0,10^{-4})$ be a given. Then there is a constant $\vartheta>0$ such that the following hold. Let $a\in\partial B(0,1)$ and $b,c\in L_0\cap\partial B(0,1)$ be such that* $\operatorname{dist}(a,(0,0,1))\leq\tau$, $\operatorname{dist}(b,(1,0,0))\leq\tau$ *and* $\operatorname{dist}(c,(-1,0,0))\leq\tau$*. Let X be the cone over $g_{a,b}\cup g_{a,c}$. Then there is a Lipschitz mapping $\varphi:\Omega_0\to\Omega_0$ with $\varphi(L_0)\subseteq L_0$, $|\varphi(z)|\leq1$ when $|z|\leq1$, and $\varphi(z)=z$ when $|z|>1$, such that*

$$\mathcal{H}^2(\varphi(X)\cap\overline{B(0,1)})\leq(1-\vartheta)\mathcal{H}^2(X\cap B(0,1))+\frac{\vartheta\pi}{2}.$$

Proof. We let b_0 a unit vector in L_0 which is perpendicular to b, and let c_0 be a unit vector in L_0 which is perpendicular to c, such that b_0+c_0 is parallel to $b+c$, and take

$$u_i = \frac{a-\langle a,i\rangle i}{|a-\langle a,i\rangle i|},\ e_i = \frac{i-\langle i,a\rangle a}{|i-\langle i,a\rangle a|},\quad \text{for } i\in\{b,c\},$$

$v_a=\lambda_a(e_b+e_c)$, $v_b=\lambda_b b_0$ and $v_c=\lambda_c c_0$, where $\lambda_j\in\mathbb{R}$, $j\in\{a,b,c\}$, will be chosen later. We let $\psi_1:\mathbb{R}\to\mathbb{R}$ be a function of class C^1 such that $0\le\psi_1\le 1$, $\psi_1(x)=0$ for $x\in(-\infty,1/4)\cup(3/4,+\infty)$, $\psi_1(x)=1$ for $x\in[2/5,3/5]$, and $|\psi_1'|\le 10$. We let $\psi_2:\mathbb{R}\to\mathbb{R}$ be a non increasing function of class C^1 such that $0\le\psi_2\le 1$, $\psi_2(x)=1$ for $x\in(-\infty,0]$, $\psi_2(x)=0$ for $x\in[1/5,+\infty)$, and $|\psi_2'|\le 10$. We let $\psi:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ be a function defined by

$$\psi(z,v)=\psi_1(\langle z,v\rangle)\psi_2(|z-\langle z,v\rangle v|). \tag{3.22}$$

We now consider the mapping $\varphi:\mathbb{R}^3\to\mathbb{R}^3$ defined by

$$\varphi(z)=z+\psi(z,a)v_a+\psi(z,b)v_b+\psi(z,c)v_c.$$

We see that $\operatorname{supp}(\psi(\cdot,a))$, $\operatorname{supp}(\psi(\cdot,b))$ and $\operatorname{supp}(\psi(\cdot,c))$ are mutually disjoint, and that

$$\overline{\{z\in\mathbb{R}^3:\varphi(z)\neq z\}}\subseteq B(0,1),\ \varphi(\Omega_0)\subseteq\Omega_0,\ \varphi(L_0)\subseteq L_0.$$

We have that

$$D\varphi(z)w=w+\langle D\psi(\cdot,a),w\rangle v_a+\langle D\psi(\cdot,b),w\rangle v_b+\langle D\psi(\cdot,c),w\rangle v_c.$$

By setting $z_v^\perp=z-\langle z,v\rangle v$ for convenient, if $w\neq 0$ and $z_v^\perp\neq 0$, we have that

$$D\psi(\cdot,v)w=\psi_1'(\langle z,v\rangle)\psi_2(|z_v^\perp|)\langle w/|w|,v\rangle+\psi_1(\langle z,v\rangle)\psi_2'(|z_v^\perp|)\langle w_v^\perp,z_v^\perp/|z_v^\perp|\rangle.$$

If w is perpendicular to v, then $w_v^\perp=w$; if w is parallel to v and $|v|=1$, then $w_v^\perp=0$. We denote by $W_j=\operatorname{supp}(\psi(\cdot,j))$ for $j\in\{a,b,c\}$. Then

$$D\psi(\cdot,v)w=\begin{cases} w, & z\notin W_a\cup W_b\cup W_c,\\ w+\langle D\psi(\cdot,v),w\rangle v_j, & z\in W_a\cup W_b\cup W_c.\end{cases}$$

But

$$\langle D\psi(\cdot,j),j\rangle=\psi_1'(\langle z,j\rangle)\psi_2(|z_j^\perp|),\ j\in\{a,b,c\},$$

$$\langle D\psi(\cdot,i),u_i\rangle=\psi_1(\langle z,i\rangle)\psi_2'(|z_i^\perp|)\langle u_i,z_i^\perp/|z_i^\perp|\rangle,\ i\in\{b,c\},$$

and

$$\langle D\psi(\cdot,a),e_i\rangle=\psi_1(\langle z,a\rangle)\psi_2'(|z_a^\perp|)\langle e_i,z_a^\perp/|z_a^\perp|\rangle,\ i\in\{b,c\},$$

by putting

$$g_j(z)=\psi_1'(\langle z,j\rangle)\psi_2(|z_j^\perp|),\ j\in\{a,b,c\},$$

$$g_{a,i}(z)=\psi_1(\langle z,a\rangle)\psi_2'(|z_a^\perp|)\langle e_i,z_a^\perp/|z_a^\perp|\rangle,\ i\in\{b,c\}$$

and

$$g_{i,i}(z)=\psi_1(\langle z,i\rangle)\psi_2'(|z_i^\perp|)\langle v_i,z_i^\perp/|z_i^\perp|\rangle,\ i\in\{b,c\},$$

and denote by X_i the cone over $g_{a,i}$, $i\in\{b,c\}$, we have that

$$D\varphi(z)a\wedge D\varphi(z)e_i=a\wedge e_i+g_a(z)v_a\wedge e_i+g_{a,i}(z)a\wedge v_a,\ z\in X_i\cap W_a$$

and

$$D\varphi(z)i\wedge D\varphi(z)u_i=i\wedge u_i+g_i(z)v_i\wedge u_i+g_{i,i}(z)i\wedge v_i,\ z\in X_i\cap W_i.$$

If $z\in X_i\cap W_a$, $i\in\{b,c\}$, we have that

$$\begin{aligned}J_2\varphi|_X(z)&=\|D\varphi(z)a\wedge D\varphi(z)e_i\|\\&\leq 1+\langle a\wedge e_i,g_a(z)v_a\wedge e_i+g_{a,i}(z)a\wedge v_a\rangle+\frac{1}{2}\|g_a(z)v_a\wedge e_i+g_{a,i}(z)a\wedge v_a\|^2\\&=1+g_a(z)\langle a,v_a\rangle+g_{a,i}(z)\langle e_i,v_a\rangle+\frac{1}{2}\left(g_a(z)^2\|v_a\wedge e_i\|^2+g_{a,i}(z)^2|v_a|^2\right)\\&\leq 1+g_{a,i}(z)\langle e_i,v_a\rangle+100|v_a|^2.\end{aligned}$$

Similarly, we have that, for $z\in X_i\cap W_i$,

$$J_2\varphi|_X(z)=\|D\varphi(z)i\wedge D\varphi(z)u_i\|\leq 1+g_{i,i}(z)\langle u_i,v_i\rangle+100|v_i|^2.$$

We see that $z_a^{\perp}/|z_a^{\perp}|=e_i$ when $z\in X_i\setminus\operatorname{span}\{a\}$, and $z_i^{\perp}/|z_i^{\perp}|=u_i$ in case $z\in X_i\setminus\operatorname{span}\{i\}$, thus

$$g_{a,i}(z)=\psi_1(\langle z,a\rangle)\psi_2'(|z_a^{\perp}|)\text{ and }g_{i,i}(z)=\psi_1(\langle z,i\rangle)\psi_2'(|z_i^{\perp}|).$$

Hence, for $j=a$ or i, we have that

$$\begin{aligned}\int_{z\in X_i\cap W_j}g_{j,i}(z)d\mathcal{H}^2(z)&=\int_{z\in X_i\cap W_j}\psi_1(\langle z,j\rangle)\psi_2'(|z_j^{\perp}|)d\mathcal{H}^2(z)\\&=\int_0^{+\infty}\int_0^{+\infty}\psi_1(t)\psi_2'(s)dtds=-\int_0^{+\infty}\psi_1(t)dt<-\frac{1}{5},\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{H}^2(\varphi(X\cap B_1))&=\int_{z\in X\cap B(0,1)}J_2\varphi|_X(z)d\mathcal{H}^2(z)\\&\leq(1+100\sum_j|v_j|^2)\mathcal{H}^2(X\cap B_1)-\frac{1}{5}(\langle v_a,e_b+e_c\rangle+\sum_i\langle u_i,v_i\rangle)\end{aligned}$$

If we take $\lambda_a=10^{-3}\mathcal{H}^2(X\cap B_1)^{-1}$ and $\lambda_i=10^{-3}\mathcal{H}^2(X\cap B_1)^{-1}\langle u_i,i_0\rangle$, $i\in\{b,c\}$, then

$$\mathcal{H}^2(\varphi(X\cap B_1))\leq\mathcal{H}^2(X\cap B_1)-10^{-4}(|e_b+e_c|^2+\langle u_b,b_0\rangle^2+\langle u_c,c_0\rangle^2).$$

Since $|\langle a,w\rangle|\leq\tau|w|$ for $w\in L_0$, and $-1\leq\langle b,c\rangle\leq -1+2\tau^2$, we get that

$$\begin{aligned}|e_b+e_c|^2=2(1+\langle e_b,e_c\rangle)&=\frac{2}{1-\langle e_b,e_c\rangle}(1-\langle e_b,e_c\rangle^2)\\&\geq 1-\frac{(\langle b,c\rangle-\langle a,b\rangle\langle a,c\rangle)^2}{(1-\langle a,b\rangle^2)(1-\langle a,c\rangle^2)}\\&\geq 1-\langle a,b\rangle^2-\langle a,c\rangle^2-\langle b,c\rangle^2+2\langle a,b\rangle\langle b,c\rangle\langle c,a\rangle\\&=(1-\langle b,c\rangle+2\langle a,b\rangle\langle a,c\rangle)(1+\langle b,c\rangle)-\langle a,b+c\rangle^2\\&\geq(1-3\tau^2)|b+c|^2.\end{aligned}$$

Since $\arcsin x=x+\sum_{n\geq 1}C_nx^{2n+1}$ for $|x|\leq 1$, where $C_n=\frac{(2n)!}{4^n(n!)^2(2n+1)}$, we have that

$$\begin{aligned}\mathcal{H}^2(X\cap B_1)-\frac{\pi}{2}&=\frac{1}{2}(\arccos\langle a,b\rangle+\arccos\langle a,c\rangle)-\frac{\pi}{2}\\&=-\frac{1}{2}(\arcsin\langle a,b\rangle+\arcsin\langle a,c\rangle)\leq\frac{1}{2}(1+\tau)|\langle a,b+c\rangle|.\end{aligned}$$

If $b+c\neq 0$, then $|b_0+c_0|\geq 1$, and we have that

$$\left\langle a, \frac{b+c}{|b+c|}\right\rangle^2 = \left\langle a, \frac{b_0+c_0}{|b_0+c_0|}\right\rangle^2 \leq 2\left(\langle a,b_0\rangle^2+\langle a,c_0\rangle^2\right).$$

We get so that in any case

$$|\langle a, b+c\rangle| \leq \frac{1}{2}\left(|b+c|^2+2\langle a,b_0\rangle^2+2\langle a,c_0\rangle^2\right).$$

Since

$$\langle u_b,b_0\rangle^2+\langle u_c,c_0\rangle^2 = \frac{\langle a,b_0\rangle^2}{1-\langle a,b\rangle^2}+\frac{\langle a,c_0\rangle^2}{1-\langle a,c\rangle^2} \geq \langle a,b_0\rangle^2+\langle a,c_0\rangle^2,$$

we get that

$$\begin{aligned}\mathcal{H}^2(\varphi(X\cap B_1)) &\leq \mathcal{H}^2(X\cap B_1) - 10^{-4}\left(\frac{1}{2}|b+c|^2+\langle a,b_0\rangle^2+\langle a,c_0\rangle^2\right)\\ &\leq \mathcal{H}^2(X\cap B_1) - 10^{-4}\left(\mathcal{H}^2(X\cap B_1)-\frac{\pi}{2}\right).\end{aligned}$$

□

Lemma 3.13. *Let* $\tau\in(0,10^{-4})$ *be a given. Then there is a constant* $\vartheta>0$ *such that the following hold. Let* $a\in\partial B(0,1)$ *and* $b,c,d\in L_0\cap\partial B(0,1)$ *be such that* $\operatorname{dist}(a,(0,0,1))\leq\tau$, $\operatorname{dist}(b,(-1/2,\sqrt{3}/2,0))\leq\tau$, $\operatorname{dist}(c,(-1/2,-\sqrt{3}/2,0))\leq\tau$ *and* $\operatorname{dist}(d,(1,0,0))\leq\tau$. *Let* X *be the cone over* $g_{a,b}\cup g_{a,c}\cup g_{a,d}$. *Then there is a Lipschitz mapping* $\varphi:\Omega_0\to\Omega_0$ *with* $\varphi(E\cap L)\subseteq L$, $|\varphi(z)|\leq 1$ *when* $|z|\leq 1$, *and* $\varphi(z)=z$ *when* $|z|>1$, *such that*

$$\mathcal{H}^2(\varphi(X)\cap\overline{B(0,1)}) \leq (1-\vartheta)\mathcal{H}^2(X\cap B(0,1))+\vartheta\frac{3\pi}{4}.$$

Proof. We let b_0, c_0 and d_0 be unit vectors in L_0 such that

$$b_0\perp b, c_0\perp c, d_0\perp d.$$

For $i\in\{b,c,d\}$, we put

$$u_i=\frac{a-\langle a,i\rangle i}{|a-\langle a,i\rangle i|},\ e_i=\frac{i-\langle i,a\rangle a}{|i-\langle i,a\rangle a|}.$$

We take $v_a=\lambda_a(e_b+e_c+e_d)$ and $v_i=\lambda_i i_0$, where $\lambda_i>0$, $i\in\{b,c,d\}$, will be chosen later. We let ψ be the same as in (3.22), and consider the mapping $\varphi:\mathbb{R}^3\to\mathbb{R}^3$ defined by

$$\varphi(z)=z+\psi(z,a)v_a+\psi(z,b)v_b+\psi(z,c)v_c+\psi(z,d)v_d.$$

We see that $\operatorname{supp}(\psi(\cdot,a))$, $\operatorname{supp}(\psi(\cdot,b))$, $\operatorname{supp}(\psi(\cdot,c))$ and $\operatorname{supp}(\psi(\cdot,d))$ are mutually disjoint, and that

$$\overline{\{z\in\mathbb{R}^3:\varphi(z)\neq z\}}\subseteq B(0,1),\ \varphi(\Omega_0)\subseteq\Omega_0,\ \varphi(L_0)\subseteq L_0.$$

By putting $W_j=\operatorname{supp}(\psi(\cdot,j))$ for $j\in\{a,b,c,d\}$, we have that

$$D\psi(\cdot,v)w=\begin{cases} w, & z\notin W_a\cup W_b\cup W_c\cup W_d,\\ w+\langle D\psi(\cdot,v),w\rangle v_j, & z\in W_a\cup W_b\cup W_c\cup W_d,\end{cases}$$

and

$$\langle D\psi(\cdot,j),j\rangle=\psi_1'(\langle z,j\rangle)\psi_2(|z_j^\perp|),\ j\in\{a,b,c,d\},$$
$$\langle D\psi(\cdot,i),u_i\rangle=\psi_1(\langle z,i\rangle)\psi_2'(|z_i^\perp|)\langle u_i,z_i^\perp/|z_i^\perp|\rangle,$$
$$\langle D\psi(\cdot,a),e_i\rangle=\psi_1(\langle z,a\rangle)\psi_2'(|z_a^\perp|)\langle e_i,z_a^\perp/|z_a^\perp|\rangle,\ i\in\{b,c,d\},$$

where $z_w=z-\langle z,w\rangle w$. By putting

$$g_j(z)=\psi_1'(\langle z,j\rangle)\psi_2(|z_j^\perp|),\ j\in\{a,b,c,d\},$$
$$g_{a,i}(z)=\psi_1(\langle z,a\rangle)\psi_2'(|z_a^\perp|)\langle e_i,z_a^\perp/|z_a^\perp|\rangle,$$
$$g_{i,i}(z)=\psi_1(\langle z,i\rangle)\psi_2'(|z_i^\perp|)\langle v_i,z_i^\perp/|z_i^\perp|\rangle,\ i\in\{b,c,d\},$$

and denote by X_i the cone over $g_{a,i}$, $i\in\{b,c,d\}$, we have that

$$D\varphi(z)a\wedge D\varphi(z)e_i=a\wedge e_i+g_a(z)v_a\wedge e_i+g_{a,i}(z)a\wedge v_a,\ z\in X_i\cap W_a,$$
$$D\varphi(z)i\wedge D\varphi(z)u_i=i\wedge u_i+g_i(z)v_i\wedge u_i+g_{i,i}(z)i\wedge v_i,\ z\in X_i\cap W_i.$$

We have that, for $i\in\{b,c,d\}$,

$$J_2\varphi|_X(z)=\|D\varphi(z)a\wedge D\varphi(z)e_i\|\le 1+g_{a,i}(z)\langle e_i,v_a\rangle+100|v_a|^2,z\in X_i\cap W_a,$$
$$J_2\varphi|_X(z)=\|D\varphi(z)i\wedge D\varphi(z)u_i\|\le 1+g_{i,i}(z)\langle u_i,v_i\rangle+100|v_i|^2,z\in X_i\cap W_i.$$

Since $z_a^\perp/|z_a^\perp|=e_i$ when $z\in X_i\setminus\operatorname{span}\{a\}$, and $z_i^\perp/|z_i^\perp|=u_i$ in case $z\in X_i\setminus\operatorname{span}\{i\}$, we have that

$$g_{a,i}(z)=\psi_1(\langle z,a\rangle)\psi_2'(|z_a^\perp|)\text{ and }g_{i,i}(z)=\psi_1(\langle z,i\rangle)\psi_2'(|z_i^\perp|).$$

Thus, for $j=a$ or i,

$$\int_{z\in X_i\cap W_j}g_{j,i}(z)d\mathcal{H}^2(z)=-\int_0^{+\infty}\psi_1(t)dt<-\frac{1}{5}.$$

Hence

$$\begin{aligned}\mathcal{H}^2(\varphi(X\cap B_1))&=\int_{z\in X\cap B_1}J_2\varphi|_X(z)d\mathcal{H}^2(z)\\&\le\left(1+100(|v_a|^2+|v_b|^2+|v_c|^2+|v_d|^2)\right)\mathcal{H}^2(X\cap B_1)\\&\quad-\frac{1}{5}\left(\langle v_a,e_b+e_c+e_d\rangle+\langle u_b,v_b\rangle+\langle u_c,v_c\rangle+\langle u_d,v_d\rangle\right).\end{aligned}$$

If we take $\lambda_a=10^{-3}\mathcal{H}^2(X\cap B_1)^{-1}$ and $\lambda_i=10^{-3}\mathcal{H}^2(X\cap B_1)^{-1}\langle u_i,i_0\rangle$, $i\in\{b,c,d\}$, then

$$\mathcal{H}^2(\varphi(X\cap B_1))\le\mathcal{H}^2(X\cap B_1)-10^{-4}\left(|e_b+e_c+e_d|^2+\sum_i\langle u_i,i_0\rangle^2\right).$$

Since $|\langle a,w\rangle|\le\tau|w|$, for $w\in L_0$, and $-1/2-\sqrt{3}\tau\le\langle i_1,i_2\rangle\le-1/2+\sqrt{3}\tau$, $i_1,i_2\in\{b,c,d\}$, $i_1\ne i_2$, we get that $\langle i,j\rangle-\langle a,i\rangle\langle a,j\rangle<0$. By putting $e=(0,0,1)$, it is evident that

$$\langle a,w\rangle^2\le 1-\langle a,e\rangle^2,\ \text{for any }w\in L_0\text{ with }|w|=1.$$

We put $N=\langle a,b\rangle^2+\langle a,c\rangle^2+\langle a,d\rangle^2$, and we claim that

$$N\le(3/2+25\tau)\left(1-\langle a,e\rangle^2\right).\tag{3.23}$$

Indeed, for any $w = \lambda b + \mu c$ with $\lambda, \mu \geq 0$, we have that

$$|w|^2 = \lambda^2 + \mu^2 + 2\lambda\mu\langle b, c\rangle \geq \lambda^2 + \mu^2 - (1+4\tau)\lambda\mu,$$
$$\langle w, d\rangle^2 \leq (1/2 + \sqrt{3}\tau)^2(\lambda+\mu)^2 \leq (1/4 + 2\tau)(\lambda+\mu)^2$$

and

$$\begin{aligned}\langle w, b\rangle^2 + \langle w, b\rangle^2 + \langle w, b\rangle^2 &= (\lambda^2+\mu^2)(1+\langle b,c\rangle^2) + 4\lambda\mu\langle b, c\rangle + \langle w, d\rangle^2 \\ &\leq (3/2+4\tau)\left(\lambda^2+\mu^2\right) - (3/2 - 10\tau)\lambda\mu \\ &\leq (3/2 + 25\tau)|w|^2.\end{aligned}$$

Hence, for any $w \in L_0$, we have that

$$\langle w, b\rangle^2 + \langle w, b\rangle^2 + \langle w, b\rangle^2 \leq (3/2 + 25\tau)|w|^2,$$

we now take $w = a - \langle a, e\rangle e$, then

$$N \leq (3/2 + 25\tau)|a - \langle a, e\rangle e|^2 = (3/2+25\tau)(1 - \langle a, e\rangle^2),$$

the claim (3.23) follows.

Since $(1-x)^{1/2} \leq 1 - x/2 - x^2/8$ for any $x \in (0,1)$, and

$$(1 - \langle a, b\rangle^2)(1 - \langle a, c\rangle^2)(1 - \langle a, d\rangle^2) \geq 1 - N,$$

we have that, for $\{i, j, k\} = \{b, c, d\}$,

$$\begin{aligned}\langle e_i, e_j\rangle =& \frac{\langle i, j\rangle - \langle a, i\rangle\langle a, j\rangle}{(1-\langle a, i\rangle^2)^{1/2}(1-\langle a, j\rangle^2)^{1/2}} \\ &\geq \frac{(\langle i, j\rangle - \langle a, i\rangle\langle a, j\rangle)(1 - \langle a, k\rangle^2/2 - \langle a, k\rangle^4/8)}{(1-N)^{1/2}}.\end{aligned}$$

Note that

$$\langle a, b\rangle^4 + \langle a, c\rangle^4 + \langle a, d\rangle^4 \geq N^2/3,$$

and

$$|\langle a, b+c+d\rangle| \leq \frac{1}{2}\left(|b+c+d|^2 + 1 - \langle a, e\rangle^2\right),$$

we get so that

$$\begin{aligned}|e_b + e_c + e_d|^2 \geq 3 + (1-N)^{-1/2}\Big(& -3 + (3/2 - \sqrt{3}\tau)N + \frac{1}{12}(1/2 - \sqrt{3}\tau)N^2 \\ &+ |b+c+d|^2 - \langle a, b+c+d\rangle^2 + \langle a, b\rangle\langle a, c\rangle\langle a, d\rangle\langle a, b+c+d\rangle \\ &+ \frac{1}{4}\langle a, b\rangle\langle a, c\rangle\langle a, d\rangle\left(\langle a, b\rangle^3 + \langle a, c\rangle^3 + \langle a, d\rangle^3\right)\Big) \\ \geq (1-N)^{-1/2}&\left((1-\tau^2)|b+c+d|^2 - 2\tau N - 2\tau^3|\langle a, b+c+d\rangle|\right) \\ \geq (1-\tau)&|b+c+d|^2 - 6\tau(1 - \langle a, e\rangle^2).\end{aligned}$$

Since $1/(1-x) = 1 + x + x^2/(1-x)$ for $x \in [0,1)$, and $\langle a, i\rangle^2 \leq 1 - \langle a, e\rangle^2$ for $i \in \{b, c, d\}$, we have that

$$\frac{\langle a, e\rangle^2}{1 - \langle a, i\rangle^2} = \langle a, e\rangle^2 + \frac{\langle a, e\rangle^2\langle a, i\rangle^2}{1 - \langle a, i\rangle^2} \leq \langle a, e\rangle^2 + \langle a, i\rangle^2$$

and

$$\begin{aligned}\langle u_b, b_0\rangle^2 + \langle u_c, c_0\rangle^2 + \langle u_d, d_0\rangle^2 &= \sum_{i\in\{b,c,d\}} \frac{1 - \langle a, e\rangle^2 - \langle a, i\rangle^2}{1 - \langle a, i\rangle^2} \\ &= 3(1 - \langle a, e\rangle^2) - N \geq (1-\tau)(1 - \langle a, e\rangle^2).\end{aligned}$$

We get so that

$$\mathcal{H}^2(\varphi(X\cap B_1)) \leq \mathcal{H}^2(X\cap B_1) - 10^{-4}(1-10\tau)\left(|b+c+d|^2 + 1 - \langle a,e\rangle^2\right)$$

Since $\arcsin x = x + \sum_{n\geq 1} C_n x^{2n+1}$ for $|x|\leq 1$, where $C_n = \frac{(2n)!}{4^n(n!)^2(2n+1)}$, we have that $\arcsin\langle a,i\rangle \geq \langle a,i\rangle - \tau\langle a,i\rangle^2$, thus

$$\begin{aligned}\mathcal{H}^2(X\cap B_1) - \frac{3\pi}{4} &= -\frac{1}{2}\left(\arcsin\langle a,b\rangle + \arcsin\langle a,c\rangle + \arcsin\langle a,c\rangle\right)\\ &\leq -\frac{1}{2}\langle a, b+c+d\rangle + \frac{\tau}{2}N\\ &\leq \frac{1}{2}\left(|b+c+d|^2 + 1 - \langle a,e\rangle^2\right) + \tau\left(1-\langle a,e\rangle^2\right).\end{aligned}$$

Thus

$$\mathcal{H}^2(\varphi(X\cap B_1)) \leq (1-10^{-4})\mathcal{H}^2(X\cap B_1) - 10^{-4}\cdot\frac{3\pi}{4}.$$

□

Let $E\subseteq\Omega_0$ be a 2-rectifiable set satisfying **(a)**, **(b)** and **(c)**. We will denote by $\mathscr{R}_2$ the set $\left\{r\in\mathscr{R}_1 : 10C(1+C\eta^{-2})\left(\varepsilon(r)+j(r)^{1/2}\right)\leq 1/2-10^{-4}\right\}$, where we take constant C to be the maximum value of the constants in Lemma 3.6 and Lemma 3.11.

Lemma 3.14. *For any $r\in(0,\mathfrak{r})\cap\mathscr{R}_2$, we have that*

$$\begin{aligned}\mathcal{H}^2(E\cap B_r) \leq (1-2\cdot 10^{-4})\frac{r}{2}\mathcal{H}^1(E\cap\partial B_r) + (2\cdot 10^{-4} - \vartheta\kappa^2)\frac{r^2}{2}\mathcal{H}^1(X\cap\partial B_1)\\ + \vartheta\kappa^2 r^2\Theta(0) + (2r)^2 h(2r).\end{aligned}$$

Proof. Let Σ, Σ_r, ξ, ψ_ξ, ϕ_ξ and $\{\varphi_t\}_{0\leq t\leq 1}$ be the same as in the proof of Lemma 3.11. We see that

$$\varphi_1(E\cap B(0,(1-\xi)r)) = p(E\cap B(0,(1-\xi)r)) \subseteq \Sigma_r,$$

and that $\Sigma\cap B(0,2\kappa) = X\cap B(0,2\kappa)$, where X is a cone defined in (3.10). We see that if $\Theta(0)=\pi/2$, then X satisfies the conditions in Lemma 3.12; if $\Theta(0)=3\pi/4$, then X satisfies the conditions in Lemma 3.13. Thus we can find a Lipschitz mapping $\Omega_0\to\Omega_0$ with $\varphi(E\cap L)\subseteq L$, $|\varphi(z)|\leq 1$ when $|z|\leq 1$, and $\varphi(z)=z$ when $|z|>1$, such that

$$\mathcal{H}^2\left(\varphi(X)\cap\overline{B(0,1)}\right) \leq (1-\vartheta)\mathcal{H}^2(X\cap B(0,1)) + \vartheta\Theta(x).$$

Let $\widetilde{\varphi}:\Omega_0\to\Omega_0$ be the mapping defined by $\widetilde{\varphi}(x) = r\varphi(x/r)$, then

$$\begin{aligned}\mathcal{H}^2(E\cap B(0,r)) &\leq \mathcal{H}^2(\widetilde{\varphi}\circ\varphi_1(E)\cap\overline{B(0,r)}) + (2r)^2h(2r)\\ &\leq \mathcal{H}^2(\widetilde{\varphi}\circ\varphi_1(E\cap B(0,(1-\xi)r))) + \mathcal{H}^2(\varphi_1(E\cap A_\xi))\\ &\leq \mathcal{H}^2(\Sigma_r\setminus\overline{B(0,\kappa r)}) + (1-\vartheta)(\kappa r)^2\mathcal{H}^2(X\cap B(0,1))\\ &\quad + \vartheta\cdot(\kappa r)^2\Theta(0) + \mathcal{H}^2(\varphi_1(E\cap A_\xi)).\end{aligned}$$

But we see that $\Sigma_r = \{rx : x\in\Sigma\}$, $\Sigma\cap B(0,2\kappa) = X\cap B(0,2\kappa)$, and

$$\lim_{\xi\to 0+}\mathcal{H}^2(\varphi_1(E\cap A_\xi)) \leq C\int_{E\cap\partial B(0,r)}\operatorname{dist}(z,\Sigma_r)\,d\mathcal{H}^1(z),$$

we get so that

$$\mathcal{H}^2(\Sigma_r \setminus \overline{B(0,\kappa r)}) = r^2 \left(\mathcal{H}^2(\Sigma) - \mathcal{H}^2(X \cap B(0,\kappa))\right),$$

and

$$\begin{aligned}\mathcal{H}^2(E \cap B(0,r)) \leq r^2\mathcal{H}^2(\Sigma) - (\kappa r)^2\mathcal{H}^2(X \cap B(0,1)) \\ + (1-\vartheta)(\kappa r)^2\mathcal{H}^2(X \cap B(0,1)) + (\kappa r)^2\vartheta \cdot \Theta(0) \\ + C\int_{E\cap\partial B(0,r)} \operatorname{dist}(z,\Sigma_r) d\mathcal{H}^1(z) + (2r)^2 h(2r).\end{aligned}$$

Since $\mathcal{M}$ is the cone over Γ_*, and X is the cone over $\mathfrak{C}$, by (3.11), we get that

$$\begin{aligned}\mathcal{H}^2(\Sigma) &\leq \mathcal{H}^2(\mathcal{M} \cap B(0,1)) - 10^{-4}\Big(\mathcal{H}^1(\Gamma_*) - \mathcal{H}^1(\mathfrak{C})\Big) \\ &= (1/2 - 10^{-4})\mathcal{H}^1(\Gamma_*) + 10^{-4}\mathcal{H}^1(\mathfrak{C}),\end{aligned}$$

and then

$$\begin{aligned}\mathcal{H}^2(E \cap B_r) \leq (1/2 - 10^{-4})r^2\mathcal{H}^1(\Gamma_*) + (10^{-4} - \vartheta\kappa^2/2)r^2\mathcal{H}^1(\mathfrak{C}) \\ + \vartheta\kappa^2 r^2\Theta(0) + C\int_{E\cap\partial B_r} \operatorname{dist}(z,\Sigma_r)d\mathcal{H}^1(z) + (2r)^2h(2r).\end{aligned} \tag{3.24}$$

By (3.13) and Lemma 3.8, we have that

$$d_{0,r}(E,\mathcal{M}) \leq d_{0,r}(E,X) + d_{0,r}(X,\mathcal{M}) \leq 5\varepsilon(r) + 10j(r)^{1/2},$$

thus for any $z \in E \cap \partial B(0,r)$,

$$\operatorname{dist}(\boldsymbol{\mu}_{1/r}(z),\mathcal{M}) = r^{-1}\operatorname{dist}(z,\mathcal{M}) \leq 5\varepsilon(r) + 10j(r)^{1/2}.$$

Since $\Sigma \setminus B(0,1-2\kappa) = \mathcal{M} \cap \overline{B(0,1)} \setminus B(0,1-2\kappa)$, we have that

$$\operatorname{dist}(z,\Sigma_r) = r\operatorname{dist}(\boldsymbol{\mu}_{1/r}(z),\Sigma) = r\operatorname{dist}(\boldsymbol{\mu}_{1/r}(z),\mathcal{M}) \leq 5r\varepsilon(r) + 10rj(r)^{1/2},$$

and we get so that

$$\int_{E\cap\partial B(0,r)} \operatorname{dist}(z,\Sigma_r)d\mathcal{H}^1(z) \leq r\big(5\varepsilon(r) + 10j(r)^{1/2}\big)\mathcal{H}^1\big((E\cap\partial B_r)\setminus(\Sigma_r\cap\partial B_r)\big). \tag{3.25}$$

By Lemma 3.6, we have that

$$\mathcal{H}^1(\Gamma_* \setminus \Gamma) \leq \mathcal{H}^1(\Gamma \setminus \Gamma_*) \leq C\eta^{-2}(\mathcal{H}^1(\Gamma) - \mathcal{H}^1(\mathfrak{C})),$$

thus

$$\mathcal{H}^1(\mathfrak{C}) \leq \mathcal{H}^1(\Gamma_*) \leq \mathcal{H}^1(\Gamma) \leq \mathcal{H}^1(\boldsymbol{\mu}_{1/r}(E\cap\partial B_r)).$$

Since $\boldsymbol{\mu}_{1/r}(\Gamma) \subseteq E \cap \partial B_r$ and $\Sigma_r \cap \partial B_r = \boldsymbol{\mu}_{1/r}(\Sigma\cap\partial B_1) = \boldsymbol{\mu}_{1/r}(\Gamma_*)$, by setting $\Gamma_r = \boldsymbol{\mu}_{1/r}(\Gamma)$ and $\Gamma_{r,*} = \boldsymbol{\mu}_{1/r}(\Gamma_*)$, we have that

$$\begin{aligned}\mathcal{H}^1\big((E\cap\partial B_r)\setminus(\Sigma_r\cap\partial B_r)\big) &\leq \mathcal{H}^1\big((E\cap\partial B_r)\setminus\Gamma_r\big) + \mathcal{H}^1\big(\Gamma_r\setminus\Gamma_{r,*}\big) \\ &\leq \mathcal{H}^1(E\cap\partial B_r) - \mathcal{H}^1(\Gamma_r) + C\eta^{-2}r(\mathcal{H}^1(\Gamma) - \mathcal{H}^1(\mathfrak{C})) \\ &\leq (1+C\eta^{-2})(\mathcal{H}^1(E\cap\partial B_r) - r\mathcal{H}^1(\mathfrak{C})).\end{aligned} \tag{3.26}$$

We obtain, from (3.24), (3.25) and (3.26), that

$$\begin{aligned}\mathcal{H}^2(E\cap B_r) \leq (1/2 - 10^{-4})r^2\mathcal{H}^1(\Gamma_*) + (10^{-4} - \vartheta\kappa^2/2)r^2\mathcal{H}^1(\mathfrak{C}) \\ + 10C(1+C\eta^{-2})(\varepsilon(r) + j(r)^{1/2})r(\mathcal{H}^1(E\cap\partial B_r) - r\mathcal{H}^1(\Gamma_*)) \\ + \vartheta\kappa^2 r^2\Theta(0) + (2r)^2h(2r).\end{aligned}$$

Since $r \in (0, \mathfrak{r}) \cap \mathscr{R}_2$, we have that $10C(1 + C\eta^{-2})\left(\varepsilon(r) + j(r)^{1/2}\right) \le 1/2 - 10^{-4}$, thus

$$\begin{aligned}\mathcal{H}^2(E \cap B_r) \le (1 - 2 \cdot 10^{-4})\frac{r}{2}\mathcal{H}^1(E \cap \partial B_r) + (2 \cdot 10^{-4} - \vartheta\kappa^2)\frac{r^2}{2}\mathcal{H}^1(\mathfrak{C}) \\ + \vartheta\kappa^2 r^2 \Theta(0) + (2r)^2 h(2r).\end{aligned}$$

□

Theorem 3.15. *There exist $\lambda, \mu \in (0, 10^{-3})$ and $\mathfrak{r}_1 > 0$ such that, for any $0 < r < \mathfrak{r}_1$,*

$$\mathcal{H}^2(E \cap B_r) \le (1 - \mu - \lambda)\frac{r}{2}\mathcal{H}^1(E \cap \partial B_r) + \mu\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1) + \lambda\Theta(0)r^2 + 4r^2 h(2r).$$

Proof. Recall that $\mathscr{R}_1 = \{r \in (0, \mathfrak{r}) \cap \mathscr{R} : j(r) \le \tau_0\}$ and

$$\mathscr{R}_2 = \left\{r \in \mathscr{R}_1 : 10C(1 + C\eta^{-2})\left(\varepsilon(r) + j(r)^{1/2}\right) \le 1/2 - 10^{-4}\right\}.$$

We put $\tau_1 = \min\{\tau_0, (100C(1 + C\eta^{-2}))^{-2}\}$, and take δ such that

$$\kappa < \delta < \kappa + (10\Theta(0)\vartheta)^{-1}(1 - 2 \cdot 10^{-4})\tau_1. \tag{3.27}$$

We see that $\varepsilon(r) \to 0$ as $r \to 0+$, there exist $\mathfrak{r}_2 \in (0, \mathfrak{r})$ such that, for any $r \in (0, \mathfrak{r}_2)$,

$$\varepsilon(r) \le 10^{-2} \min\{\tau_1, \vartheta(\delta^2 - \kappa^2)\}. \tag{3.28}$$

If $r \in (0, \mathfrak{r}_2)$ and $j(r) \le \tau_1$, then $r \in \mathscr{R}_2$, then by Lemma 3.14, we have that

$$\begin{aligned}\mathcal{H}^2(E \cap B_r) \le (1 - 2 \cdot 10^{-4})\frac{r}{2}\mathcal{H}^1(E \cap \partial B_r) + (2 \cdot 10^{-4} - \vartheta\kappa^2)\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1) \\ + \vartheta\kappa^2 r^2 \Theta(0) + (2r)^2 h(2r).\end{aligned}$$

We only need to consider the case $r \in (0, \mathfrak{r}_2)$, $j(r) > \tau_1$ and $\mathcal{H}^1(E \cap \partial B_r) < +\infty$, thus

$$\mathcal{H}^1(X \cap \partial B_1) + \tau_1 \le \frac{1}{r}\mathcal{H}^1(E \cap \partial B_r). \tag{3.29}$$

By the construction of X, we see that $X \cap B(0,1)$ is local Lipschitz neighborhood retract, let U be a neighborhood of $X \cap B(0,1)$ and $\varphi_0 : U \to X \cap B(0,1)$ be a retraction such that $|\varphi_0(x) - x| \le r/2$. We put $U_1 = \boldsymbol{\mu}_{8r/9}(U)$, $\varphi_1 = \boldsymbol{\mu}_{8r/9} \circ \varphi_0 \circ \boldsymbol{\mu}_{9/(8r)}$, and let $s : [0, \infty) \to [0,1]$ be a function given by

$$s(t) = \begin{cases} 1, & 0 \le t \le 3r/4, \\ -(8/r)(t - 7r/8), & 3r/4 < t \le 7r/8, \\ 0, & t > 7r/8. \end{cases}$$

We see, from Lemma 3.8, that there exist sliding minimal cone Z such that $d_{0,r}(E, X) \le 5\varepsilon(r)$, then for any $x \in E \cap B(0,r) \setminus B(0, 3r/4)$,

$$\operatorname{dist}(x, X) \le 5\varepsilon(r)r \le \frac{20\varepsilon(r)}{3}|x| \le 7\varepsilon(r)|x|.$$

We consider the mapping $\psi : \Omega_0 \to \Omega_0$ defined by

$$\psi(x) = s(|x|)\varphi_1(x) + (1 - s(|x|))x,$$

then $\psi(L) = L$ and $\psi(x) = x$ for $|x| \ge 8r/9$.

Since $\varepsilon(r) \to 0$ and U is a neighborhood of $X \cap B(0,1)$, we can find $\mathfrak{r}_1 \in (0, \mathfrak{r}_2)$ such that, for any $r \in (0, \mathfrak{r}_1)$, $\{x \in \Omega_0 \cap B(0,1) : \operatorname{dist}(x, X) \leq 7\varepsilon(r)\} \subseteq U$. Then we get that $\psi(x) \in X$ for any $x \in E \cap B(0, 3r/4)$;

$$\operatorname{dist}(\psi(x), X) \leq 7\varepsilon(r)|x| \text{ for any } x \in E \cap B(0,r) \setminus B(0, 3r/4);$$

and $\psi(E \cap B_r) \cap B(0, r/4) = X \cap B(0, r/4)$. We now consider the mapping $\Pi_1 : \Omega_0 \to \Omega_0$ defined by

$$\Pi_1(x) = s(4|x|)x + (1 - s(4|x|))\frac{x}{|x|},$$

and the mapping $\psi_1 : \Omega_0 \to \Omega_0$ defined by

$$\psi_1(x) = \begin{cases} \Pi_1 \circ \psi(x), & |x| \leq r, \\ x, & |x| \geq r. \end{cases}$$

We have that ψ_1 is Lipschitz, $\psi_1(L_0) = L_0$ and $\psi_1(B(0,r)) \subseteq \overline{B(0,r)}$,

$$\psi_1(E \cap B(0,r)) \subseteq (X \cap B(0,r)) \cup \{x \in \partial B_r : \operatorname{dist}(x, X) \leq 7r\varepsilon(r)\}.$$

Let φ be the same as in Lemma 3.12 and Lemma 3.13, and let $\psi_2 = \boldsymbol{\mu}_\delta \circ \varphi \circ \boldsymbol{\mu}_{1/\delta} \circ \psi_1$. Then we have that

(3.30)

$$\begin{aligned}
\mathcal{H}^2(E \cap \overline{B(0,r)}) &\leq \mathcal{H}^2\left(\psi_2\left(E \cap \overline{B(0,r)}\right)\right) + (2r)^2 h(2r) \\
&\leq (1 - \vartheta\delta^2)\mathcal{H}^2(X \cap B(0,r)) + \vartheta\delta^2\Theta(0)r^2 \\
&\quad + \mathcal{H}^2(\{x \in \partial B_r : \operatorname{dist}(x, X) \leq 7r\varepsilon(r)\}) + 4r^2 h(2r) \\
&\leq (1 - \vartheta\delta^2)\mathcal{H}^2(X \cap B(0,r)) + \vartheta\delta^2\Theta(0)r^2 \\
&\quad + 8r\varepsilon(r)\mathcal{H}^1(X \cap \partial B_r) + 4r^2 h(2r) \\
&\leq (1 - \vartheta\delta^2 + 16\varepsilon(r))\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1) + \vartheta\delta^2\Theta(0)r^2 + 4r^2 h(2r)
\end{aligned}$$

We take $\mu = 2 \cdot 10^{-4} - \vartheta\kappa^2$ and $\lambda = \vartheta\kappa^2$, then by (3.27) and (3.28), we have that

$$16\varepsilon(r) < \vartheta(\delta^2 - \kappa^2) \text{ and } \vartheta(\delta^2 - \kappa^2)\Theta(0) \leq (1 - 2 \cdot 10^{-4})\frac{\tau_1}{2}.$$

We obtain from (3.29) and (3.30) that

$$\begin{aligned}
\mathcal{H}^2(E \cap \overline{B_r}) &\leq (1 - 2 \cdot 10^{-4})\frac{r^2}{2}(\mathcal{H}^1(X \cap \partial B_1) + \tau_1) - (1 - 2 \cdot 10^{-4})\frac{\tau_1 r^2}{2} \\
&\quad + \mu\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1) + \vartheta\kappa^2\Theta(0)r^2 + 4r^2 h(2r) \\
&\quad + (16\varepsilon(r) - \vartheta\delta^2 + \vartheta\kappa^2)\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1) + (\vartheta\delta^2 - \vartheta\kappa^2)\Theta(0)r^2 \\
&\leq (1 - \lambda - \mu)\frac{r}{2}\mathcal{H}^1(E \cap \partial B_r) + \mu\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1) + \lambda\Theta(0)r^2 + 4r^2 h(2r).
\end{aligned}$$

□

For convenient, we put $\lambda_0 = \lambda/(1 - \lambda)$, $f(r) = \Theta(0,r) - \Theta(0)$ and $u(r) = \mathcal{H}^1(E \cap B(0,r))$ for $r > 0$. Since $f(r) = r^{-2}u(r) - \Theta(0)$ and u is a nondecreasing function, we have that, for any $\lambda_1 \in \mathbb{R}$ and $0 < r \leq R < +\infty$,

$$R^{\lambda_1} f(R) - r^{\lambda_1} f(r) \geq \int_r^R \left(t^{\lambda_1} f(t)\right)' dt,$$

thus

$$f(r) \le r^{-\lambda_1} R^{\lambda_1} f(R) + r^{-\lambda_1} \int_r^R \left(t^{\lambda_1} f(t)\right)' dt. \tag{3.31}$$

Corollary 3.16. *If the gauge function h satisfy*

$$h(t) \le C_h t^{\alpha},\ 0 < t \le \mathfrak{r}_1 \text{ for some } C_h > 0,\ \alpha > 0,$$

then for any $0 < \beta < \min\{\alpha, 2\lambda_0\}$, there is a constant $C = C(\lambda_0, \alpha, \beta, \mathfrak{r}_1, C_h) > 0$ such that

$$|\Theta(0,\rho) - \Theta(0)| \le C\rho^{\beta},\ \text{for any } 0 < \rho \le \mathfrak{r}_1. \tag{3.32}$$

Proof. For any $r > 0$, we put $u(r) = \mathcal{H}^2(E \cap B(0,r))$. Then u is differentiable for $\mathcal{H}^1$-a.e. $r \in (0,\infty)$. By Theorem 3.15 and Lemma 2.1, we have that for any $r \in (0,\mathfrak{r}_1) \cap \mathscr{R}$,

$$\begin{aligned} u(r) &\le (1-\lambda)\frac{r}{2}\mathcal{H}^1(E \cap \partial B(0,r)) + \lambda\Theta(0)r^2 + 4r^2 h(2r) \\ &\le (1-\lambda)\frac{r}{2}u'(r) + \lambda\Theta(0)r^2 + 4r^2 h(2r), \end{aligned}$$

thus

$$rf'(r) \ge \frac{2\lambda}{1-\lambda} f(r) - \frac{8}{1-\lambda} h(2r) = 2\lambda_0 f(r) - 8(1+\lambda_0)h(2r),$$

and

$$\left(r^{-2\lambda_0} f(r)\right)' = r^{-1-2\lambda_0}\left(rf'(r) - 2\lambda_0\right) \ge -8(1+\lambda_0) r^{-1-2\lambda_0} h(2r).$$

Recall that $\mathcal{H}^1((0,\infty)\setminus\mathscr{R}) = 0$. We get so that, from (3.31), for any $0 < r < R \le \mathfrak{r}_1$,

$$f(r) \le r^{2\lambda_0} R^{-2\lambda_0} f(R) + 8(1+\lambda_0) r^{2\lambda_0} \int_r^R t^{-1-2\lambda_0} h(2t)dt. \tag{3.33}$$

Since $h(t) \le C_h t^{\alpha}$, we have that

$$f(r) \le (r/R)^{-2\lambda_0} f(R) + 2^{3+\alpha}(1+\lambda_0) C_h r^{2\lambda_0} \int_r^R t^{\alpha - 2\lambda_0 - 1} dt.$$

If $\alpha > 2\lambda_0$, then

$$f(r) \le \left(f(R) + 2^{3+\alpha}(1+\lambda_0)(1+\lambda_0)(\alpha - 2\lambda_0)^{-1} C_h R^{\alpha}\right)(r/R)^{2\lambda_0}; \tag{3.34}$$

if $\alpha = 2\lambda_0$, then

$$f(r) \le f(R)(r/R)^{\alpha} + 2^{\alpha+3}(1+\lambda_0) C_h r^{\alpha} \ln(R/r),$$

thus, for any $\beta \in (0,\alpha)$,

$$\begin{aligned} f(r) &\le f(R) r^{\alpha} + 2^{\alpha+3}(1+\lambda_0) C_h r^{\beta} R^{\alpha-\beta} \frac{\ln(R/r)}{(R/r)^{\alpha-\beta}} \\ &\le \left(f(R) + 2^{\alpha+3}(1+\lambda_0) C_h (\alpha-\beta)^{-1} e^{-1} R^{\alpha}\right)(r/R)^{\beta}; \end{aligned} \tag{3.35}$$

if $\alpha < 2\lambda_0$, then

(3.36)

$$\begin{aligned} f(r) &\le f(R)(r/R)^{2\lambda_0} + 2^{\alpha+3}(1-\lambda_0) C_h r^{2\lambda_0} \cdot (2\lambda_0 - \alpha)^{-1}\left(r^{\alpha-2\lambda_0} - R^{\alpha-2\lambda_0}\right) \\ &\le \left((r/R)^{2\lambda_0-\alpha} f(R) + 2^{\alpha+3}(1-\lambda_0) C_h (2\lambda_0-\alpha)^{-1} R^{\alpha}\right)(r/R)^{\alpha}. \end{aligned}$$

Hence (3.32) follows from (3.34), (3.35), (3.36) and Theorem 2.3. Indeed, there is a constant $C_1(\alpha,\beta,\lambda_0)>0$ such that

$$r^{2\lambda_0}\int_r^R t^{\alpha-2\lambda_0-1}dt\le C_1(\alpha,\beta,\lambda_0)R^\alpha\cdot(r/R)^\beta, \tag{3.37}$$

and there is a constant $C_2(\alpha,\beta,\lambda_0)>0$ such that

$$f(r)\le\left(f(R)+C_2(\alpha,\beta,\lambda_0)C_h\cdot R^\alpha\right)(r/R)^\beta.$$

□

Remark 3.17. If the gauge function h satisfy that

$$h(t)\le C\left(\ln\left(\frac{A}{t}\right)\right)^{-b}$$

for some $A,b,C>0$, then (3.33) implies that there exist $R>0$ and constant $C(R,\lambda,b)$ such that

$$f(r)\le C(R,\lambda,b)\left(\ln\left(\frac{A}{r}\right)\right)^{-b}\ \text{ for } 0<r\le R.$$

4. Approximation of E by cones at the boundary

In the previous section, we get a power decay of the almost density, and in this section we will use that to get the uniqueness of blow-up limit of E at 0, and also the estimation $d_{0,r}(E,Z)\le Cr^\beta$ for r small, where Z is the unique blow-up limit, see Theorem 4.14.

We also assume that $E\subseteq\Omega_0$ is a 2-rectifiable set satisfying **(a)**, **(b)** and **(c)**. We let $\varepsilon(r)=\varepsilon_P(r)$ if E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{P}_+$; and let $\varepsilon(r)=\varepsilon_Y(r)$ if E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{Y}_+$.

For any $r>0$, we put

$$f(r)=\Theta(0,r)-\Theta(0),\ F(r)=f(r)+8h_1(r),\ F_1(r)=F(r)+8h_1(r),$$

and for $r\in\mathscr{R}$, we put

$$\Xi(r)=rf'(r)+2f(r)+16h(2r)+32h_1(r).$$

We see from Theorem 2.3 that F is nondecreasing, and $\lim_{r\to0+}F(r)=0$, thus $\Xi(r)\ge0$.

We denote by $X(r)$ and $\Gamma(r)$, respectively, the cone X and the set Γ which are defined in (3.10), and by $\gamma(r)$ the set $\boldsymbol{\mu}_r(\Gamma(r))$. Let $\Pi:\mathbb{R}^3\setminus\{0\}\to\partial B(0,1)$ be the mapping defined by $\Pi(x)=x/|x|$. For any $r_2>r_1>0$, we put $A(r_1,r_2)=\{x\in\mathbb{R}^3:r_1\le|x|\le r_2\}$. Let λ, μ and $\mathfrak{r}_1$ be the constants in Theorem 3.15.

Lemma 4.1. *For any* $0<r<R<\infty$ *with* $\mathcal{H}^2(E\cap\partial B_r)=\mathcal{H}^2(E\cap\partial B_R)=0$*, we have that*

$$\int_{E\cap A(r,R)}\frac{1-\cos\theta(x)}{|x|^2}d\mathcal{H}^2(x)\le F(R)-F(r), \tag{4.1}$$

and

$$\mathcal{H}^2\left(\Pi(E\cap A(r,R))\right)\le\int_{E\cap A(r,R)}\frac{\sin\theta(x)}{|x|^2}d\mathcal{H}^2(x). \tag{4.2}$$

Proof. We see that for $\mathcal{H}^2$-a.e. $x \in E$, the tangent plane $\mathrm{Tan}(E,x)$ exists, we will denote by $\theta(x)$, the angle between the line $[0,x]$ and the plane $\mathrm{Tan}(E,x)$. For any $t > 0$, we put $u(t) = \mathcal{H}^2(E \cap B(0,t))$, then $u : (0,\infty) \to [0,\infty]$ is a nondecreasing function. By Lemma 2.2, we have that

$$u(t) \le \frac{t}{2}\mathcal{H}^1(E \cap \partial B(0,t)) + 4t^2 h(2t),$$

for $\mathcal{H}^1$-a.e. $t \in (0,\infty)$. Considering the mapping $\phi : \mathbb{R}^3 \to [0,\infty)$ given by $\phi(x) = |x|$, we have, by (2.2), that for $\mathcal{H}^2$-a.e. $x \in E$,

$$\mathrm{ap}\, J_1(\phi|_E)(x) = \cos\theta(x).$$

Apply Theorem 3.2.22 in [10], we get that

$$\begin{aligned}
&\int_{E\cap A(r,R)} \frac{1}{|x|^2}\cos\theta(x) d\mathcal{H}^2(x) = \int_r^R \frac{1}{t^2}\mathcal{H}^1(E \cap \partial B(0,t) dt \\
&\ge 2\int_r^R \frac{u(t)}{t^3}dt - 8\int_r^R \frac{h(2t)}{t}dt = 2\int_r^R \frac{1}{t^3}\left(\int_{E\cap B(0,t)} d\mathcal{H}^2(x)\right)dt - 8(h_1(R) - h_1(r)) \\
&= 2\int_{E\cap B(0,R)}\left(\int_{\max\{r,|x|\}}^R \frac{1}{t^3}dt\right)d\mathcal{H}^2(x) - 8(h_1(R) - h_1(r)) \\
&= \int_{E\cap A(r,R)} \frac{1}{|x|^2}d\mathcal{H}^2(x) + r^{-2}u(r) - R^{-2}u(R) - 8(h_1(R) - h_1(r)),
\end{aligned}$$

thus (4.1) holds.

By a simple computation, we get that

$$\mathrm{ap}\, J_2\Pi(x) = \frac{\sin\theta(x)}{|x|^2},$$

then applying Theorem 3.2.22 in [10], we will get that (4.2) hold. □

For any $0 < r < R$, if $\mathcal{H}^2(E \cap \partial B_r) = \mathcal{H}^2(E \cap \partial B_R) = 0$, by Cauchy-Schwarz inequality, we get from above Lemma that

$$\mathcal{H}^2(\Pi(E \cap A(r,R))) \le \frac{R}{r}(2\Theta(0,R))^{1/2}(F(R) - F(r))^{1/2} \le \frac{R}{r}(2\Theta(0,R))^{1/2}F(R)^{1/2}.$$

Lemma 4.2. *For any $r \in (0,\mathfrak{r}_1) \cap \mathscr{R}$, if $\Xi(r) \le \mu\tau_0$, then*

$$d_H(\Gamma(r), X(r) \cap \partial B(0,1)) \le 10\mu^{-1/2}\Xi(r)^{1/2}.$$

Proof. By lemma 2.1, we get that

$$\frac{1}{r}\mathcal{H}^1(E \cap \partial B(0,r)) \le 2\Theta(0) + rf'(r) + 2f(r),$$

By Theorem 3.15, we get that

$$\begin{aligned}
&r^2\Theta(0,r) \\
&\quad \le (1-\lambda-\mu)\frac{r}{2}\mathcal{H}^1(E \cap \partial B_r) + \mu\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1) + \lambda\Theta(0)r^2 + 4r^2h(2r) \\
&\quad \le \frac{(1-\lambda-\mu)r^2}{2}(2\Theta(0) + rf'(r) + 2f(r)) + \frac{\mu r^2}{2}\mathcal{H}^1(X \cap \partial B_1) + \lambda\Theta(0)r^2 + 4r^2h(2r),
\end{aligned}$$

thus

$$\mathcal{H}^1(X \cap \partial B_1) \ge 2\Theta(0) + \frac{2(\lambda+\mu)}{\mu}f(r) - \frac{1-\lambda-\mu}{\mu}rf'(r) - \frac{\mu}{8}h(2r).$$

By Theorem 2.3, we see that $f(r)+8h_1(r)$ is nondecreasing, thus $f(r)+8h_1(r)\geq 0$ and $rf'(r)+8h(2r)\geq 0$. Hence

$$j(r)=\frac{1}{r}\mathcal{H}^1(E\cap B_r)-\mathcal{H}^1(X\cap\partial B_1)\leq\frac{1-\lambda}{\mu}rf'(r)-\frac{2\lambda}{\mu}f(r)+\frac{8}{\mu}h(2r)$$
$$\leq\frac{1}{\mu}(rf'(r)+8h_1(r)+16h(2r))\leq\frac{1}{\mu}\Xi(r).$$

Since

$$\mathcal{H}^1(X\cap\partial B_1)\leq\mathcal{H}^1(\Gamma_*(r))\leq\mathcal{H}^1(\Gamma(r))\leq\mathcal{H}^1(\boldsymbol{\mu}_{1/r}(E\cap\partial B_r)),$$

we have that

$$0\leq\mathcal{H}^1(\Gamma(r))-\mathcal{H}^1(X\cap B_1)\leq j(r)\leq\frac{1}{\mu}\Xi(r),$$

by Lemma 3.5, we get that for any $z\in\Gamma(r)$,

$$\operatorname{dist}(z,X\cap\partial B(0,1))\leq 10\left(\frac{\Xi(r)}{\mu}\right)^{1/2}.$$

□

Lemma 4.3. *For any $0<r_1<r_2<(1-\tau)\mathfrak{r}_1$, if P is a plane such that $\mathcal{H}^1(E\cap P\cap B_{\mathfrak{r}_1})<\infty$ and $P\cap\mathcal{X}_r=\emptyset$ for any $r\in[r_1,r_2]$, then there is a compact path connected set*

$$\mathcal{C}_{P,r_1,r_2}\subseteq E\cap P\cap A(r_2,r_1)$$

such that

$$\mathcal{C}_{P,r_1,r_2}\cap\gamma(t)\neq\emptyset \text{ for } r_1\leq t\leq r_2.$$

Proof. We let ϱ be the same as in 3.1. Since $\|\Phi-\mathrm{id}\|_\infty\leq\tau\varrho$, we get that

$$\Phi^{-1}\left(E\cap\overline{B(0,r_2)}\right)\subseteq Z_{0,\varrho}\cap\overline{B(0,r_2+\tau\varrho)}.$$

We put

$$\mathbb{X}=Z_{0,\varrho}\cap\overline{B(0,r_2+\tau\varrho)},\ F=\mathbb{X}\cap\Phi^{-1}(E\cap P_z).$$

We take $x_1,x_2\in\mathcal{X}_r$, $x_2\neq x_1$, such that $\Phi^{-1}(x_1)$ and $\Phi^{-1}(x_2)$ are contained in two different connected components of $\mathbb{X}\backslash F$. By Lemma 3.2, there is a connected closed subset F_0 of F such that $\Phi^{-1}(x)$ and $\Phi^{-1}(x_2)$ are still contained in two different connected components of $\mathbb{X}\setminus F_0$. Then $F_0\cap\phi^{-1}(\gamma(t))\neq\emptyset$ for $0<t\leq r_2$; otherwise, if $F_0\cap\phi^{-1}(\gamma(t_0))=\emptyset$, then x_1 and x_2 are in the same connected component of $\Phi(\mathbb{X})\setminus\Phi(F_0)$, thus $\Phi^{-1}(x_1)$ and $\Phi^{-1}(x_2)$ are in the same connected component of $\mathbb{X}\setminus F_0$, absurd!

Since $\mathcal{H}^1(\Phi(F_0))\leq\mathcal{H}^1(E\cap P_z\cap B_\varrho)<\infty$, we get that $\Phi(F_0)$ is path connected. We take $z_1\in\Phi(F_0)\cap\gamma(r_1)$ and $z_2\in\Phi(F_0)\cap\gamma(r_2)$, and let $g:[0,1]\to\Phi(F_0)$ be a path such that $g(0)=z_1$ and $g(1)=z_2$. We take $t_1=\sup\{t\in[0,1]:|g(t)|\leq r_1\}$ and $t_2=\inf\{t\in[t_1,1]:|g(t)|\geq r_2\}$. Then $\mathcal{C}_{z,r_1,r_2}=g([t_1,t_2])$ is our desire set. □

Lemma 4.4. *Let $T\in[\pi/4,3\pi/4]$ and $\varepsilon\in(0,1/2)$ be given. Suppose that F a 2-rectifiable set satisfying*

$$F\subseteq\partial B(0,1)\cap\{(t\cos\theta,t\sin\theta,x_3)\in\mathbb{R}^3\mid t\geq 0,|\theta|\leq T/2,|x_3|\leq\varepsilon\}.$$

Then we have, by putting $\mathcal{P}_\theta=\{(t\cos\theta,t\sin\theta,x_3)\mid t\geq 0,x_3\in\mathbb{R}\}$, that

$$\int_{-T/2}^{T/2}\mathcal{H}^1(F\cap\mathcal{P}_\theta)d\theta\leq(1+\varepsilon)\mathcal{H}^2(F)$$

Proof. For any $x = (x_1, x_2, x_3) \in F$, we have that $x_1^2 + x_2^2 + x_3^2 = 1$ and $|x_3| \le \varepsilon$, thus $x_1^2 + x_2^2 \ge 1 - \varepsilon^2$. Since $|\theta| \le T/2 \le 3\pi/8$, we get that the mapping $\phi : F \to \mathbb{R}$ given by

$$\phi(x_1, x_2, x_3) = \arctan \frac{x_2}{x_1}$$

is well defined and Lipschitz. Moreover, we have that

$$\operatorname{ap} J_1\phi(x) = (x_1^2 + x_2^2)^{-1/2} \le (1 - \varepsilon^2)^{-1/2} \le 1 + \varepsilon.$$

Hence

$$\int_{-T/2}^{T/2} \mathcal{H}^1(F \cap \mathcal{P}_\theta) d\theta = \int_F \operatorname{ap} J_1\phi(x) d\mathcal{H}^2(x) \le (1+\varepsilon)\mathcal{H}^2(F).$$

□

For any $0 < t_1 \le t_2$, we put $E_{t_1,t_2} = \Pi\left(\{x \in E : t_1 \le |x| \le t_2\}\right)$. For any $t > 0$, we put

$$\bar{\varepsilon}(t) = \sup\{\varepsilon(r) : r \le t\}.$$

Lemma 4.5. *If $r_2 > r_1 > 0$ satisfy that $10(1 + r_2/r_1)\bar{\varepsilon}(r_2) < 1/2$, then we have that*

$$\int_{X(t)\cap \partial B(0,1)} \mathcal{H}^1\left(P_z \cap E_{r_1,r_2}\right) d\mathcal{H}^1(z) \le 2\mathcal{H}^2\left(E_{r_1,r_2}\right), \ \forall r_1 \le t \le r_2.$$

Proof. By Lemma 3.8, we have that, for any $r > 0$, if $\varepsilon(r) < 1/2$, then

$$d_{0,r}(E, X(r)) \le 5\varepsilon(r).$$

We get so that

$$\begin{aligned} d_{0,1}(X(t), X(r_2)) =& d_{0,t}(X(t), X(r_2)) \le d_{0,t}(E, X(t)) + d_{0,t}(E, X(r_2)) \\ &\le 5\bar{\varepsilon}(r_2) + 5\frac{r_2}{t}\bar{\varepsilon}(r_2). \end{aligned}$$

Since

$$\operatorname{dist}(x, X(r_2)) \le 5r_2\varepsilon(r_2), \text{ for any } x \in E \cap B(0, r_2),$$

we have that

$$\operatorname{dist}(\Pi(x), X(r_2)) \le \frac{5r_2\varepsilon(r_2)}{|x|}, \text{ for any } x \in E \cap A(r_1, r_2),$$

we get so that

$$\operatorname{dist}(\Pi(x), X(t)) \le \frac{5r_2\varepsilon(r_2)}{|x|} + 5\bar{\varepsilon}(r_2) + 5\frac{r_2}{t}\bar{\varepsilon}(r_2) \le 10(r_2/r_1 + 1)\bar{\varepsilon}(r_2) < \frac{1}{2}.$$

Applying Lemma 4.4, we will get the result. □

Lemma 4.6. *Let $\varepsilon \in (0, 1/2)$ be given. Let $A \subseteq \partial B(0,1)$ be an arc of a great circle such that $0 < \mathcal{H}^1(A) \le \pi$ and*

$$\operatorname{dist}(x, L_0) \le \varepsilon, \forall x \in A.$$

Then

$$\operatorname{dist}(x, L_0) \le \frac{\pi^2}{2\mathcal{H}^1(A)^2} \int_A \operatorname{dist}(x, L_0) d\mathcal{H}^1(x), \ \forall x \in A.$$

Proof. We let P be the plane such that $A \subseteq P$, let $v_0 \in P \cap L_0 \cap \partial B(0,1)$ and $v_2 \in P \cap \partial B(0,1)$ be two vectors such that v_0 is perpendicular to v_1. Then A can be parametrized as $\gamma : [\theta_1, \theta_2] \to A$ given by

$$\gamma(t) = v_0 \cos t + v_1 \sin t,$$

where $\theta_2 - \theta_1 = \mathcal{H}^1(A)$. We write $v_1 = w + w^\perp$ with $w \in L_0$ and $w^\perp$ perpendicular to L_0. Since $\operatorname{ap} J_1 \gamma(t) = 1$ for any $t \in [\theta_1, \theta_2]$, by Theorem 3.2.22 in [10], we have that

$$\begin{aligned}\int_A \operatorname{dist}(x, L_0)\mathcal{H}^1(x) &= \int_{\theta_1}^{\theta_2} \operatorname{dist}(\gamma(t), L_0) dt = \int_{\theta_1}^{\theta_2} |w^\perp \sin t| dt \\ &\geq 2|w^\perp| \left(1 - \cos \frac{\theta_2 - \theta_1}{2}\right) \geq \frac{2(\theta_2 - \theta_1)^2}{\pi^2} |w^\perp|,\end{aligned}$$

and that

$$\operatorname{dist}(x, L_0) \leq |w^\perp| \leq \frac{\pi^2}{2\mathcal{H}^1(A)^2} \int_A \operatorname{dist}(x, L_0) d\mathcal{H}^1(x).$$

□

Lemma 4.7. *Let r_1 and r_2 be the same as in Lemma 4.3. If $\Xi(r_i) \leq \mu\tau_0$, $10(1 + r_2/r_1)\bar{\varepsilon}(r_2) \leq 1$, then we have that*

$$d_{0,1}(X(r_1), X(r_2)) \leq \frac{30 r_2}{r_1} \Theta(0, r_2)^{1/2} \cdot F(r_2)^{1/2} + 20\pi\mu^{-1/2} \cdot \left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right).$$

Proof. For $z \in X(r_2) \cap \partial B_1$, if $z \notin \{y_r\} \cup \mathcal{X}_r$, we will denote by P_z the plane which is through 0 and z and perpendicular to $\operatorname{Tan}(X(r_2) \cap \partial B_1, z)$. By Lemma 4.2, we have that

$$|z - a| \leq 10\mu^{-1/2} \Xi(r_1)^{1/2}, \forall a \in \Gamma(r_2) \cap P_z.$$

Since $\mathcal{C}_{P_z, r_1, r_2} \cap \gamma(r_i) \neq \emptyset$, $i = 1, 2$, we take $b_i \in \mathcal{C}_{P_z, r_1, r_2} \cap \gamma(r_i)$, then

$$|\Pi(b_1) - \Pi(b_2)| \leq \mathcal{H}^1(\Pi(\mathcal{C}_{P_z, r_1, r_2})) \leq \mathcal{H}^1(P_z \cap E_{r_1, r_2}),$$

thus

$$\begin{aligned}\operatorname{dist}(z, X(r_1) \cap \partial B_1) &\leq |z - \Pi(b_2)| + |\Pi(b_2) - \Pi(b_1)| + \operatorname{dist}(\Pi(b_1), X(r_1) \cap \partial B_1) \\ &\leq \mathcal{H}^1(P_z \cap E_{r_1, r_2}) + 10\mu^{-1/2} \left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right).\end{aligned}$$

For any $x \in \mathcal{X}_r$, we let A_x be the arc in $\partial B(0,1)$ which join $\Pi(x)$ and $\Pi(y_r)$, We see that $X(r_2) \cap \partial B(0,1) = \cup_{x \in \mathcal{X}_r} A_x$, and $\mathcal{H}^1(A_x) \geq (1/2 - \bar{\varepsilon}(r_2))\pi \geq \pi/4$. Suppose $z \in A_x$, then

$$\begin{aligned}&\operatorname{dist}(z, X(r_1)) \\ &\leq \frac{\pi^2}{2\mathcal{H}^1(A_x)^2} \int_{A_x} \operatorname{dist}(z, X(r_1)) d\mathcal{H}^1(x) \\ &\leq \frac{2\pi}{\mathcal{H}^1(A_x)} \int_{A_x} \mathcal{H}^1(P_z \cap E_{r_1, r_2}) d\mathcal{H}^1(x) + 20\pi\mu^{-1/2} \left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right) \\ &\leq 16\mathcal{H}^2(E_{r_1, r_2}) + 20\pi\mu^{-1/2} \left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right) \\ &\leq \frac{16 r_2}{r_1} \left(2\Theta(0, r_2)\right)^{1/2} F(r_2)^{1/2} + 20\pi\mu^{-1/2} \left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right).\end{aligned}$$

□

Remark 4.8. It is easy to see that, for any cones X_1 and X_2,

$$d_H(X_1 \cap \partial B(0,1), X_2 \cap \partial B(0,1)) \le 2d_{0,1}(X_1, X_2).$$

Since $\Xi(r) = rf'(r) + 2f(r) + 16h(2r) + 32h_1(r)$ and $F_1(r) = f(r) + 16h_1(r)$, we see that $\Xi(r) = [rF_1(r)]'$ for any $r \in \mathscr{R}$, we get so that

$$\int_{r_1}^{r_2} \Xi(t)dt \le r_2F_1(r_2) - r_1F_1(r_1).$$

For any $\zeta > 2$, if $r_1 \le r_2 \le r$, then by Chebyshev's inequality, we get that,

$$\mathcal{H}^1\left(\left\{t \in [r_1, r_2] \middle| \Xi(t) \le \zeta F_1(r)^{2/3}\right\}\right) \ge r_2 - r_1 - \frac{1}{\zeta} rF_1(r)^{1/3},$$

thus $\left\{t \in [r_1, r_2] \middle| \Xi(t) \le \zeta F_1(r)^{2/3}\right\} \ne \emptyset$ when $r_2 - r_1 > (1/\zeta)rF_1(r)^{1/3}$.

Lemma 4.9. *Let $R_0 < (1-\tau)\mathfrak{r}_1$ be a positive number such that $F(R_0) \le \mu\tau_0/4$ and $\bar{\varepsilon}(R_0) \le 10^{-4}$. For any $r \in \mathscr{R} \cap (0, R_0)$, if $\Xi(r) \le \mu\tau_0$, then there is a constant $C = C(\mu, \Theta(0))$ such that*

$$\operatorname{dist}(x, E) \le Cr\left(F_1(r)^{1/3} + \Xi(r)^{1/2}\right),\ x \in X(r) \cap B_r.$$

Proof. For any $k \ge 0$, we take $r_k = 2^{-k}r$. Then there exists $t_k \in [r_k, r_{k-1}]$ such that

$$\Xi(t_k) \le \frac{\int_{r_k}^{r_{k-1}} \Xi(t)dt}{r_{k-1} - r_k} \le \frac{r_{k-1}F_1(r_{k-1})}{r_{k-1}/2} = 2F_1(r_{k-1}).$$

We let $X_k = X(t_k)$, then for any $j > i \ge 1$, we have that

(4.3)

$$\begin{aligned}
&d_{0,1}(X_i, X_j)\\
&\le \sum_{k=i}^{j-1} d_{0,1}(X_k, X_{k+1})\\
&\le 60\left(\Theta(0) + \mu\tau_0/4\right)^{1/2} \sum_{k=i}^{j-1} F_1(t_k)^{1/2} + 20\pi\mu^{-1/2} \sum_{k=i}^{j-1} \left(\Xi(t_k)^{1/2} + \Xi(t_{k+1})^{1/2}\right)\\
&\le \left(60\left(\Theta(0) + \mu\tau_0/4\right)^{1/2} + 40\pi\mu^{-1/2}\right) \sum_{k=i}^{j-1} 2F_1(t_k)^{1/2} + F_1(t_{k-1})^{1/2}\\
&\le C_1(\mu, \Theta(0))(j-i)F_1(r_{i-1})^{1/2} = C_1(\mu, \Theta(0))F_1(r_{i-1})^{1/2} \log_2(r_i/r_j),
\end{aligned}$$

where $C_1(\mu, \Theta(0)) = 3\left(60\left(\Theta(0) + \mu\tau_0/4\right)^{1/2} + 40\pi\mu^{-1/2}\right)$.

For any $x \in X(r) \cap B_r$ with $\Xi(|x|) \le \mu\tau_0$, we assume that $t_{k+1} \le |x| < t_k$, then

$$\begin{aligned}
&\operatorname{dist}(x, E)\\
&\le d_H(X(r) \cap B_{|x|}, X(|x|) \cap B_{|x|}) + d_H(X(|x|) \cap B_{|x|}, \gamma(|x|))\\
&\le 2|x| d_{0,1}(X(r), X(|x|)) + 10\mu^{-1/2}|x|\Xi(|x|)^{1/2}\\
&\le 2|x|(d_{0,1}(X(|x|), X_k) + d_{0,1}(X_k, X_1) + d_{0,1}(X_1, X(r))) + 10\mu^{-1/2}|x|\Xi(|x|)^{1/2}\\
&\le (40\pi + 10)\mu^{-1/2}|x|\left(\Xi(|x|)^{1/2} + \Xi(r)^{1/2}\right) + C_2(\mu, \Theta(0))|x|F_1(r)^{1/2}\log_2(r/|x|)\\
&\le (40\pi + 10)\mu^{-1/2}|x|\Xi(|x|)^{1/2} + C_3(\mu, \Theta(0))r\left(\Xi(r)^{1/2} + F_1(r)^{1/2}\right)
\end{aligned}$$

For any $0 \le a \le b \le r$, we put

$$I(a,b) = \left\{ t \in [a,b] \middle| \Xi(t) \le F_1(r)^{2/3} \right\},$$

then $I(a,b) \neq \emptyset$ when $b-a > rF_1(r)^{1/3}$. If $|x| \in I(0,r)$, then

$$\operatorname{dist}(x,E) \le C_4(\mu,\Theta(0))r\left(F_1(r)^{1/3} + \Xi(r)^{1/2}\right).$$

We let $\{s_i\}_{i=0}^{m+1} \subseteq [0,r]$ be a sequence such that

$$0 = s_0 < s_1 < \cdots < s_m < s_{m+1} = r,\ \ s_i \in I(0,r),$$

and

$$s_{i+1} - s_i \le 2rF_1(r)^{1/3}.$$

For any $x \in X(r) \cap B_r$, if $s_i \le |x| < s_{i+1}$ for some $0 \le i \le m$, we have that

$$\begin{aligned}\operatorname{dist}(x,E) &\le \left| x - \frac{s_i}{|x|}x \right| + \operatorname{dist}\left(\frac{s_i}{|x|}x, E\right)\\ &\le (s_{i+1}-s_i) + C_4(\mu,\Theta(0))r\left(F_1(r)^{1/3} + \Xi(r)^{1/2}\right)\\ &\le (C_4(\mu,\Theta(0))+2)r\left(F_1(r)^{1/3} + \Xi(r)^{1/2}\right).\end{aligned}$$

□

Definition 4.10. Let $U \subseteq \mathbb{R}^3$ be an open set, $E \subseteq \mathbb{R}^3$ be a set of Hausdorff dimension 2. E is called Ahlfors-regular in U if there is a $\delta > 0$ and $\xi_0 \ge 1$ such that, for any $x \in E \cap U$, if $0 < r < \delta$ and $B(x,r) \subseteq U$, we have that

$$\xi_0^{-1}r^2 \le \mathcal{H}^2(E \cap B(x,r)) \le \xi_0 r^2.$$

Lemma 4.11. *Let R_0 be the same as in Lemma 4.9. If E is Ahlfors-regular, and $r \in \mathscr{R} \cap (0,R_0)$ satisfies $\Xi(r) \le \mu\tau_0$, then there is a constant $C = C(\mu,\xi_0,\Theta(0))$ such that*

$$\operatorname{dist}(x,X(r)) \le Cr\left(F_1(r)^{1/4} + \Xi(r)^{1/2}\right),\ \ x \in E \cap B(0,9r/10).$$

Proof. Let $\{X_k\}_{k\ge1}$ be the same as in (4.3). For any $t \in \mathscr{R}$ with $t_{k+1} \le t < t_k$, $\Xi(t) \le \mu\tau_0$ and $x \in \gamma(t)$, we have that

$$\begin{aligned}\operatorname{dist}(x,X(r)) &\le d_H(\gamma(t), X(|x|)\cap B_{|x|}) + d_H(X(|x|)\cap B_{|x|}, X(r))\\ &\le (40\pi+10)\mu^{-1/2}|x|\Xi(|x|)^{1/2} + C_3(\mu,\Theta(0))r\left(\Xi(r)^{1/2} + F_1(r)^{1/2}\right)\end{aligned}$$

We put

$$J(0,r) = \{t \in [0,r] : \Xi(t) > F_1(r)^{1/2}\}.$$

For any $x \in \gamma(t)$ with $t \in (0,r) \setminus J(0,r)$, we have that

$$\operatorname{dist}(x,X(r)) \le C_5(\mu,\Theta(0))r\left(\Xi(r)^{1/2} + F_1(r)^{1/4}\right).$$

We put

$$E_1 = \bigcup_{t \in J(0,r)} (E \cap \partial B_t),\ \ E_2 = \bigcup_{t \in (0,r)\setminus J(0,r)} (E \cap B_t \setminus \gamma(t)),$$

and

$$E_3 = E \cap B_r \setminus (E_1 \cup E_2) = \bigcup_{t \in (0,r)\setminus J(0,r)} \gamma(t).$$

Then

$$\begin{aligned}\mathcal{H}^2(E_1\cup E_2) &= \int_{E\cap B_r} d\mathcal{H}^2(x) - \int_{E_3} d\mathcal{H}^2(x) \le \int_{E\cap B_r} d\mathcal{H}^2(x) - \int_{E_3}\cos\theta(x)d\mathcal{H}^2(x)\\ &= \int_{E\cap B_r}(1-\cos\theta(x))d\mathcal{H}^2(x) + \int_{E_1\cup E_2}\cos\theta(x)d\mathcal{H}^2(x)\\ &\le r^2F(r) + \int_0^r \mathcal{H}^1(E_1\cap\partial B_t)dt + \int_0^r \mathcal{H}^1(E_2\cap\partial B_t)dt\\ &\le r^2F(r) + \int_{J(0,r)}(2\Theta(0)+tf'(t)+2f(t))tdt + \mu^{-1}\int_0^r t\Xi(t)dt\\ &\le (2+\mu^{-1})r^2F_1(r) + 2\Theta(0)\int_{\{t\in[0,r]:\Xi(t)>F_1(r)^{1/2}\}} tdt\\ &\le (2+\mu^{-1})r^2F_1(r) + \frac{2\Theta(0)}{F_1(r)^{1/2}}\int_0^r t\Xi(t)dt \le C_6(\mu,\Theta(0))r^2F_1(r)^{1/2},\end{aligned}$$

where $C_6(\mu,\Theta(0)) = (2+\mu^{-1})(\mu\tau_0/4)^{1/2} + 2\Theta(0)$.

We see that, for any $x\in E_3$,

$$\operatorname{dist}(x,X(r)) \le C_5(\mu,\Theta(0))r\left(\Xi(r)^{1/2}+F_1(r)^{1/4}\right).$$

If $x\in E\cap B(0,9r/10)$ with

$$\operatorname{dist}(x,X(r)) > C_5(\mu,\Theta(0))r\left(\Xi(r)^{1/2}+F_1(r)^{1/4}\right) + s$$

for some $s\in(0,r/10)$, then $E\cap B(x,s)\subseteq E_1\cup E_2$, thus

$$\mathcal{H}^2(E\cap B(x,s)) \le C_6(\mu,\Theta(0))r^2F_1(r)^{1/2}.$$

But on the other hand, by Ahlfors-regular property of E, we have that

$$\mathcal{H}^2(E\cap B(x,s)) \ge \xi_0^{-1}s^2.$$

We get so that

$$s \le C_6(\mu,\Theta(0))^{1/2}\cdot\xi_0^{1/2}\cdot rF_1(r)^{1/4}.$$

Therefore, for $x\in E\cap B(0,9r/10)$,

$$\operatorname{dist}(x,X(r)) \le \left(C_6(\mu,\Theta(0))^{1/2}\cdot\xi_0^{1/2} + C_5(\mu,\Theta(0))\right)\left(\Xi(r)^{1/2}+F_1(r)^{1/4}\right).$$

□

For any $k\ge 0$, we take $R_k = 2^{-k}R_0$ and $s_k\in[R_{k+1},R_k]$ such that

$$\Xi(s_k) \le \frac{\int_{R_{k+1}}^{R_k}\Xi(t)dt}{R_k-R_{k+1}} \le 2F_1(R_k).$$

We put $X_k = X(s_k)$. Then for any $j\ge i\ge 2$, we have that

$$\begin{aligned}&d_{0,1}(X_i,X_j)\\ &\le \frac{C_1(\mu,\Theta(0))}{3}\sum_{k=i}^{j-1}\left(2F_1(s_k)^{1/2}+F_1(s_{k-1})^{1/2}\right) \le C_1(\mu,\Theta(0))\sum_{k=i-1}^{j-1}F_1(R_k)^{1/2}\\ &\le \frac{C_1(\mu,\Theta(0))}{\ln 2}\sum_{k=i-1}^{j-1}\int_{R_k}^{R_{k-1}}\frac{F_1(t)^{1/2}}{t}dt = \frac{C_1(\mu,\Theta(0))}{\ln 2}\int_{R_{i-2}}^{R_{j-1}}\frac{F_1(t)^{1/2}}{t}dt.\end{aligned}$$

If the gauge function h satisfy that

$$\int_0^{R_0} \frac{F_1(t)^{1/2}}{t} dt < +\infty, \tag{4.4}$$

then X_k converges to a cone $X(0)$, and

$$d_{0,1}(X(0), X_k) \le \frac{C_1(\mu, \Theta(0))}{\ln 2} \int_0^{R_{k-2}} \frac{F_1(t)^{1/2}}{t} dt.$$

Remark 4.12. If $h(r) \le C(\ln(A/r))^{-b}$, $0 < r \le R_0$, for some $A > R_0$, $C > 0$ and $b > 3$, then (4.4) holds.

Indeed,

$$h_1(r) = \int_0^r \frac{h(2t)}{t} dt \le \frac{C}{b-1} \left(\ln \left(\frac{A}{r} \right) \right)^{-b+1},$$

and then Remark 3.17 implies that

$$F(r) \le C_1 \left(\ln \left(\frac{A}{r} \right) \right)^{-b} + \frac{C}{b-1} \left(\ln \left(\frac{A}{r} \right) \right)^{-b+1} \le C_2 \left(\ln \left(\frac{A}{r} \right) \right)^{-b+1},$$

thus (4.4) holds.

Lemma 4.13. *If* (4.4) *holds, then* $X(0)$ *is a minimal cone.*

Proof. By Lemma 3.8, for any $r \in (0, \mathfrak{r}_1) \cap \mathscr{R}$, there exist sliding minimal cone $Z(r)$ such that $d_{0,1}(X(r), Z(r)) \le 4\varepsilon(r)$. But $\varepsilon(r) \to 0$ as $r \to 0+$, we get that

$$d_{0,1}(Z(s_k), X(0)) \to 0.$$

Since $Z(s_k)$ is sliding minimal for any k, we get that $X(0)$ is also sliding minimal. □

For any $r \in \mathscr{R} \cap (0, R_0)$ with $\Xi(r) \le \mu\tau_0$, we assume $R_{k+1} \le r < R_k$, by Lemma 4.7, we have that

$$\begin{aligned} d_{0,1}(X(0), X(r)) &\le d_{0,1}(X(0), X_{k+3}) + d_{0,1}(X_{k+3}, X(r)) \\ &\le \frac{C_1(\mu, \Theta(0))}{\ln 2} \int_0^{R_{k+1}} \frac{F_1(t)^{1/2}}{t} dt \\ &\quad + \frac{30r}{s_{k+3}} \Theta(0, r)^{1/2} F_1(r)^{1/2} + 20\pi\mu^{-1/2} \left(\Xi(s_{k+3})^{1/2} + \Xi(r)^{1/2} \right) \\ &\le 10 C_1(\mu, \Theta(0)) \left(\Xi(r)^{1/2} + F_1(r)^{1/2} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right). \end{aligned} \tag{4.5}$$

Theorem 4.14. *If* (4.4) *holds, and* E *is Ahlfors-regular, then* E *has unique blow-up limit* $X(0)$ *at* 0, *and there is a constant* $C = C_{10}(\mu, \Theta, \xi_0)$ *such that*

$$d_{0,9r/10}(E, X(0)) \le C \left(F_1(r)^{1/4} + \int_0^r \frac{F(t)^{1/2}}{t} dt \right), \quad 0 < r < R_0, \tag{4.6}$$

where $0 < R_0 < (1-\tau)\mathfrak{r}_1$ *satisfying that* $F(R_0) \le \mu\tau_0/4$ *and* $\bar{\varepsilon}(R_0) \le 10^{-4}$. *In particular,*

- *if* $h(r) \le C_h(\ln(A/r))^{-b}$ *for some* $A, C_h > 0$, $b > 3$ *and* $0 < r \le R_0 < A$, *then*

$$d_{0,r}(E, X(0)) \le C'(\ln(A_1/r))^{-(b-3)/4},\ 0 < r \le 9R_0/10,\ A_1 \le 10A/9;$$

- *if* $h(r) \le C_h r^{\alpha_1}$ *for some* $C_h, \alpha_1 > 0$, *and* $0 < r \le r_0$, $0 < r_0 \le \min\{1, R_0\}$, *then*

$$d_{0,r}(E, X(0)) \le C(r/r_0)^{\beta},\ 0 < r \le 9r_0/10,\ 0 < \beta < \alpha_1,$$

where

$$C \le C_{11}(\mu, \lambda_0, \alpha_1, \beta, C_h, \xi_0, \Theta(0)) \left(F(r_0)^{1/4} + r_0^{\alpha_1/4}\right).$$

Proof. From (4.5) and Lemma 4.9, we get that, for any $x \in X(0) \cap B_r$ where $r \in \mathscr{R} \cap (0, R_0)$ such that $\Xi(r) \le \mu\tau_0$,

$$\operatorname{dist}(x, E) \le C_7(\mu, \xi_0, \Theta(0)) r \left(\Xi(r)^{1/2} + F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt\right).$$

Similarly to the proof of Lemma 4.9, we still consider

$$I(a,b) = \left\{t \in [a,b] \middle| \Xi(t) \le F_1(r)^{2/3}\right\},\ 0 \le a \le b \le r,$$

we have that $I(a,b) \ne \emptyset$ whenever $b - a > rF_1(r)^{1/3}$. We let $\{s_i\}_0^{m+1} \subseteq [0, r]$ be a sequence such that

$$0 = s_0 < s_1 < \cdots < s_m < s_{m+1} = r,\ s_i \in I(0, r),$$

and

$$s_{i+1} - s_i \le 2rF_1(r)^{1/3}.$$

For any $r \in (0, R_0)$, we assume that $s_i \le r < s_{i+1}$, $x \in X(0) \cap \partial B_r$.

$$\begin{aligned} \operatorname{dist}(x, E) &\le \left| x - \frac{s_i}{|x|} x \right| + \operatorname{dist}\left(\frac{s_i}{|x|} x, E\right) \\ &\le C_8(\mu, \xi_0, \Theta(0)) r \left(F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt\right) \end{aligned} \tag{4.7}$$

From (4.5) and Lemma 4.11, we have that, for any $x \in X(0) \cap B(0, 9r/10)$ where $r \in \mathscr{R} \cap (0, R_0)$ such that $\Xi(r) \le \mu\tau_0$,

$$\operatorname{dist}(x, X(0)) \le C_9(\mu, \xi_0, \Theta(0)) \left(\Xi(r)^{1/2} + F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt\right).$$

Similarly to the proof of (4.7), we can get that

$$\operatorname{dist}(x, X(0)) \le C_{10}(\mu, \xi_0, \Theta(0)) \left(F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt\right). \tag{4.8}$$

We get, from (4.7) and (4.8), that (4.6) holds.

If $h(r) \le C_h(\ln(A/r))^{-b}$ for some $A, C_h > 0$ and $b > 3$ and $0 < r \le R_0 < A$, then

$$h_1(r) = \int_0^r \frac{h(2t)}{t} dt \le \frac{C_h}{b-1} \left(\ln\left(\frac{A}{r}\right)\right)^{-b+1},$$

and by Remark 3.17 we have that

$$F(r) \leq C'' \left(\ln \frac{A}{r}\right)^{-b+1}$$

where

$$C'' \leq C(R_0, \lambda, b) \left(\ln \frac{A}{r}\right)^{-1} + \frac{C_1}{b-1} \leq C(R_0, \lambda, b) \left(\ln \frac{A}{R_0}\right)^{-1} + \frac{C_1}{b-1}$$

is bounded, thus

$$\int_0^r \frac{F_1(t)^{1/2}}{t} dt \leq C''' \left(\ln \frac{A}{r}\right)^{(-b+3)/2}$$

Hence we get that

$$d_{0,9r/10}(E, X(0)) \leq C_{10}(\mu, \xi_0, \Theta(0)) \left(F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt\right) \leq C' \left(\ln \frac{A}{r}\right)^{-\frac{b-3}{4}}.$$

If $h(r) \leq C_h r^{\alpha_1}$ for some $C_h, \alpha_1 > 0$ and $0 < r \leq r_0$, then

$$h_1(r) = \int_0^r \frac{h(2t)}{t} dt \leq \frac{C_h}{\alpha_1}(2r)^{\alpha_1}.$$

We see, from the proof of Corollary 3.16, that

$$f(r) \leq (f(r_0) + C_2(\alpha_1, \beta, \lambda_0) C_h r_0^{\alpha_1}) (r/r_0)^\beta, \ \forall 0 < \beta < \alpha_1,$$

thus

$$F_1(r) = f(r) + 16 h_1(r) \leq (f(r_0) + C_2'(\alpha_1, \beta, \lambda_0) C_h r_0^{\alpha_1})(r/r_0)^\beta.$$

Then

$$d_{0,9r/10}(E, X(0)) \leq C_{10}(\mu, \xi_0, \Theta(0)) \left(F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt\right) \leq C(r/r_0)^{\beta/4},$$

where

$$C \leq C_{10}'(\mu, \xi_0, \Theta(0))(F(r_0)^{1/4} + C_2''(\alpha_1, \beta, \lambda_0, C_h) r_0^{1/4}).$$

□

5. Parameterization of well approximate sets

Recall that a cone in $\mathbb{R}^3$ is called of type $\mathbb{P}$ if it is a plane; a cone is called of type $\mathbb{Y}$ if it is the union of three half planes with common boundary line and that make $120°$ angles along the boundary line; a cone of type $\mathbb{T}$ if it is the cone over the union of the edges of a regular tetrahedron.

Theorem 5.1. *Let $E \subseteq \Omega_0$ be a set with $0 \in E$. Suppose that there exist $C > 0$, $r_0 > 0$, $\beta > 0$ and $0 < \eta \leq 1$ such that, for any $x \in E \cap B(0, r_0)$ and $0 < r \leq 2r_0$, we can find cone $Z_{x,r}$ through x such that*

$$d_{x,r}(E, Z_{x,r}) \leq C r^\beta,$$

where $Z_{x,r}$ is a minimal cone in $\mathbb{R}^3$ of type $\mathbb{P}$ or $\mathbb{Y}$ when $x \notin \partial\Omega_0$ and $0 < r < \eta \operatorname{dist}(x, \partial\Omega_0)$, and otherwise, $Z_{x,r}$ is a sliding minimal cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$ in Ω_0 with sliding boundary $\partial\Omega_0$ centered at some point in $\partial\Omega_0$. Then there exist a radius $r_1 \in (0, r_0/2)$, a sliding minimal cone Z centered at 0 and a mapping $\Phi : \Omega_0 \cap B(0, r_1) \to \Omega_0$, which is a $C^{1,\beta}$-diffeomorphism between its domain and image, such that $\Phi(0) = 0$, $\Phi(\partial\Omega_0 \cap B(0, 2r_1)) \subseteq \partial\Omega_0$, $\|\Phi - \mathrm{id}\,\|_\infty \leq 10^{-2} r_1$ and

$$E \cap B(0, r_1) = \Phi(Z) \cap B(0, r_1).$$

Proof. Let $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\sigma(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. By setting $E_1 = E \cup \sigma(E)$, we have that, for any $x \in E_1 \cap B(0, r_0)$ and $0 < r \leq 2r_0$, there exist minimal cone $Z(x, r)$ in $\mathbb{R}^3$ centered at x of type $\mathbb{P}$ or $\mathbb{Y}$ such that $Z(\sigma(x), r) = \sigma(Z(x, r))$ and

$$d_{x,r}(E, Z(x, r)) \leq Cr^{\beta}.$$

By Theorem 4.1 in [9], there exist $r_1 \in (0, r_0)$, $\tau \in (0, 1)$, a cone Z centered at 0 of type $\mathbb{P}$ or $\mathbb{Y}$, and a mapping $\Phi_1 : B(0, 3r_1/2) \to B(0, 2r_1)$ such that

$$\begin{gathered}\sigma(Z) = Z,\ \sigma \circ \Phi_1 = \Phi_1 \circ \sigma,\ \|\Phi_1 - \mathrm{id}\,\| \leq r_0\tau,\\ C_1|x - y|^{1+\tau} \leq |\Phi(x) - \Phi(y)| \leq C_1^{-1}|x - y|^{1/(1+\tau)},\\ E_1 \cap B(0, r_1) \subseteq \Phi_1(Z \cap B(0, 3r_1/2)) \subseteq E_1 \cap B(0, 2r_1).\end{gathered}$$

Using the same argument as in Section 10 in [3], we get that Φ_1 is of class $C^{1,\beta}$. □

6. Approximation of E by cones away from the boundary

In this section, we let $\Omega \subseteq \mathbb{R}^3$ be a closed set. Let $E \in SAM(\Omega, \partial\Omega, h)$ be a sliding almost minimal set, $x_0 \in E \setminus \partial\Omega$. Then $E \cap B(x, r)$ is almost minimal with gauge function h for any $0 < r < \mathrm{dist}(x_0, \partial\Omega)$. We put

$$F(x, r) = \Theta(x, r) - \Theta(x) + 8h_1(r).$$

We see from Theorem 2.3 that $F(x, r) \geq 0$ and $F(x, \cdot)$ is nondecreasing for $0 < r < \mathrm{dist}(x_0, L)$.

Theorem 6.1. *If $\int_0^{R_0} r^{-1}F(x, r)^{1/3}dr < \infty$ for some $R_0 > 0$, then E has unique blow-up limit T at x. Moreover there is a constant $C > 0$ and a radius $\rho_0 = \rho_0(x) > 0$ such that*

$$d_{x,r}(E, T) \leq C\int_0^{200r} \frac{F(x, t)^{1/3}}{t}dt,\ 0 < r \leq \rho_0.$$

In particular, if the gauge function h satisfies that

$$h(t) \leq C_h t^{\alpha_1} \text{ for some } \alpha_1 > 0 \text{ and } 0 < t \leq R_0,$$

then there exists $\beta_0 > 0$ such that, for any $0 < \beta < \beta_0$,

$$d_{x,r}(E, T) \leq C(\alpha_1, \beta)\left(F(x, \rho_0) + C_h\rho_0^{\alpha_1}\right)^{1/3}(r/\rho_0)^{\beta/3}.$$

Proof. By Theorem 16.1 in [4], we get that E is a locally C^{0,α_2}-equivalent to a two dimensional minimal cone for some $0 < \alpha_2 < 1$. Let ϱ be the radius defines as in (3.2). We take $\rho_0 = 10^{-3}\min\{R_0, \mathrm{dist}(x_0, \partial\Omega), \varrho\}$. By Theorem 11.4 in [5], there is a constant $C > 0$ and cone Z_r for each $0 < r < \rho_0$ such that

$$d_{x,r}(E, Z_r) \leq CF(x, 110r)^{1/3}.$$

We put $\rho_k = 2^{-k}\rho_0$, and $Z_k = Z_{\rho_k}$. Then

$$\begin{aligned}d_{x,1}(Z_k, Z_{k+1}) = d_{x,\rho_{k+1}}(Z_k, Z_{k+1}) &\leq d_{x,\rho_{k+1}}(Z_k, E) + d_{x,\rho_{k+1}}(E, Z_{k+1})\\ &\leq CF(x, 110\rho_{k+1})^{1/3} + 2CF(x, 110\rho_k)^{1/3}.\end{aligned}$$

For any $1 \le i < j$, we have that

$$\begin{aligned} d_{x,1}(Z_i, Z_j) &\le 2C\sum_{k=i}^{j-1} F(x,110\rho_k)^{1/3} + C\sum_{k=i+1}^{j} F(x,110\rho_k)^{1/3} \\ &\le 3C\sum_{k=i}^{j} F(x,110\rho_k)^{1/3} \\ &\le \frac{3C}{\ln 2}\int_{\rho_j}^{\rho_{i-1}} \frac{F(x,110t)^{1/3}}{t}dt. \end{aligned}$$

Let Z_0 be the limit of $\{Z_k\}_{k=1}^{\infty}$. Then we have that

$$d_{x,1}(Z_0, Z_i) \le \frac{3C}{\ln 2}\int_{0}^{\rho_{i-1}} \frac{F(x,110t)^{1/3}}{t}dt.$$

For any $0 < r < \rho_0$, we assume that $\rho_{k+1} \le r < \rho_k$, then

$$\begin{aligned} d_{x,1}(Z_r, Z_0) &\le d_{x,\rho_{k+1}}(Z_r, Z_{k+1}) + d_{x,1}(Z_{k+1}, Z_0) \\ &\le d_{x,1}(Z_{k+1}, Z_0) + d_{x,\rho_{k+1}}(Z_r, E) + d_{x,\rho_{k+1}}(E, Z_{k+1}) \\ &\le d_{x,1}(Z_{k+1}, Z_0) + \frac{r}{\rho_{k+1}} d_{x,r}(Z_r, E) + d_{x,\rho_{k+1}}(E, Z_{k+1}) \\ &\le 3CF(x,110r)^{1/3} + \frac{3C}{\ln 2}\int_0^{\rho_k} \frac{F(x,110t)^{1/3}}{t}dt. \end{aligned}$$

Hence

$$d_{x,r}(E, Z_0) \le d_{x,r}(E, Z_r) + d_{x,r}(Z_r, Z_0) \le \frac{10C}{\ln 2}\int_0^{200r} \frac{F(x,t)^{1/3}}{t}dt \tag{6.1}$$

and $T = \boldsymbol{\tau}_x(Z_0)$ is the only blow up limit of E at x, which is a minimal cone.

By Theorem 4.5 in [5], we have that

$$\Theta_E(x,r) \le \left(\frac{1}{2} - \alpha_0\right)\frac{\mathcal{H}^1(E\cap \partial B(x,r))}{r} + 2\alpha_0\Theta_E(x) + 4h(r),$$

where we take α_0 the constant α in Theorem 4.5 in [5]. For our convenient, we denote $u(r) = \mathcal{H}^2(E\cap B(x,r))$ and $f(r) = \Theta_E(x,r) - \Theta_E(x)$, then we have $\mathcal{H}^1(E\cap\partial B(x,r)) \le u'(r)$ and

$$\begin{aligned} f(r) + \Theta_E(x) &\le \left(\frac{1}{2} - \alpha_0\right)\frac{u'(r)}{r} + 2\alpha_0\Theta_E(x) + 4h(r) \\ &= \left(\frac{1}{2} - \alpha_0\right)(2f(r) + rf'(r) + 2\Theta_E(x)) + 2\alpha_0\Theta_E(x) + 4h(r), \end{aligned}$$

thus

$$rf'(r) \ge \frac{4\alpha_0}{1-2\alpha_0}f(r) - \frac{8}{1-2\alpha_0}h(r),$$

and

$$\left(r^{-\frac{4\alpha_0}{1-2\alpha_0}}f(r)\right)' \ge -\frac{8}{1-2\alpha_0}r^{-\frac{1+2\alpha_0}{1-2\alpha_0}}h(r).$$

We take $\beta_0 = \min\{4\alpha_0/(1-2\alpha_0), \alpha_1\}$. Then for any $0 < \beta < \beta_0$, we have that

$$\begin{aligned} f(r) &\le (r/\rho_0)^{\frac{4\alpha_0}{1-2\alpha_0}}f(\rho_0) + \frac{8}{1-2\alpha_0}r^{\frac{4\alpha_0}{1-2\alpha_0}}\int_r^{\rho_0} t^{-\frac{1+2\alpha_0}{1-2\alpha_0}}h(t)dt \\ &\le (r/\rho_0)^{\frac{4\alpha_0}{1-2\alpha_0}}f(\rho_0) + C_1'(\alpha_1,\beta,\alpha_0)\rho_0^{\alpha_1}\cdot(r/\rho_0)^{\beta}. \end{aligned}$$

We get so that

$$F(x,r) \le C(\alpha_1, \beta, \alpha_0)(F(x,\rho_0) + C_h \rho_0^{\alpha_1})(r/\rho_0)^{\beta},$$

combine this with (6.1), we get the conclusion. □

7. Parameterization of sliding almost minimal sets

Let n, $d \le n$ and k be nonnegative integers, $\alpha \in (0,1)$. By a d-dimensional submanifold of class $C^{k,\alpha}$ of $\mathbb{R}^n$ we mean a subset M of $\mathbb{R}^n$ satisfying that for each $x \in M$ there exist s neighborhood U of x in $\mathbb{R}^n$, a mapping $\Phi : U \to \mathbb{R}^n$ which is a diffeomorphism of class $C^{k,\alpha}$ between its domain and image, and a d dimensional vector subspace Z of $\mathbb{R}^n$ such that

$$\Phi(M \cap U) = Z \cap \Phi(U).$$

In Section 4, we get the estimation $d_{x,r}(E, Z) \le Cr^{\beta}$ for $x \in E \cap \partial\Omega$ and $0 < r < r(x)$, where Ω is a half space, E is locally sliding almost minimal at x, and $r(x) > 0$ depends on x. In Section 6, we get the estimation $d_{x,r}(E, Z) \le Cr^{\beta}$ for $x \in E \setminus \partial\Omega$ and $0 < r < r(x)$.

In this section, we assume that $\Omega \subseteq \mathbb{R}^3$ is a closed set whose boundary $\partial\Omega$ is a 2-dimensional submanifold of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$, and suppose that Ω has tangent cone a half space at any point in $\partial\Omega$. We will show that Ω is locally $C^{1,\alpha}$ diffeomorphic to a half space at any point $x_0 \in \partial\Omega$, see Lemma 7.1, and after the diffeomorphism Ψ, $\Psi(E)$ become a locally sliding almost set at 0, see Lemma 7.2, so we can apply the results in Section 4 to see that the estimation $d_{x,r}(E, Z) \le Cr^{\beta}$ for $x \in E \cap \partial\Omega$ and $0 < r < r(x)$ is still valid, see Theorem 7.4. But the problem is that $r(x)$ depends on x. In fact, we need a uniform control of radius $r(x)$ to apply the Reifenberg's parameterization theorem, Theorem 5.1, to get our main result Theorem 1.2, and that will be done in Lemma 7.9 and Lemma 7.10.

Let $E \subseteq \Omega$ be a closed set such that $E \in SAM(\Omega, \partial\Omega, h)$ and $\partial\Omega \subseteq E$, $x_0 \in \partial\Omega$. We always assume that the gauge function h satisfies that

$$\int_0^{R_0} \frac{1}{r} \left(\int_0^r \frac{h(2t)}{t} dt \right)^{1/2} dr < +\infty \tag{7.1}$$

and

$$\int_0^{R_0} r^{-1+\frac{\lambda}{1-\lambda}} \left(\int_r^{R_0} t^{-1-\frac{2\lambda}{1-\lambda}} h(2t) dt \right)^{1/2} dr < +\infty, \tag{7.2}$$

for some $R_0 > 0$, where λ is the same constant as in Theorem 3.15. It is easy to see that if $h(t) \le Ct^{\alpha_1}$ for some $\alpha_1 > 0$, $C > 0$ and $0 < t \le R_0$, then (7.1) and (7.2) hold. For our convenient, we still put $\lambda_0 = \lambda/(1-\lambda)$, and put

$$h_2(\rho) = \int_0^{\rho} \frac{1}{r} \left(\int_0^r \frac{h(2t)}{t} dt \right)^{1/2} dr,$$

$$h_3(\rho) = \int_0^{\rho} r^{-1+\lambda_0} \left(\int_r^{R_0} t^{-1-2\lambda_0} h(2t) dt \right)^{1/2} dr.$$

We see, from Proposition 4.1 in [6], that E is Ahlfors-regular in $B(x_0, R_0)$, i.e. there exist $\delta_1 > 0$ and $\xi_1 \ge 1$ such that for any $x \in E \cap B(x_0, R_0)$, if $0 < r < \delta_1$

and $B(x, r) \subseteq B(x_0, R_0)$, we have that

$$\xi_1^{-1} r^2 \leq \mathcal{H}^2(E \cap B(x, r)) \leq \xi_1 r^2.$$

We see from Theorem 3.10 in [9] that there only there kinds of possibility for the blow-up limits of E at x_0, they are the plane $\mathrm{Tan}(\partial\Omega, x_0)$, cones of type $\mathbb{P}_+$ union $\mathrm{Tan}(\partial\Omega, x_0)$, and cones of type $\mathbb{Y}_+$ union $\mathrm{Tan}(\partial\Omega, x_0)$. By Proposition 29.53 in [6], we get so that

$$\Theta_E(x_0) = \pi, \ \frac{3\pi}{2}, \text{ or } \frac{7\pi}{4}.$$

If $\Theta_E(x_0) = \pi$, then there is a neighborhood U_0 of x_0 in $\mathbb{R}^3$ such that $E \cap U_0 = \partial\Omega \cap U_0$, see Lemma 5.2 in [9]. In the next content of this section, we put ourself in the case $\Theta_E(x_0) = 3\pi/2$ or $7\pi/4$.

By Theorem 4.14 and Theorem 1.15 in [5], we see that, for any $x \in E$, there is unique blow-up limit of E at x, which coincide with the tangent cone $\mathrm{Tan}(E, x)$.

Lemma 7.1. *For any $R_0 > 0$, there exist $r_0 = r_0(x_0) > 0$ and a mapping $\Psi = \Psi_{x_0} : B(0, r_0) \to \mathbb{R}^3$, which is a diffeomorphism of class $C^{1,\alpha}$ from $B(0, r_0)$ to $\Psi(B(0, r_0))$, such that*

$$\Psi(0) = x_0, \Psi(\Omega_0 \cap B_{r_0}) \subseteq \Omega \cap B(x_0, R_0), \Psi(L_0 \cap B_{r_0}) \subseteq \partial\Omega \cap B(x_0, R_0),$$

and that $D\Psi(0)$ is a rotation satisfying that

$$D\Psi(0)(\Omega_0) = \mathrm{Tan}(\Omega, x_0) \text{ and } D\Psi(0)(L_0) = \mathrm{Tan}(\partial\Omega, x_0).$$

Proof. By definition, there exist open sets $U, V \subseteq \mathbb{R}^3$ and a diffeomorphism $\Phi : U \to V$ of class $C^{1,\alpha}$ such that $x_0 \in U$, $0 = \Phi(x_0) \in V$ and

$$\Phi(U \cap \partial\Omega) = Z \cap V,$$

where Z is a plane through 0. Indeed, we have that

$$Z = D\Phi(x_0)\, \mathrm{Tan}(\partial\Omega, x_0)$$

and

$$\Phi(U \cap \Omega) = V \cap D\Phi(x_0)\, \mathrm{Tan}(\Omega, x_0).$$

We will denote by A the linear mapping given by $A(v) = D\Phi(x_0)^{-1} v$, and assume that $A(V) = B(0, r)$ is a ball. Let Φ_1 be a rotation such that $\Phi_1(\mathrm{Tan}(\partial\Omega, x_0)) = L_0$ and $\Phi_1(\mathrm{Tan}(\Omega, x_0)) = \Omega_0$. Then we get that $\Phi_1 \circ A \circ \Phi$ is also $C^{1,\alpha}$ mapping which is a diffeomorphism between U and $B(0, r)$,

$$D(\Phi_1 \circ A \circ \Phi)(x_0)\, \mathrm{Tan}(\Omega, x_0) = \Phi_1(\mathrm{Tan}(\Omega, x_0)) = \Omega_0,$$
$$D(\Phi_1 \circ A \circ \Phi)(x_0)\, \mathrm{Tan}(\partial\Omega, x_0) = \Phi_1(\mathrm{Tan}(\partial\Omega, x_0)) = L_0,$$

and

$$\Phi_1 \circ A \circ \Phi(U \cap \partial\Omega) = \Phi_1 \circ A(Z \cap V) = L_0 \cap B(0, r),$$
$$\Phi_1 \circ A \circ \Phi(U \cap \partial\Omega) = \Phi_1 \circ A(V \cap D\Phi(x_0)\, \mathrm{Tan}(\Omega, x_0)) = \Omega_0 \cap B(0, r).$$

We now take $r_0 = r$ and $\Psi = (\Phi_1 \circ A \circ \Phi)^{-1}|_{B(0,r)}$ to get the result. □

Let $U \subseteq \mathbb{R}^n$ be an open set. For any mapping $\Psi : U \to \mathbb{R}^n$ of class $C^{1,\alpha}$, we will denote by C_Ψ the constant $C_\Psi = \sup\{\|D\Psi(x) - D\psi(y)\|/|x - y|^\alpha : x, y \in U, x \neq y\}$. Then we have that

$$\Psi(x) - \Psi(y) = \left\langle x - y, \int_0^1 D\Psi(y + t(x - y))dt \right\rangle,$$

and thus

$$(7.3)\quad |\Psi(x)-\Psi(y)-D\Psi(y)(x-y)| \le |x-y| \int_0^1 C_\Psi (t|x-y|)^\alpha dt \le \frac{C_\Psi}{\alpha+1}|x-y|^{1+\alpha}.$$

For any $0<\rho\le r_0$, we set $U_\rho = \Psi(B_\rho)$, $M_\rho = \Psi^{-1}(E\cap U_\rho)$ and

$$(7.4)\qquad \Lambda(\rho) = \max\left\{ \mathrm{Lip}\left(\Psi_{B_\rho}\right), \mathrm{Lip}\left(\Psi^{-1}_{U_\rho}\right)\right\}.$$

Then

$$\|D\Psi(0)\| - \|D\Psi(x)-D\Psi(0)\| \le \|D\Psi(x)\| \le \|D\Psi(0)\| + \|D\Psi(x)-D\Psi(0)\|,$$

thus $1-C_\Psi\rho^\alpha \le \|D\Psi(x)\| \le 1+C_\Psi\rho^\alpha$ for $x\in B_\rho$, and we have that

$$(7.5)\qquad \Lambda(\rho)\le 1/(1-C_\Psi\rho^\alpha) \text{ whenever } C_\Psi\rho^\alpha<1.$$

Lemma 7.2. *For any $1<\rho\le\min\{r_0, C_\Psi^{-1/\alpha}\}$, M_ρ is local almost minimal in B_ρ at 0 with gauge function H satisfying that*

$$H(2r)\le 4\Lambda(r)^2 h(2\Lambda(r)r) + 4\xi_1 C_\Psi \Lambda(\rho) r^\alpha \text{ for } 0<r<(1-C_\Psi\rho^\alpha)\delta_1.$$

Proof. For any open set $U\subseteq\mathbb{R}^3$, $M\ge 1$, $\delta>0$ and $\epsilon>0$, we let $GSAQ(U,M,\delta,\epsilon)$ be the collection of generalized sliding Almgren quasiminimal sets which is defined in Definition 2.3 in [6]. We see that

$$\mathrm{diam}(U_\rho)\le 2\rho\,\mathrm{Lip}\left(\Psi|_{B_\rho}\right)\le 2\rho\Lambda(\rho)$$

and

$$E\cap U_\rho\in GSAQ(U_\rho, 1, \mathrm{diam}(U_\rho), h(2\,\mathrm{diam}(U_\rho))),$$

By Proposition 2.8 in [6], we have that

$$M_\rho\in GSAQ\left(B_\rho, \Lambda(\rho)^4, 2\rho, \Lambda(\rho)^4 h\left(2\rho\Lambda(\rho)\right)\right)$$

By Proposition 4.1 in [6], we get that M_ρ is Ahlfors-regular in B_ρ. Indeed, we can get a little more, that is, for any $x\in M_\rho$ with $0<r\Lambda(\rho)<\delta_1$ and $B(x,r)\subseteq B(0,\rho)$, we have that

$$(7.6)\qquad (\xi_1\Lambda(\rho))^{-1} r^2 \le \mathcal{H}^2(M_\rho\cap B(x,r)) \le (\xi_1\Lambda(\rho))\, r^2.$$

Let $\{\varphi_t\}_{0\le t\le 1}$ be any sliding deformation of M_ρ in B_r. Then

$$\left\{\Psi\circ\varphi_t\circ\Psi^{-1}\right\}_{0\le t\le 1}$$

is a sliding deformation of E in U_r. Hence we get that

$$(7.7)\qquad \mathcal{H}^2(E\cap U_r)\le \mathcal{H}^2(\Psi\circ\varphi_1\circ\Psi^{-1}(E\cap U_r)) + h(2\,\mathrm{diam}(U_r))^2\,\mathrm{diam}(U_r)^2$$

For any 2-rectifiable set $A\subseteq B_\rho$, by Theorem 3.2.22 in [10], we have that

$$\mathrm{ap}\, J_2(\Psi|_A)(x) = \left\|\wedge_2\left(D\Psi(x)|_{\mathrm{Tan}(A,x)}\right)\right\|$$

and

$$\mathcal{H}^2(\Psi(A\cap B_r)) = \int_{A\cap B_r} \mathrm{ap}\, J_2(\Psi|_A)(x) d\mathcal{H}^2(x)$$

By (7.5), we get that

$$\int_{A\cap B_r}(1-C_\Psi|x|^\alpha)^2 d\mathcal{H}^2 \le \mathcal{H}^2(\Psi(A\cap B_r)) \le \int_{A\cap B_r}(1+C_\Psi|x|^\alpha)^2 d\mathcal{H}^2.$$

Thus, by taking $A = M_\rho$, we have that $M_r = M_\rho \cap B_r$, $\Psi(M_r) = E \cap U_r$ and

$$\mathcal{H}^2(\Psi(M_r)) \geq (1 - C_\Psi \rho^\alpha)^2 \mathcal{H}^2(M_r);$$

by taking $A = \varphi_1(M_\rho)$, we have that

$$\mathcal{H}^2(\Psi(\varphi_1(M_\rho) \cap B_r)) \leq (1 + C_\Psi r^\alpha)^2 \mathcal{H}^2(\varphi_1(M_\rho) \cap B_r).$$

Combine these two equations with (7.7) and (7.6), we get that

$$\begin{aligned}
\mathcal{H}^2(\varphi_1(M_\rho) \cap B_r) &\geq (1 + C_\Psi r^\alpha)^{-2} \mathcal{H}^2(\Psi(\varphi_1(M_\rho) \cap B_r)) \\
&\geq (1 + C_\Psi r^\alpha)^{-2} \left(\mathcal{H}^2(E \cap U_r) - h(4r\Lambda(r))(2r\Lambda(r))^2\right) \\
&\geq \left(\frac{1 - C_\Psi \rho^\alpha}{1 + C_\Psi r^\alpha}\right)^2 \mathcal{H}^2(M_r) - \left(\frac{2r\Lambda(r)}{1 + C_\Psi r^\alpha}\right)^2 h(4r\Lambda(r)) \\
&\geq \mathcal{H}^2(M_r) - H(2r) r^2.
\end{aligned}$$

□

Lemma 7.3. *Let $E_1 \subseteq \Omega_0$ be a 2-rectifiable set, $x \in E_1$, X a cone centered at 0, $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ a diffeomorphism of class $C^{1,\alpha}$. Then there exist $C > 0$ such that, for any $r > 0$ and $\rho > 0$ with $B(\Phi(x), \rho) \subseteq \Phi(B(x, r))$,*

$$d_{\Phi(x),\rho}\left(\Phi(E_1), \Phi(x) + D\Phi(x)X\right) \leq \left(Cr^\alpha + \|D\Phi(x)\| d_{x,r}(E_1, x + X)\right) \frac{r}{\rho}.$$

Proof. Since Φ is of class $C^{1,\alpha}$, by (7.3), we have that

$$|\Phi(y) - \Phi(x) - D\Phi(x)(y - x)| \leq \frac{C_\Phi}{\alpha + 1} |x - y|^{1+\alpha},$$

by putting $C_1 = C_\Phi/(\alpha + 1)$, we get that

$$\operatorname{dist}(\Phi(y), \Phi(x) + D\Phi(x)X) \leq C_1 |y - x|^{1+\alpha} \text{ for } y \in x + X.$$

For any $z \in E_1 \cap B_r$ and $y \in x + X$, we have that

$$\begin{aligned}
|\Phi(z) - \Phi(y)| &\leq |\Phi(z) - \Phi(y) - D\Phi(x)(z - y)| + \|D\Phi(x)\| \cdot |z - y| \\
&\leq \|D\Phi(x)\| \cdot |z - y| + C_1 |z - x|^{1+\alpha} + C_1 |y - x|^{1+\alpha},
\end{aligned}$$

thus

$$\operatorname{dist}(\Phi(z), \Phi(x + X)) \leq \|D\Phi(x)\| r d_{x,r}(E_1, x + X) + 2C_1 r^{1+\alpha},$$

hence

$$\operatorname{dist}(\Phi(z), \Phi(x) + D\Phi(x)X) \leq \|D\Phi(x)\| r d_{x,r}(E_1, x + X) + 3C_1 r^{1+\alpha}. \tag{7.8}$$

For any $z \in X \cap B_r$, $\Phi(x) + D\Phi(x)z \in \Phi(x) + D\Phi(x)X$, and

$$\begin{aligned}
\operatorname{dist}(\Phi(x) + D\Phi(x)z, \Phi(E_1)) &= \inf\{|\Phi(y) - \Phi(x) - D\Phi(x)z| : y \in E_1\} \\
&\leq \inf\{C_1 r^{1+\alpha} + \|D\Phi(x)\| \cdot |y - x - z| : y \in E_1\} \\
&\leq \|D\Phi(x)\| r d_{x,r}(x + X, E_1) + C_1 r^{1+\alpha}.
\end{aligned} \tag{7.9}$$

We get from (7.8) and (7.9) that

$$d_{\Phi(x),\rho}(\Phi(E_1), \Phi(x) + D\Phi(x)X) \leq \frac{r}{\rho} \left(3C_1 r^\alpha + \|D\Phi(x)\| \cdot d_{x,r}(E_1, x + X)\right)$$

□

Theorem 7.4. *Let Ω, $E \subseteq \Omega$, $x_0 \in \partial\Omega$ and h be the same as in the beginning of this section. Then there is a unique blow-up limit X of E at x_0; moreover, if the gauge function h satisfy that*

$$h(t) \leq C_h t^{\alpha_1} \text{ for some } C_h > 0, \alpha_1 > 0 \text{ and } 0 < t < t_0, \tag{7.10}$$

then there exists $\rho_0 > 0$ such that, for any $0 < \beta < \min\{\alpha, \alpha_1, 2\lambda_0\}$,

$$d_{x_0,\rho}(E, x_0 + X) \leq C(\rho/\rho_0)^{\beta/4},\ 0 < \rho \leq 9\rho_0/20,$$

where C is a constant satisfying that

$$C \leq C_{20}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1)(F_E(x_0, 2\rho_0) + C_\Psi \rho_0^\alpha + C_h \rho_0^{\alpha_1})^{1/4},$$

and $F_E(x_0, r) = r^{-2}\mathcal{H}^2(E \cap B(x_0, r)) - \Theta_E(x_0) + 16h_1(r)$.

Proof. We take $R_0 > 0$ such that $R_0 < (1-\tau)\mathfrak{r}_1$ and $\bar{\varepsilon}(R_0) \leq 10^{-4}$, let Ψ, r_0 be the same as in Lemma 7.1. Let $r \in (0, r_0)$ be such that $C_\Psi r^\alpha \leq 1/2$ and $2r \leq R_0$. Then $\Lambda(r) \leq 2$, see (7.4) and (7.5). By Lemma 7.2, we have that M_r is local almost minimal at 0 with gauge function H satisfying that

$$H(t) \leq 16h(2t) + C_r t^\alpha,\ 0 < t < r, \tag{7.11}$$

where $C_r \in (0, 2^{3-\alpha}\xi_1 C_\Psi)$ is a constant.

We put $f_{M_r}(\rho) = \Theta_{M_r}(0, \rho) - \Theta_{M_r}(0)$, $0 < \rho \leq r$. From (3.33) and (3.37), we get that

$$\begin{aligned}
f_{M_r}(\rho) &\leq \left(r^{-2\lambda_0} f_{M_r}(r)\right)\rho^{2\lambda_0} + 8(1+\lambda_0)\rho^{2\lambda_0}\int_\rho^r t^{-1-2\lambda_0} H(2t)dt \\
&\leq \left(r^{-2\lambda_0} f_{M_r}(r)\right)\rho^{2\lambda_0} + 2^{7+2\lambda_0}(1+\lambda_0)\rho^{2\lambda_0}\int_{2\rho}^{2r} \frac{h(2t)}{t^{1+2\lambda_0}}dt \\
&\quad + 2^{\alpha+3}(1+\lambda_0)C_r \cdot C_1(\alpha, \beta, \lambda_0) r^\alpha \cdot (\rho/r)^\beta,
\end{aligned}$$

where $C_1(\alpha, \beta, \lambda_0)$ is the constant in (3.37).

We get from (7.11) that

$$H_1(\rho) = \int_0^\rho \frac{H(2s)}{s}ds \leq 16h_1(2\rho) + \frac{C_r}{\alpha}(2\rho)^\alpha,$$

by setting $F_1(\rho) = f_{M_r}(\rho) + 16H_1(\rho)$, we have that

$$\begin{aligned}
F_1(\rho) &\leq C_{12}(\lambda_0, \alpha, \beta, r)(\rho/r)^\beta + 2^8 h_1(2\rho) + 2^{4+\alpha}C_r\alpha^{-1}\rho^\alpha \\
&\quad + 2^{7+2\lambda_0}(1+\lambda_0)\rho^{2\lambda_0}\int_{2\rho}^{2r}\frac{h(2t)}{t^{1+2\lambda_0}}dt,
\end{aligned}$$

where

$$C_{12}(\lambda_0, \alpha, \beta, r) \leq f_{M_r}(r) + 2^{\alpha+3}(1+\lambda_0)C_r C_1(\alpha, \beta, \lambda_0) r^\alpha.$$

Hence

$$\begin{aligned}
\int_0^t \frac{F_1(\rho)^{1/2}}{\rho}d\rho &\leq C_{12}(\lambda_0, \alpha, \beta, r)^{1/2}(\beta/2)(t/r)^\beta + 16h_2(2t) + C_{13}(\alpha, r)t^{\alpha/2} \\
&\quad + 2^{4+\lambda_0}(1+\lambda_0)^{1/2}\int_0^t \rho^{-1+\lambda_0}\left(\int_{2\rho}^{2r}\frac{h(2s)}{s^{1+2\lambda_0}}ds\right)^{1/2} d\rho,
\end{aligned}$$

where $C_{13}(\alpha, r) \le 2^{3+\alpha/2}\alpha^{-3/2}C_r^{1/2}$, thus

$$\int_0^t \frac{F_1(\rho)^{1/2}}{\rho} d\rho < +\infty, \text{ for } 0 < t \le r.$$

We now apply Theorem 4.14, there is a unique blow-up limit T of M_r at 0, thus there is a unique blow-up limit X of E at x_0.

For any $R \in (0, R_0)$, we put

$$f_E(x_0, R) = R^{-2}\mathcal{H}^2(E \cap B(x_0, R)) - \Theta_E(x_0)$$

and

$$F_E(x_0, R) = f_E(x_0, R) + 16h_1(R),$$

where $h_1(r) = \int_0^r t^{-1}h(2t)dt$. From (7.7) and $B(x_0, \rho/\Lambda(\rho)) \subseteq U_\rho \subseteq B(x_0, \rho\Lambda(\rho))$, we see that

$$(1-C_\Psi\rho^\alpha)^2(f_{M_r}(\rho)+\Theta_E(x_0)) \le \rho^{-2}\mathcal{H}^2(E\cap U_\rho) \le (1+C_\Psi\rho^\alpha)^2(f_{M_r}(\rho)+\Theta_E(x_0)),$$

since $\Lambda(\rho) \le 1/(1 - C_\Psi\rho^\alpha)$, we get so that

$$f_{M_r}(\rho) \le (1 - C_\Psi\rho^\alpha)^{-4}f_E(x_0, \rho\Lambda(\rho)) + 4\Theta_E(x_0)C_\Psi\rho^\alpha,$$

and

$$f_{M_r}(\rho) \ge (1 - C_\Psi^2\rho^{2\alpha})^2 f_E(x_0, \rho/\Lambda(\rho)) + 2\Theta_E(x_0)C_\Psi^2\rho^{2\alpha}.$$

Since $\rho < r$, $C_\Psi r^\alpha \le 1/2$, $h_1 \ge 0$, $\Theta_E(x_0) \le 7\pi/4$ and $\Lambda(r) \le 2$, we get that

$$f_{M_r}(\rho) \le (1-C_\Psi\rho^\alpha)^{-4}F_E(x, \rho\Lambda(\rho))+4\Theta_E(x_0)C_\Psi\rho^\alpha \le 16F_E(x, 2\rho)+(7\pi/2)(\rho/r)^\alpha,$$

and

$$C_{12}(\lambda_0, \alpha, \beta, r) \le 16F_E(x_0, 2r) + 9\xi_1 \cdot 2^{\alpha+2}(1 + \lambda_0)C_1(\alpha, \beta, \lambda_0) + 2\Theta_E(x_0).$$

If h satisfy (7.10), we take $0 < \rho_0 \le \min\{r, t_0, r_0(x_0), R_0/2, (2C_\Psi)^{-1/\alpha}, 1\}$ such that

$$F_E(x_0, 2\rho_0) \le 10^{-2}\mu\tau_0,\ h_1(2\rho) \le 10^{-2}\mu\tau_0 \text{ and } (\rho_0/r)^\alpha \le 10^{-2}\mu\tau_0, \tag{7.12}$$

then

$$h_1(\rho) \le \frac{C_h}{\alpha_1}(2\rho)^{\alpha_1},\ H_1(\rho) \le \frac{2^{4+2\alpha_1}C_h}{\alpha_1}\rho^{\alpha_1} + \frac{2^\alpha C_r}{\alpha}\rho^\alpha,\ 0 < \rho \le \rho_0,$$

and

$$F_1(\rho) \le C_{13}(\lambda_0, \alpha, \beta, \rho_0, C_h)(\rho/\rho_0)^\beta + 2^{8+\alpha_1}\alpha_1^{-1}C_h\rho^{\alpha_1} + C_{14}(\alpha, \xi_1, C_\Psi)\rho^\alpha, \tag{7.13}$$

where $C_{13}(\lambda_0, \alpha_1, \beta, \rho_0, C_h)$ and $C_{14}(\alpha, \xi_1, C_\Psi)$ are constant satisfying that

$$C_{13}(\lambda_0, \alpha_1, \beta, \rho_0, C_h) \le C_{12}(\lambda_0, \alpha, \rho_0) + 2^{7+4\alpha_1}(1 + \lambda_0)C_1(\alpha_1, \beta, \lambda_0)C_h\rho_0^{\alpha_1}$$

and

$$C_{14}(\alpha, \xi_1, C_\Psi) \le 2^{8+\alpha}\alpha^{-1}\xi_1 C_\Psi.$$

We get so that (7.13) can be rewrite as

$$F_1(\rho) \le C_{15}(\lambda_0, \alpha, \alpha_1, \beta, \xi_1)(F_E(x_0, 2\rho_0) + C_\Psi\rho_0^\alpha + C_h\rho_0^{\alpha_1})(\rho/\rho_0)^{\beta/4}.$$

By Theorem 4.14, we have that

$$\begin{aligned} d_{0,9\rho/10}(M_r, T) &\le C_{16}(\mu, \xi_0)\left(F_1(\rho)^{1/4} + \int_0^\rho \frac{F_1(t)^{1/2}}{t}dt\right) \\ &\le C_{17}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1)G_E(x_0, \rho_0)(\rho/\rho_0)^{\beta/4}, \end{aligned}$$

where

$$G_E(x_0,\rho_0) = (F_E(x_0, 2\rho_0) + C_\Psi \rho_0^\alpha + C_h\rho_0^{\alpha_1})^{1/4}.$$

Applying Lemma 7.3 with $\Phi = \Psi$, by setting $X = D\Psi(0)T$, we get that for any $\rho \in (0, 9\rho_0/10)$,

$$\begin{aligned} d_{x_0,\rho/2}(E, x_0 + X) &\le d_{x_0,\rho/\Lambda(\rho)}(E, x_0 + D\Psi(0)T) \\ &\le 6C_\Psi\rho^\alpha + 2d_{x,\rho}(M_r, T) \\ &\le 6C_\Psi\rho^\alpha + C_{18}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1)G_E(x_0, \rho_0)(\rho/\rho_0)^{\beta/4} \\ &\le C_{19}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1)G_E(x_0, \rho_0)(\rho/\rho_0)^{\beta/4}. \end{aligned}$$

□

Lemma 7.5. *For any $\tau > 0$ small enough, there exists $\varepsilon_2 = \varepsilon_2(\tau) > 0$ such that the following hold: E is an sliding almost minimal set in Ω with sliding boundary $\partial\Omega$ and gauge function h, $x_0 \in E \cap \partial\Omega$, Ψ is a mapping as in Lemma 7.1 and C_Ψ is the constant as in (7.5), if $r_1 > 0$ satisfy that $C_\Psi r_1^\alpha \le \varepsilon_2$, $h(2r_1) \le \varepsilon_2$ and $F_E(x_0, r_1) \le \varepsilon_2$, then for any $r \in (0, 9r_1/10)$, we can find sliding minimal cone $Z_{x_0,r}$ in $\mathrm{Tan}(\Omega, x_0)$ with sliding boundary $\mathrm{Tan}(\partial\Omega, x_0)$ such that*

$$\mathrm{dist}(x, Z_{x_0,r}) \le \tau r,\ x \in E \cap B(x_0, (1-\tau)r)$$
$$\mathrm{dist}(x, E) \le \tau r,\ x \in Z_{x_0,r} \cap B(x_0, (1-\tau)r),$$

and for any ball $B(x, t) \subseteq B(x_0, (1-\tau)r)$,

$$|\mathcal{H}^2(Z_{x_0,r} \cap B(x,t)) - \mathcal{H}^2(E \cap B(x,t))| \le \tau r^2.$$

Moreover, if $E \supseteq \partial\Omega$, then $Z_{x_0,r} \supseteq \mathrm{Tan}(\partial\Omega, x_0)$.

Proof. It is a consequence of Proposition 30.19 in [6]. □

Corollary 7.6. *Let Ω, $E \subseteq \Omega$, $x_0 \in \partial\Omega$, h and F_E be the same as in Theorem 7.4. Suppose that the gauge function h satisfying*

$$h(t) \le C_h t^{\alpha_1} \text{ for some } C_h > 0, \alpha_1 > 0 \text{ and } 0 < t < t_0.$$

Then there exists $\delta > 0$ and constant $C = C_{20}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1) > 0$ for $0 < \beta < \min\{\alpha, \alpha_1, 2\lambda_0\}$ such that, whenever

$$0 < \rho_0 \le \min\left\{r, t_0, r_0(x_0), R_0/2, (2C_\Psi)^{-1/\alpha}, 1, (1-\tau)\mathfrak{r}_1\right\}$$

satisfying

$$F_E(x_0, 2\rho_0) + C_\Psi\rho_0^\alpha + C_h\rho_0^{\alpha_1} \le \delta,$$

we have that, for $0 < \rho \le 9\rho_0/20$,

$$d_{x_0,\rho}(E, x_0 + \mathrm{Tan}(E, x_0)) \le C(F_E(x_0, 2\rho_0) + C_\Psi\rho_0^\alpha + C_h\rho_0^{\alpha_1})^{1/4}(\rho/\rho_0)^{\beta/4}.$$

Proof. By Theorem 7.4, there exist $\rho_0 > 0$ such that

$$d_{x_0,\rho}(E, x_0 + \mathrm{Tan}(E, x_0)) \le C(\rho/\rho_0)^{\beta/4},\ 0 < \rho \le 9\rho_0/20,$$

where $\rho_0 > 0$ is chosen to be as in Theorem 7.4.

By Lemma 7.5, there exists $\delta > 0$ such that if $F_E(x_0, 2\rho_0) + C_\Psi\rho_0^\alpha + C_h\rho_0^{\alpha_1} \le \delta$, then (7.12) holds, and we get the result. □

Lemma 7.7. *Let Ω, E and h be the same as in Theorem 7.4. We have that*

$$\overline{E \setminus \partial\Omega} \in SAM(\Omega, \partial\Omega, h).$$

Proof. We will put $E_1 = \overline{E \setminus \partial\Omega}$ for convenient. We first show that $\mathcal{H}^2(E_1 \cap \partial\Omega) = 0$. Indeed, for any $x \in E_1 \cap \partial\Omega$, $\Theta_E(x) \geq 3\pi/2$. It follows from the fact that for $\mathcal{H}^2$-a.e. $x \in E$, $\Theta_E(x) = \pi$ that $\mathcal{H}^2(E_1 \cap \partial\Omega) = 0$.

Let $\{\varphi_t\}_{0\leq t\leq 1}$ be any sliding deformation in some ball $B = B(y, r)$. Since $E \supseteq \partial\Omega$ and $E \in SAM(\Omega, \partial\Omega, h)$, we have that

$$\begin{aligned}\mathcal{H}^2(E_1) &= \mathcal{H}^2(E \setminus \partial\Omega) \leq \mathcal{H}^2(\varphi_1(E) \setminus \partial\Omega) + 4h(2r)r^2\\ &= \mathcal{H}^2(\varphi_1(E_1) \setminus \partial\Omega) + 4h(2r)r^2\\ &\leq \mathcal{H}^2(\varphi_1(E_1)) + 4h(2r)r^2.\end{aligned}$$

Thus $E_1 \in SAM(\Omega, \partial\Omega, h)$. □

Lemma 7.8. *Let Ω, E, x_0 and h be the same as in Theorem 7.4. For any $\varepsilon > 0$ small enough, there exists a $\rho_0 > 0$ such that for any $0 < \rho < \rho_0$ and $x \in E \cap B(x_0, \rho)$, there exists $x_1 \in B(x_0, 5\rho) \cap \partial\Omega$ with $x_1 \in \overline{E \setminus \partial\Omega}$ such that*

$$|x - x_1| \leq (1 + \varepsilon) \operatorname{dist}(x, \partial\Omega).$$

Proof. If $\Theta_E(x_0) = \pi$, then there is an open ball $B = B(x_0, r)$ such that $E \cap B = \partial\Omega \cap B$, and we have nothing to prove.

We assume that $\Theta_E(x_0) = 3\pi/2$ or $7\pi/4$. We put $E_1 = \overline{E \setminus \partial\Omega}$. Then $x_0 \in E_1$ and $\Theta_E(x_0) = \pi/2$ or $3\pi/4$, and by Lemma 7.7, we have that $E_1 \in SAM(\Omega, \partial\Omega, h)$. By Lemma 7.5, for any $\varepsilon \in (0, 10^{-3})$, there exists $\rho_0 \in (0, r_0)$ such that, for any $0 < \rho < \rho_0$, we can find sliding minimal cone Z_ρ centered at x_0 of type $\mathbb{P}_+$ or $\mathbb{Y}_+$ satisfying that

$$d_{x_0,\rho}(E_1, Z_\rho) \leq \varepsilon.$$

Let $\Psi : B(0, r_0) \to \mathbb{R}^3$ be the mapping defined in Lemma 7.1, and let Λ be the same as in (7.4). We put $U_\rho = \Psi(B_\rho)$, $A_1 = \Psi^{-1}(E_1 \cap U_{\rho_0})$. By Lemma 7.3, for any $0 < r \leq \rho/\Lambda(\rho)$, there exist sliding minimal cone X_r in Ω_0 such that

$$d_{0,r}(A_1, X_r) \leq (C\rho^\alpha + \varepsilon)\frac{\rho}{r}.$$

Thus there exists $\rho_1 > 0$ such that for any $0 < r \leq \rho_1$, we can find sliding minimal cone X_r of type $\mathbb{P}_+$ or $\mathbb{Y}_+$ such that

$$d_{0,r}(A_1, X_r) \leq 2\varepsilon.$$

Using the same argument as in the proof Lemma 5.4 in [9], we get that there exists $\rho_2 > 0$ such that for any $x \in A_1 \cap B(0, \rho)$ with $0 < \rho \leq \rho_2$, we can find $a \in A_1 \cap L_0 \cap B(0, 3\rho)$ such that

$$|P_{L_0}(x) - a| \leq 8\varepsilon|x - a|,$$

where we denote by P_{L_0} the orthogonal projection from $\mathbb{R}^3$ to L_0. Thus

$$|x - a| \leq |x - P_{L_0}(x)| + |P_{L_0}(x) - a| \leq \operatorname{dist}(x, L_0) + 8\varepsilon|x - a|,$$

and we get that

$$\operatorname{dist}(x, A_1 \cap L_0 \cap B(0, 3\rho)) \leq \frac{1}{1 - 8\varepsilon} \operatorname{dist}(x, L_0 \cap B(0, 3\rho)).$$

We take $\rho_3 = \mathrm{dist}(x_0, \mathbb{R}^3 \setminus U_{\rho_2})/10$. Then, for any $0 < \rho \leq \rho_3$ and $z \in E_1 \cap B(x_0, \rho)$,

$$\begin{aligned}\mathrm{dist}(z, E_1 \cap \partial\Omega \cap B(x_0, 5\rho)) &\leq \mathrm{Lip}\left(\Psi|_{B(0,3\rho_2)}\right) \mathrm{dist}(\Psi^{-1}(z), A_1 \cap L_0 \cap B(0, 3\rho)) \\ &\leq (1-8\varepsilon)^{-1}\Lambda(3\rho)\,\mathrm{dist}(\Psi^{-1}(z), A_1 \cap L_0 \cap B(0, 3\rho)) \\ &\leq (1-8\varepsilon)^{-1}\Lambda(3\rho)^2\,\mathrm{dist}(z, \partial\Omega \cap B(x_0, 5\rho)).\end{aligned}$$

We assume ρ_2 to be small enough such that $(1-8\varepsilon)^{-1}\Lambda(3\rho_2)^2 < 1 + 10\varepsilon$, then

$$\mathrm{dist}(z, E_1 \cap \partial\Omega \cap B(x_0, 5\rho)) \leq (1+10\varepsilon)\,\mathrm{dist}(z, \partial\Omega \cap B(x_0, 5\rho)).$$

□

Lemma 7.9. *Let Ω, E, x_0 and h be the same as in Theorem 7.4. Suppose that $\Theta_E(x_0) = 3\pi/2$. Then, by putting $E_1 = \overline{E \setminus \partial\Omega}$, there exist a radius $r > 0$, a number $\beta > 0$ and a constant $C > 0$ such that, for any $x \in B(x_0, r) \cap E_1$ and $0 < \rho < 2r$, we can find cone $Z_{x,\rho}$ such that*

$$d_{x,\rho}(E_1, Z_{x,\rho}) \leq C\rho^\beta,$$

where $Z_{x,\rho} = y + \mathrm{Tan}(E_1, y)$, $y \in E_1 \cap B(x, C\rho)$, and $y \in E_1 \cap \partial\Omega \cap B(x, C\rho)$ in case $\rho \geq \mathrm{dist}(x, \partial\Omega)/10$.

Proof. We see that $E = E_1 \cup \partial\Omega$, and $F_E(x_0, \rho) = F_{E_1}(x, \rho) + F_{\partial\Omega}(x_0, \rho)$. By Corollary 7.6, there exist $\delta > 0$ and $C > 0$ such that whenever $0 < \rho_0 \leq \min\{1, t_0, r_0(x_0)\}$ satisfying

$$F_{E_1}(x_0, 2\rho_0) + C_{\Psi_{x_0}}\rho_0^\alpha + C_h\rho_0^{\alpha_1} \leq \delta.$$

we have that, for $0 < \rho \leq 9\rho_0/20$,

$$d_{x_0,\rho}(E_1, x_0 + \mathrm{Tan}(E_1, x_0)) \leq C\delta^{1/4}(\rho/\rho_0)^\beta,$$

where $0 < \beta < \min\{\alpha, \alpha_1, 2\lambda_0, \beta_0\}/4$. We take $\rho_1 \in (0, \rho_0)$ such that

$$F_{E_1}(x_0, 2\rho) + C_{\Psi_{x_0}}\rho^\alpha + C_h\rho^{\alpha_1} \leq \min\{\delta/2, \varepsilon_2(\tau)\}, \forall 0 < \rho \leq \rho_1.$$

If $x \in \partial\Omega \cap B(x_0, \rho_1/10)$, we take $t = \rho_1/2$, then apply Lemma 7.5 with $r = |x - x_0| + t$ to get that

$$\mathcal{H}^2(E_1 \cap B(x, t)) \leq \mathcal{H}^2(Z_{x,r} \cap B(x, t)) + \tau r^2,$$

thus

$$\Theta_{E_1}(x, t) \leq \frac{1}{t^2}\mathcal{H}^2(Z_{x,r} \cap B(x, t)) + 4\tau \leq \frac{\pi}{2} + C_{\Psi_{x_0}}r^\alpha + 4\tau,$$

and

$$F_{E_1}(x, t) \leq C_{\Psi_{x_0}}r^\alpha + 4\tau + 16h_1(t).$$

We get that $F_{E_1}(x, 2\rho) + C_{\Psi_x}\rho^\alpha + C_h\rho^{\alpha_1} \leq \delta$ for $0 < \rho \leq t/2$. Thus

$$d_{x,r}(E_1, x + \mathrm{Tan}(E_1, x)) \leq C\delta^{1/4}(r/t)^\beta,\ 0 < r < 9t/20. \tag{7.14}$$

By Lemma 7.8, we assume that for any $x \in E_1 \cap B(x_0, \rho_1/10)$, there exists $x_1 \in E_1 \cap B(x_0, \rho_1/2) \cap \partial\Omega$ such that

$$|x - x_1| \leq 2\,\mathrm{dist}(x, \partial\Omega).$$

If $x \in E_1 \cap B(x_0, \rho_1/10) \setminus \partial\Omega$, we take $t = t(x) = 10^{-3}\,\mathrm{dist}(x, \partial\Omega)$, then apply Lemma 7.5 with $r = |x - x_1| + t$ to get that

$$\mathcal{H}^2(E_1 \cap B(x, t)) \leq \mathcal{H}^2(Z_{x_1,r} \cap B(x, t)) + \tau r^2,$$

thus

$$\Theta_{E_1}(x,t) \le \frac{1}{t^2}\mathcal{H}^2(Z_{x_1,r}\cap B(x,t)) + (1+2\cdot 10^3)^2\tau \le \pi/2 + (1+2\cdot 10^3)^2\tau,$$

and

$$F(x,t) \le (1+2\cdot 10^3)^2\tau + 8h_1(t).$$

By Theorem 6.1, there is a constent $C_1 > 0$ such that

$$d_{x,r}(E_1, x+\operatorname{Tan}(E_1,x)) \le C_1(r/t)^\beta,\ 0<r<t. \tag{7.15}$$

Hence we get from (7.14) and (7.15) that

$$d_{x,r}(E_1, x+\operatorname{Tan}(E_1,x)) \le C_2(r/t_0)^\beta, \forall x\in E_1\cap B(x_0,\rho_1/10), 0<r<t_0, \tag{7.16}$$

where

$$t_0 = \begin{cases}\rho_1/10, & x\in\partial\Omega,\\ 10^{-3}\operatorname{dist}(x,\partial\Omega), & x\notin\partial\Omega.\end{cases}$$

We take $0<a<\beta/(1+\beta)$. For any $x\in B(x_0,\rho_1/10)\setminus\partial\Omega$, if $r\le t_0^{1/(1-a)}$, then we get from (7.16) that

$$d_{x,r}(E_1, x+\operatorname{Tan}(E_1,x)) \le C_2 r^{a\beta}; \tag{7.17}$$

if $t_0^{1/(1-a)} < r < \rho_1/5$, then by (7.16), we have that

$$\begin{aligned} d_{x,r}(E_1,x_1+\operatorname{Tan}(E_1,x_1)) &\le \frac{|x-x_1|+r}{r} d_{x_1,|x-x_1|+r}(E_1,x_1+\operatorname{Tan}(E_1,x_1))\\ &\le C_2\left(1+\frac{2\cdot 10^3 t_0}{r}\right)\left(\frac{r+2\cdot 10^3 t_0}{\rho_1/10}\right)^\beta\\ &\le C_5(1+C_6 r^{-a})^{\beta+1} r^\beta \le C_7 r^{\beta-a\beta-a}. \end{aligned} \tag{7.18}$$

From (7.17) and (7.18), we get so that, for any $0<\beta_1<\min\{a\beta, \beta-a\beta-a\}$ there is a constant $C_8>0$ such that for any $x\in E_1\cap B(x_0,\rho_1/10)$ and $0<\rho<\rho_1/5$, we can find cone $Z_{x,\rho}$ such that

$$d_{x,\rho}(E_1, Z_{x,\rho}) \le C_8\rho^{\beta_1},$$

where $Z_{x,\rho} = y+\operatorname{Tan}(E_1,y)$, $y\in E_1\cap B(x,C_8\rho)$, and $y\in E_1\cap\partial\Omega\cap B(x,C_8\rho)$ in case $\rho\ge t_0^{1/(1-a)}$. □

Lemma 7.10. *Let Ω, E, x_0 and h be the same as in Theorem 7.4. Suppose that $\Theta_E(x_0) = 7\pi/4$. Then, by putting $E_1 = \overline{E\setminus\partial\Omega}$, there exist a radius $r>0$, a number $\beta>0$ and a constant $C>0$ such that, for any $x\in B(x_0,r)\cap E_1$ and $0<\rho<2r$, we can find a cone $Z_{x,\rho}$ such that*

$$d_{x,\rho}(E_1, Z_{x,\rho}) \le C\rho^\beta,$$

where $Z_{x,\rho} = y+\operatorname{Tan}(E_1,y)$, $y\in E_1\cap B(x_0,C\rho)$, and $y\in E_1\cap\partial\Omega\cap B(x_0,C\rho)$ in case $\rho\ge\operatorname{dist}(x,\partial\Omega)/10$.

Proof. By Corollary 7.6, there exist $\delta>0$ and $C>0$ such that whenever $0<\rho_0\le\min\{1,t_0,r_0(x_0)\}$ satisfying

$$F_{E_1}(x_0,2\rho_0) + C_{\Psi_{x_0}}\rho_0^\alpha + C_h\rho_0^{\alpha_1} \le \delta,$$

we have that, for $0<\rho\le 9\rho_0/20$,

$$d_{x_0,\rho}(E_1, x_0+\operatorname{Tan}(E_1,x_0)) \le C\delta^{1/4}(\rho/\rho_0)^\beta,$$

where $0 < \beta < \min\{\alpha, \alpha_1, 2\lambda_0\}/4$. We take $\rho_1 \in (0, \rho_0)$ such that

$$F_{E_1}(x_0, 2\rho) + C_{\Psi_{x_0}} \rho^\alpha + C_h \rho^{\alpha_1} \leq \min\{\delta/2, \varepsilon_2(\tau)\}, \forall 0 < \rho \leq \rho_1.$$

If $x \in \partial\Omega \cap B(x_0, \rho_1/10)$, we take $t = |x - x_0|/2$, then apply Lemma 7.5 with $r = |x - x_0| + t$ to get that

$$\mathcal{H}^2(E_1 \cap B(x,t)) \leq \mathcal{H}^2(Z_{x,r} \cap B(x,t)) + \tau r^2,$$

thus

$$\Theta_{E_1}(x,t) \leq \frac{1}{t^2}\mathcal{H}^2(Z_{x,r} \cap B(x,t)) + 9\tau \leq \frac{\pi}{2} + C_{\Psi_{x_0}} r^\alpha + 9\tau,$$

and

$$F_{E_1}(x,t) \leq C_{\Psi_{x_0}} r^\alpha + 9\tau + 16 h_1(t).$$

We get that $F_{E_1}(x, 2\rho) + C_{\Psi_x}\rho^\alpha + C_h \rho^{\alpha_1} \leq \delta$ for $0 < \rho \leq t/2$. Thus

$$d_{x,r}(E_1, x + \mathrm{Tan}(E_1, x)) \leq C\delta^{1/4}(r/t)^\beta,\ 0 < r < 9t/20. \tag{7.19}$$

By Lemma 7.8, we assume that for any $x \in E_1 \cap B(x_0, \rho_1/10)$, there exists $x_1 \in E_1 \cap B(x_0, \rho_1/5) \cap \partial\Omega$ such that

$$|x - x_1| \leq 2\,\mathrm{dist}(x, \partial\Omega). \tag{7.20}$$

If $x \in E_1 \cap B(x_0, \rho_1/10) \setminus \partial\Omega$, then $\Theta_{E_1}(x) = \pi$ or $3\pi/2$. We put $t(x) = \mathrm{dist}(x, \partial\Omega)$. If $\Theta_{E_1}(x) = 3\pi/2$, we take $t = 10^{-3} t(x)$, then apply Lemma 7.5 with $r = |x - x_1| + t$ to get that

$$\mathcal{H}^2(E_1 \cap B(x,t)) \leq \mathcal{H}^2(Z_{x_1,r} \cap B(x,t)) + \tau r^2,$$

thus

$$\Theta_{E_1}(x,t) \leq \frac{1}{t^2}\mathcal{H}^2(Z_{x_1,r} \cap B(x,t)) + (1 + 2\cdot 10^3)^2\tau \leq \frac{3\pi}{2} + (1 + 2\cdot 10^3)^2 \tau.$$

and

$$F_{E_1}(x,t) \leq (1 + 2\cdot 10^3)^2\tau + 8h_1(t).$$

By Theorem 6.1, we have that

$$d_{x,\rho}(E_1, x + \mathrm{Tan}(E_1, x)) \leq C_1(\rho/t)^\beta,\ 0 < \rho < t. \tag{7.21}$$

We put $E_Y = \{x_0\} \cup \{x \in E \backslash \partial\Omega : \Theta_{E_1}(x) = \pi\}$. If $\Theta_{E_1}(x) = \pi$ and $\mathrm{dist}(x, E_Y) \leq 10^{-2}\,\mathrm{dist}(x, \partial\Omega)$, we take $x_2 \in E_Y$ such that $|x - x_2| \leq 2\,\mathrm{dist}(x, E_Y)$ and $t = 10^{-1}\,\mathrm{dist}(x, E_Y)$, then apply Lemma 7.24 in [4] with $r = |x - x_2| + t$ to get that

$$\mathcal{H}^2(E_1 \cap B(x,t)) \leq \mathcal{H}^2(Z_{x_2,r} \cap B(x,t)) + \tau r^2,$$

thus

$$\Theta_{E_1}(x,t) \leq \frac{1}{t^2}\mathcal{H}^2(Z_{x_2,r} \cap B(x,t)) + 400\tau \leq \pi + 400\tau,$$

and

$$F_{E_1}(x,t) \leq 4\tau + 8h_1(t).$$

By Theorem 6.1, we have that

$$d_{x,\rho}(E_1, x + \mathrm{Tan}(E_1, x)) \leq C_2(\rho/t)^\beta,\ 0 < \rho < t. \tag{7.22}$$

If $\Theta_{E_1}(x) = \pi$ and $\mathrm{dist}(x, E_Y) > 10^{-2}\,\mathrm{dist}(x, \partial\Omega)$, we take $t = 10^{-3}\,\mathrm{dist}(x, \partial\Omega)$, then apply Lemma 7.5 with $r = |x - x_1| + t$ to get that

$$\mathcal{H}^2(E_1 \cap B(x,t)) \leq \mathcal{H}^2(Z_{x_1,r} \cap B(x,t)) + \tau r^2,$$

thus

$$\Theta_{E_1}(x,t) \leq \frac{1}{t^2}\mathcal{H}^2(Z_{x_1,r}\cap B(x,t)) + (1+2\cdot 10^3)^2\tau \leq \pi + (1+2\cdot 10^3)^2\tau.$$

and

$$F_{E_1}(x,t) \leq (1+2\cdot 10^3)^2\tau + 8h_1(t).$$

By Theorem 6.1, we have that

$$d_{x,\rho}(E_1, x+\mathrm{Tan}(E_1,x)) \leq C_3(\rho/t)^\beta,\ 0<\rho<t. \tag{7.23}$$

We get, from (7.19), (7.21), (7.22) and (7.23), so that

$$d_{x,\rho}(E_1, x+\mathrm{Tan}(E_1,x)) \leq C_4(\rho/t_0)^\beta,\ x\in E_1\cap B(x_0,\rho_1/10),\ 0<\rho<t_0, \tag{7.24}$$

where

$$t_0 = \begin{cases} \rho_1/2, & x=x_0,\\ |x-x_0|/10, & x\in\partial\Omega\setminus\{x_0\},\\ 10^{-3}\,\mathrm{dist}(x,\partial\Omega), & x\notin\partial\Omega, \Theta_{E_1}(x)=3\pi/2\\ 10^{-1}\min\{10^{-2}\,\mathrm{dist}(x,\partial\Omega),\mathrm{dist}(x,E_Y)\}, & x\notin\partial\Omega, \Theta_{E_1}(x)=\pi. \end{cases}$$

Claim. $E_Y\cap B(x_0,\rho_1/2)$ is a C^1 curve which is perpendicular to $\mathrm{Tan}(\Omega,x_0)$. Indeed, by biHölder regularity at the boundary, we see that $E_Y\cap B(x_0,\rho_1/2)$ is a curve, and by J. Taylor's regularity theorem [13], we get that $E_Y\cap B(x_0,\rho_1/2)$ is of class C^1.

By the claim, we can assume that, there is a constant $\eta_3>0$ such that

$$\mathrm{dist}(x,\partial\Omega)\geq \eta_3|x-x_0|,\ \forall x\in E_Y\cap B(x_0,\rho_1/10). \tag{7.25}$$

We fix $0<\beta_1<\beta_2<\beta/(1+\beta)$ such that $\beta_1\leq\beta_2\beta/(1+\beta)$. By (7.24), we have that, for any $x\in\partial\Omega\cap B(x_0,\rho_1/10)\setminus\{x_0\}$, and any $0<\rho<|x-x_0|/10$,

$$d_{x,\rho}(E_1,x+\mathrm{Tan}(E_1,x))\leq C_4(10\rho/|x-x_0|)^\beta.$$

If $0<\rho\leq(|x-x_0|/10)^{1/(1-\beta_1)}$, then

$$d_{x,\rho}(E_1,x+\mathrm{Tan}(E_1,x))\leq C_4(10\rho/|x-x_0|)^\beta = C_6\rho^{\beta_1\beta};$$

if $(|x-x_0|/10)^{1/(1-\beta_1)}<\rho\leq\rho_1/5$, then

$$\begin{aligned} d_{x,\rho}(E_1,x_0+\mathrm{Tan}(E_1,x_0)) &\leq \frac{|x-x_0|+\rho}{\rho} d_{x_0,|x-x_0|+\rho}(E_1,x_0+\mathrm{Tan}(E_1,x_0))\\ &\leq \left(1+10\rho^{-\beta_1}\right)C_4\left(\frac{10\rho^{1-\beta_1}+\rho}{\rho_1/2}\right)^\beta \leq C_7\rho^{\beta-\beta_1-\beta\beta_1}. \end{aligned}$$

Thus we get that, for any $0<\beta_3\leq\min\{\beta\beta_1,\beta-\beta_1-\beta\beta_1)\}$, there is a constant C_8 such that for any $x\in\partial\Omega\cap B(x_0,\rho_1/10)$ and $0<\rho\leq\rho_1/5$ we can find cone $Z_{x,\rho}$ satisfying that

$$d_{x,\rho}(E_1,Z_{x,\rho})\leq C_8\rho^{\beta_3}. \tag{7.26}$$

If $x\in E_1\cap B(x_0,\rho_1/10)\setminus\partial\Omega$ and $\Theta_{E_1}(x)=3\pi/2$, then for any $0<\rho\leq(10^{-3}\eta_3|x-x_0|)^{1/(1-\beta_1)}$, from (7.24) and (7.25), we get that

$$0<\rho<10^{-3}\eta_3|x-x_0|\leq t_0$$

and

$$d_{x,\rho}(E_1,x+\mathrm{Tan}(E_1,x))\leq C_4(10^3\rho/\mathrm{dist}(x,\partial\Omega))^\beta = C_9\rho^{\beta_1\beta};$$

for any $(10^{-3}\eta_3|x-x_0|)^{1/(1-\beta_1)} < \rho \le \rho_1/5$, we have that

$$\begin{aligned} d_{x,\rho}(E_1, x_0+\operatorname{Tan}(E_1,x_0)) &\le \frac{|x-x_0|+\rho}{\rho} d_{x_0,|x-x_0|+\rho}(E_1, x_0+\operatorname{Tan}(E_1,x_0)) \\ &\le \left(1+10^3\eta_3^{-1}\rho^{-\beta_1}\right) C_4 \left(\frac{10^3\eta_3^{-1}\rho^{1-\beta_1}+\rho}{\rho_1/2}\right)^{\beta} \\ &\le C_{10}\rho^{\beta-\beta_1-\beta\beta_1}. \end{aligned}$$

Thus we get that, for any $0 < \beta_4 \le \min\{\beta\beta_1, \beta-\beta_1-\beta\beta_1)\}$, there is a constant C_{11} such that for any $x \in E_1 \cap B(x_0, \rho_1/10) \setminus \partial\Omega$ with $\Theta_{E_1}(x) = 3\pi/2$, and $0 < \rho \le \rho_1/5$ we can find cone $Z_{x,\rho}$ satisfying that

$$d_{x,\rho}(E_1, Z_{x,\rho}) \le C_{11}\rho^{\beta_4}. \tag{7.27}$$

If $x \in E_1 \cap B(x_0, \rho_1/10) \setminus \Omega$, $\Theta_{E_1}(x) = \pi$ and $\operatorname{dist}(x, \partial\Omega) < 100 \operatorname{dist}(x, E_Y)$, then

$$t_0 = 10^{-3}\operatorname{dist}(x, \partial\Omega).$$

From (7.24), we get that, for any $0 < \rho < 10^{-3}\operatorname{dist}(x,\partial\Omega)^{1/(1-\beta_1)}$,

$$d_{x,\rho}(E_1, x+\operatorname{Tan}(E_1,x)) \le C_4(10^3\rho/\operatorname{dist}(x,\partial\Omega))^{\beta} = 10^{3\beta_1\beta}C_4\cdot\rho^{\beta_1\beta}. \tag{7.28}$$

For any $10^{-3}\operatorname{dist}(x,\partial\Omega)^{1/(1-\beta_1)} \le \rho \le \rho_1/5$, we take $C_{12} = 10^{-3}\cdot 10^{2/(1-\beta_1)}$, then we see that $10^{-3}\operatorname{dist}(x,\partial\Omega)^{1/(1-\beta_1)} \le C_{12}|x-x_0|^{1/(1-\beta_1)}$. If $\rho \le C_{12}|x-x_0|^{1/(1-\beta_2)}$, we let x_1 be the point chose in (7.20), then we have that $|x-x_1| \le 2\operatorname{dist}(x,\partial\Omega) \le 2(10^3\rho)^{1-\beta_1}$ and

$$|x-x_0| \ge \operatorname{dist}(x, E_Y) \ge 100\operatorname{dist}(x,\partial\Omega) \ge 50|x-x_1|,$$

thus

$$|x_1-x_0| \ge |x-x_0| - |x-x_1| \ge (1-1/50)|x-x_0| \ge \frac{49}{50}(C_{12}^{-1}\rho)^{1-\beta_2},$$

From (7.24), we get that

$$\begin{aligned} d_{x,\rho}(E_1, x_1+\operatorname{Tan}(E_1,x_1)) &\le \frac{|x-x_1|+\rho}{\rho} d_{x_1,|x-x_1|+\rho}(E_1, x_1+\operatorname{Tan}(E_1,x_1)) \\ &\le \left(1+2\cdot 10^{3-3\beta_1}\rho^{-\beta_1}\right) C_4 \left(\frac{2\cdot 10^{3-3\beta_1}\rho^{1-\beta_1}+\rho}{|x_0-x_1|/10}\right)^{\beta} \\ &\le 101\cdot C_4\cdot C_{12}^{1-\beta_2}\cdot(2\cdot 10^{3-3\beta_1}+\rho^{\beta_1})^{1+\beta}\rho^{\beta\beta_2-\beta\beta_1-\beta_1} \\ &\le C_{13}\rho^{\beta\beta_2-\beta\beta_1-\beta_1}. \end{aligned} \tag{7.29}$$

If $\rho > C_{12}|x-x_0|^{1/(1-\beta_2)}$, we have that

$$\begin{aligned} d_{x,\rho}(E_1, x_0+\operatorname{Tan}(E_1,x_0)) &\le \frac{|x-x_0|+\rho}{\rho} d_{x_0,|x-x_0|+\rho}(E_1, x_0+\operatorname{Tan}(E_1,x_0)) \\ &\le \left(1+C_{12}^{-1+\beta_2}\rho^{-\beta_2}\right) C_4 \left(\frac{C_{12}^{-1+\beta_2}\rho^{1-\beta_2}+\rho}{\rho_1/2}\right)^{\beta} \\ &= 2C_4\rho_1^{-1}(C_{12}^{-1+\beta_2}+\rho)^{1+\beta}\rho^{\beta-\beta_2-\beta\beta_2} \le C_{14}\rho^{\beta-\beta_2-\beta\beta_2}. \end{aligned} \tag{7.30}$$

If $x \in E_1 \cap B(x_0, \rho_1/10) \setminus \partial\Omega$, $\Theta_{E_1}(x) = \pi$ and $\operatorname{dist}(x, \partial\Omega) \geq 100 \operatorname{dist}(x, E_Y)$, then from (7.24), we have that for any $0 < \rho < 10^{-1} \operatorname{dist}(x, E_Y)^{1/(1-\beta_1)}$,

$$d_{x,\rho}(E_1, x + \operatorname{Tan}(E_1, x)) \leq C_4 (10\rho / \operatorname{dist}(x, E_Y))^{\beta} \leq 10^{\beta\beta_1} \cdot C_4 \cdot \rho^{\beta_1 \beta}. \tag{7.31}$$

For any $10^{-1} \operatorname{dist}(x, E_Y)^{1/(1-\beta_1)} \leq \rho \leq \rho_1/5$, we take $y \in E_Y$ such that $|x - y| \leq 2 \operatorname{dist}(x, E_Y)$. We put $C_{16} = 10^{-1-2/(1-\beta_1)}$. We see that $10^{-1} \operatorname{dist}(x, E_Y)^{1/(1-\beta_1)} \leq C_{16} \operatorname{dist}(x, \partial\Omega)^{1/(1-\beta_1)}$. If $\rho \leq C_{16} \operatorname{dist}(y, \partial\Omega)^{1/(1-\beta_2)}$, then $|x-y| \leq 2 \operatorname{dist}(x, E_Y) \leq 2(10\rho)^{1-\beta_1}$. From (7.24), we get that

$$\begin{aligned} d_{x,\rho}(E_1, y + \operatorname{Tan}(E_1, y)) &\leq \frac{|x-y| + \rho}{\rho} d_{y,|x-y|+\rho}(E_1, y + \operatorname{Tan}(E_1, y)) \\ &\leq \left(1 + 2 \cdot 10^{1-\beta_1} \rho^{-\beta_1}\right) C_4 \left(\frac{2 \cdot 10^{1-\beta_1} \rho^{1-\beta_1} + \rho}{10^{-3} \operatorname{dist}(y, \partial\Omega)} \right)^{\beta} \\ &= 10^{3\beta} C_4 C_{16}^{\beta(1-\beta_1)} \left(2 \cdot 10^{1-\beta_1} + \rho^{\beta_1}\right)^{1+\beta} \rho^{\beta(\beta_2 - \beta_1) - \beta_1} \\ &\leq C_{17} \rho^{\beta\beta_2 - \beta_1 - \beta\beta_1}. \end{aligned} \tag{7.32}$$

If $\rho > C_{16} \operatorname{dist}(y, \partial\Omega)^{1/(1-\beta_2)}$, we have that $|x - x_0| \geq \operatorname{dist}(x, \partial\Omega) \geq 100 \operatorname{dist}(x, E_Y) \geq 50|x - y|$. Since $y \in E_y$, we see from (7.25) that

$$\operatorname{dist}(y, \partial\Omega) \geq \eta_3 |y - x_0| \geq \eta_3 (|x - x_0| - |x - y|) \geq \eta_3 \cdot \frac{49}{50} |x - x_0|,$$

and $|x - x_0| \leq \frac{50}{49} \eta_3^{-1} \operatorname{dist}(y, \partial\Omega) \leq 2\eta_3^{-1} (C_{16}^{-1} \rho)^{1-\beta_2}$. We get from (7.24) that

$$\begin{aligned} &d_{x,\rho}(E_1, x_0 + \operatorname{Tan}(E_1, x_0)) \\ &\quad \leq \frac{|x - x_0| + \rho}{\rho} d_{x_0, |x-x_0|+\rho}(E_1, x_0 + \operatorname{Tan}(E_1, x_0)) \\ &\quad \leq \left(1 + 2\eta_3^{-1} C_{16}^{-1+\beta_2} \rho^{-\beta_2}\right) C_4 \left(\frac{2\eta_3^{-1} C_{16}^{-1+\beta_2} \rho^{1-\beta_2} + \rho}{\rho_1/2} \right)^{\beta} \\ &\quad = 2C_4 \rho_1^{-1} (2\eta_3^{-1} C_{16}^{-1+\beta_2} + \rho^{\beta_2})^{1+\beta} \rho^{\beta(1-\beta_2)-\beta_2} \\ &\quad \leq C_{18} \rho^{\beta - \beta_2 - \beta\beta_2}. \end{aligned} \tag{7.33}$$

We get, from (7.28), (7.29), (7.30), (7.31),(7.32) and (7.33), that for any $0 < \beta_5 \leq \min\{\beta\beta_1, \beta\beta_2 - \beta_1 - \beta\beta_1, \beta - \beta_2 - \beta\beta_2\}$, there is a constant $C_{19} > 0$ such that for any $x \in E_1 \cap B(x_0, \rho_1/10) \setminus \partial\Omega$ with $\Theta_{E_1}(x) = \pi$ and $0 < \rho \leq \rho_1/5$, we can find cone $Z_{x,\rho}$ such that

$$d_{x,\rho}(E_1, Z_{x,\rho}) \leq C_{19} \rho^{\beta_5}. \tag{7.34}$$

Hence we get, from (7.26), (7.27) and (7.34), there is a constant $C_{20} > 0$ and $C_{21} > 0$ such that for any $x \in E_1 \cap B(x_0, \rho_1/10)$ and $0 < \rho \leq \rho_1/5$, we can find cone $Z_{x,\rho}$ such that

$$d_{x,\rho}(E_1, Z_{x,\rho}) \leq C_{20} \rho^{\beta_6},$$

where $Z_{x,\rho} = z + \operatorname{Tan}(E_1, z)$ for some $z \in E_1 \cap B(x, C_{21}\rho)$, and $z \in E_1 \cap \partial\Omega \cap B(x, C_{21}\rho)$ in case

$$\rho \geq \max\{(10^{-3}\eta_3 |x - x_0|)^{1/(1-\beta_1)}, 10^{-3} \operatorname{dist}(x, \partial\Omega)^{1/(1-\beta_1)}, C_{16} \operatorname{dist}(y, \partial\Omega)^{1/(1-\beta_2)}\}.$$

□

Corollary 7.11. *Let Ω, E and h be the same as in Theorem 7.4. Let $E_1 = \overline{E \setminus \partial\Omega}$ and $x_0 \in E_1 \cap \partial\Omega$. Then there exist a radius $r > 0$, a number $\beta > 0$ and a constant $C > 0$ such that, for any $x \in E_1 \cap B(x_0, r)$ and $0 < \rho < 2r$, we can find cone $Z_{x,\rho}$ such that*

$$d_{x,\rho}(E_1, Z_{x,\rho}) \leq C\rho^\beta,$$

where $Z_{x,\rho} = y + \mathrm{Tan}(E_1, y)$, $y \in E_1 \cap B(x, C\rho)$, and $y \in E_1 \cap \partial\Omega \cap B(x, C\rho)$ in case $\rho \geq \mathrm{dist}(x, \partial\Omega)/10$.

Proof. It is follow from Lemma 7.9 and Lemma 7.10. □

Lemma 7.12. *Let Ω, E, x_0 and h be the same as in Corollary 7.11. Let $\Psi : B(0, r_0) \to \mathbb{R}^3$ be the mapping defined in Lemma 7.1. Let $R > 0$ be such that $\Psi(B(0, R)) \subseteq B(x_0, r)$, where $B(x_0, r)$ is the ball considered as in Corollary 7.11. By putting $U = \Psi(B(0, R))$, $M_1 = \Psi^{-1}(E_1 \cap U)$, we have that there exist $\rho_3 > 0$, $\beta > 0$, and constant $C > 0$ such that for any $z \in M_1 \cap B(0, \rho_3)$ and $0 < t < 2\rho_3$, we can find cone $Z(z, t)$ through z such that*

$$d_{z,t}(M_1, Z(z, t)) \leq Ct^\beta,$$

where $Z(z, t)$ is a minimal cone of type $\mathbb{P}$ or $\mathbb{Y}$ in case $z \in M_1 \setminus L_0$ and $0 < t < \mathrm{dist}(z, L_0)/2$; and in case $t \geq \mathrm{dist}(z, L_0)/2$ or $z \in L_0$, $Z(z, t)$ is a sliding minimal cone in Ω_0 with sliding boundary L_0, if $Z(z, t) \setminus L_0 \neq \emptyset$, we can be written as $Z(z, t) = L_0 \cup Z'(z, t)$, $Z'(z, t)$ is a sliding minimal cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$.

Proof. For any $x \in B(x_0, r) \cap E_1$ and $0 < \rho < 2r$, we let $Z_{x,\rho}$ be the same cone considered as in Corollary 7.11. We put $\Phi = \Psi^{-1}|_{B(x_0,r)}$ for convenient. For any $y \in E_1 \cap B(x_0, r)$, and $z \in E_1 \cap B(x, \rho)$, we put $X = \mathrm{Tan}(E_1, y)$, then

$$\mathrm{dist}(\Phi(z), \Phi(y + X)) \leq \mathrm{Lip}(\Phi)\,\mathrm{dist}(z, y + X) \leq C\,\mathrm{Lip}(\Phi)\rho^{1+\beta}. \tag{7.35}$$

Since

$$|\Phi(z_1) - \Phi(z_2) - D\Phi(z_2)(z_1 - z_2)| \leq C_1|z_1 - z_2|^{1+\alpha},$$

we have that, for any $z_1 \in y + X$,

$$\mathrm{dist}(\Phi(z_1), \Phi(y) + D\Phi(y)X) \leq C_1|z_1 - y|^{1+\alpha}. \tag{7.36}$$

Hence, from (7.35) and (7.36), we have that

$$\mathrm{dist}(\Phi(z), \Phi(y) + D\Phi(y)X) \leq C\,\mathrm{Lip}(\Phi)\rho^{1+\beta} + C_1(\rho + C\rho + C\rho^{1+\beta})^{1+\alpha} \leq C_2\rho^{1+\beta}. \tag{7.37}$$

For any $v \in X$, we see that $\Phi(y) + D\Phi(y)v \in \Phi(y) + D\Phi(y)X$, and we have that

$$\begin{aligned}\mathrm{dist}(\Phi(y) + D\Phi(y)v, M_1) &\leq \mathrm{dist}(\Phi(y) + D\Phi(y)v, \Phi(E_1 \cap B(x, \rho)))\\ &= \inf\{|\Phi(z) - \Phi(y) - D\Phi(y)v| : z \in E_1 \cap B(x, \rho)\}\\ &\leq \inf\{C_1|z - y|^{1+\alpha} + \mathrm{Lip}(\Phi)|z - y - v| : z \in E_1 \cap B(x, \rho)\}\\ &\leq C_1(\rho + C\rho)^{1+\alpha} + \mathrm{Lip}(\Phi)\,\mathrm{dist}(y + v, E_1).\end{aligned}$$

Thus there exist $C_3 > 0$ such that, for any $v \in X$ with $|y + v - x| \leq \rho$,

$$\mathrm{dist}(\Phi(y) + D\Phi(y)v, M_1) \leq C_3\rho^{1+\beta}. \tag{7.38}$$

We take $0 < C_5 < C_4 < 1$ small enough, for example $C_4 < (10\,\mathrm{Lip}(\Phi))^{-1}$, then for any $C_5\rho \leq t \leq C_4\rho \leq \rho/\mathrm{Lip}(\Phi) - C_1(C\rho)^{1+\alpha}$, we have that $M_1 \cap B(\Phi(x), t) \subseteq \Phi(E_1 \cap B(x, \rho))$ and

$$[\Phi(y) + D\Phi(y)X] \cap B(\Phi(x), t) \subseteq \{\Phi(y) + D\Phi(y)v : v \in X, y + v \in B(x, \rho)\}.$$

From (7.37) and (7.38), we get so that

$$d_{\Phi(x),t}(M_1, \Phi(y) + D\Phi(y)X) \leq C_6\rho^\beta \leq C_7 t^\beta,$$

and

$$|\Phi(x) - \Phi(y)| \leq \mathrm{Lip}(\Phi)|x - y| \leq (\mathrm{Lip}(\Phi)CC_5^{-1})t.$$

Hence

$$d_{\Phi(x),t}(M_1, \Phi(y) + D\Phi(y)X) \leq C_7 t^\beta, \quad \text{for any } 0 < t < C_4\rho_1,$$

where $\rho_1 \in (0, 2r)$ satisfy that $C_1 C^{1+\alpha}\rho_1 \leq \mathrm{Lip}(\Phi)^{-1} - C_4$.

We take $\rho_2 > 0$ such that, for any $x \in E_1 \cap \Phi(B(x_0, \rho_2))$ and $0 < \rho < 2\rho_2$, $Z_{x,\rho}$ can be expressed as $Z_{x,\rho} = y + \mathrm{Tan}(E_1, y)$ with $y \in E_1 \cap U$. Since $D\Phi(y)X = D\Phi(y)\,\mathrm{Tan}(E_1, y) = \mathrm{Tan}(M_1, \Phi(y))$ in case $y \in E_1 \cap U$, by putting $\rho_3 = \min\{\rho_2, C_4\rho_1/2, R\}$, we have that, for any $z \in M_1 \cap B(0, \rho_3)$ and $0 < t < 2\rho_3$, there exist cone $Z'(z, t)$ in Ω_0 with sliding boundary $L_0 = \partial\Omega_0$, such that

$$d_{x,t}(M_1, Z'(z, t)) \leq C_7 t^\beta.$$

For such cone $Z'(z, t)$, we have that $Z'(z, t) = w + \mathrm{Tan}(M_1, w)$, $w \in M_1$, $|w - z| \leq C_8 t$, and $w \in L_0 \cap B(z, C_8 t)$ in case $t \geq \mathrm{dist}(z, L_0)/2$. $Z'(z, t)$ may not pass through z, but the cone $Z(z, t) = Z'(z, t) - w + z$ pass through z, and

$$d_{x,t}(M_1, Z(z, t)) \leq C_7 t^\beta + C_8 t \leq C_9 t^\beta.$$

□

Proof of Theorem 1.2. Let M_1 be the same as in Lemma 7.12, and let $M = \Psi^{-1}(E \cap U)$. Then by Lemma 7.12, we have that for any $x \in M_1 \cap B(0, \rho_3)$ and $0 < r < 2\rho_3$, there exist cone $Z(x, r)$ such that

$$d_{x,r}(M_1, Z(x, r)) \leq Cr^\beta,$$

where $Z(x, r)$ is a minimal cone in $\mathbb{R}^3$ of type $\mathbb{P}$ or $\mathbb{Y}$ in case $x \notin L_0$ and $t \leq \mathrm{dist}(x, L_0)/2$; and $Z(x, r)$ is a sliding minimal cone in Ω_0 with sliding boundary L_0 of type $\mathbb{P}_+$ or $\mathbb{Y}_+$ in other case. We apply Theorem 5.1 to get that there exist $\rho_4 > 0$, a sliding minimal cone Z' centered at 0, and a mapping $\Phi_1 : \Omega_0 \cap B(0, \rho_4) \to \Omega_0$, which is a $C^{1,\beta}$-diffeomorphism such that $\Phi_1(0) = 0$, $\Phi_1(\partial\Omega_0 \cap B(0, \rho_4)) \subseteq L_0$, $\|\Phi - \mathrm{id}\| \leq 10^{-1}\rho_4$ and

$$M_1 \cap B(0, \rho_4) = \Phi(Z') \cap B(0, \rho_4).$$

We take $Z = Z' \cup L_0$, then we get that

$$M \cap B(0, \rho_4) = \Phi(Z) \cap B(0, \rho_4).$$

□

8. Existence of the plateau problem with sliding boundary conditions

The Plateau Problem with sliding boundary conditions arise in [7], proposed by Guy David. That is, given an initial set E_0, and boundary Γ, to find the minimizers among all competitors. The author of the paper [7] also gives the sketch to the existence in Section 6, and later on in [6], he pave the way. We will give an existence result in case the boundary is nice enough.

Let $\Omega \subseteq \mathbb{R}^3$ be a closed domain such that the boundary $\partial\Omega$ is a 2-dimensional manifold of class $C^{1,\alpha}$ for some $\alpha > 0$. Let $E_0 \subseteq \Omega$ be a closed set with $E_0 \supseteq \partial\Omega$. We denote by $\mathscr{C}(E_0)$ the collection of all competitors of E_0.

Theorem 8.1. *If there is a bounded minimizing sequence of competitors, then there exists $E \in \mathscr{C}(E_0)$ such that*

$$\mathcal{H}^2(E \setminus \partial\Omega) = \inf\{\mathcal{H}^2(S \setminus \partial\Omega) : S \in \mathscr{C}(E_0)\}$$

Proof. We put

$$m_0 = \inf\{\mathcal{H}^2(S \setminus \partial\Omega) : S \in \mathscr{C}(E_0)\}.$$

If $m_0 = +\infty$, we have nothing to do. We now assume that $0 \leq m_0 < +\infty$.

Let $\{S_i\} \subseteq \mathscr{C}_0$ be a sequence of competitors bounded by $B(0, R)$ such that

$$\lim_{i\to\infty} \mathcal{H}^2(S_i \setminus \partial\Omega) = m_0.$$

Apply Lemme 5.2.6 in [11], we can fined a sequence of open sets $\{U_i\}$ and a sequence of competitors $\{E_i\} \subseteq \mathscr{C}(E_0)$ of E_0 bounded by $B(0, R+1)$ such that

- $U_i \subseteq U_{i+1}$, $\cup_{i\geq 1} U_i = B(0, R+2) \setminus \partial\Omega$;
- $E_i \cap U_i \in QM(U_i, M, \operatorname{diam}(U_i))$ for constant $M > 0$;
- $\mathcal{H}^2(E_i) \leq \mathcal{H}^2(S_i) + 2^{-i}$.

We assume that E_i converge locally to E_∞ in $B(0, R+2)$, pass to subsequence if necessary, then by Corollary 21.15 in [6], we get that E_∞ is sliding minimal.

Since $E_i \cap U_i \in QM(U_i, M, \operatorname{diam}(U_i))$, by Lemma 3.3 in [4], we have that

$$\mathcal{H}^2(E_\infty \cap U_i) \leq \liminf_{k\to\infty} \mathcal{H}^2(E_k \cap U_i) \leq m_0,$$

thus

$$\mathcal{H}^2(E_\infty \setminus \partial\Omega) \leq m_0.$$

By Theorem 1.2 and Theorem 1.15 in [5], we get that E_∞ is local Lipschitz neighborhood retract. We denote by φ a Lipschitz neighborhood retraction of E_∞, since E_i converges to E_∞, we get that $\varphi(E_k) \subseteq E_\infty$ for k large enough. Thus $\varphi(E_k)$ are minimizers. □

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