COHOMOLOGY RING OF TREE BRAID GROUPS AND EXTERIOR FACE RINGS

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ABSTRACT. For a tree T and a positive integer n, let $B_n T$ denote the n-strand braid group on T. We use discrete Morse theory techniques to show that the cohomology ring $H^*(B_n T)$ is encoded by an explicit abstract simplicial complex $K_n T$ that measures n-local interactions among essential vertices of T. We show that, in many cases (for instance when T is a binary tree), $H^*(B_n T)$ is the exterior face ring determined by $K_n T$.

1. MAIN RESULTS

For a finite graph Γ and a positive integer n, let $\mathrm{Conf}_n \Gamma$ denote the configuration space of n ordered points on Γ,

$$\mathrm{Conf}_n \Gamma := \{(x_1, \ldots, x_n) \in \Gamma^n : x_i \neq x_j \text{ for } i \neq j\}.$$

The usual right action of the n-symmetric group Σ_n on $\mathrm{Conf}_n \Gamma$ is given by $(x_1, \ldots, x_n) \cdot \sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, and $\mathrm{UConf}_n \Gamma$ stands for the corresponding orbit space, the configuration space of n unlabelled points on Γ. Both $\mathrm{Conf}_n \Gamma$ and $\mathrm{UConf}_n \Gamma$ are known to be aspherical [1, 10]; their corresponding fundamental groups are denoted by $P_n \Gamma$ (the pure n-braid group on Γ) and $B_n \Gamma$ (the full n-braid group or, simply, the n-braid group on Γ). We focus on the case of a tree $\Gamma = T$.

Besides its central role in geometric group theory, graph braid groups have applications in areas outside pure mathematics such as robotics, topological quantum computing and data science. Yet, there is a relatively limited knowledge of the algebraic topology properties of a graph braid group (or, for that matter, of a tree braid group), particularly concerning its cohomology ring structure.

Using discrete Morse theory techniques on Abrams' cubical model $\mathrm{UD}_n T$ for $\mathrm{UConf}_n T$ (reviewed below), D. Farley gave in [4] an efficient description of the *additive* structure of the cohomology of $B_n T$. Later, and in order to get at the *multiplicative* structure, the Morse theoretic methods were replaced in [5] by the use of a Salvetti complex $\mathcal{S}$ obtained by identifying opposite faces of cells in $\mathrm{UD}_n T$. Being a union of tori, $\mathcal{S}$ has a well understood cohomology ring. Yet more importantly, the projection map $q: \mathrm{UD}_n T \to \mathcal{S}$ induces a surjection in cohomology. Farley's main result in [5] is a description of a set of generators for $\mathrm{Ker}(q^*)$, which yields a presentation for the cohomology ring of $B_n T$.

Although [5] includes an algorithm for performing computations mod $\mathrm{Ker}(q^*)$, the price of not working at the Morse theoretic level is that Farley's presentation includes many non-essential generators. As a result, calculations are hard to work

with, in concrete examples, as well as in theoretical developments (cf. Remark 1.9). In particular, Farley-Sabalka's conjecture [7, Conjecture 5.7] that $H^*(B_nT; \mathbb{Z}_2)$ is an exterior face ring, suggested on the basis of extensive concrete calculations, was left open.

In this paper we combine Farley-Sabalka's original Morse theoretic approach with Forman's Morse-theoretic description of cup products to prove the integral version of Farley-Sabalka's conjecture for a large family of trees. The statement in Theorem 1.1, which focuses on binary trees, i.e., on trees all of whose essential vertices have degree three, disproves Conjecture 5.17 in [15] by exhibiting an infinite family of non-linear trees T all of whose braid group cohomology rings are exterior face rings.

Theorem 1.1. *Assume T is a binary tree. For a commutative ring R with 1, the cohomology ring $H^*(B_nT; R)$ is the exterior face ring $\Lambda_R(K_nT)$ determined by a simplicial complex K_nT. Explicitly, $H^*(B_nT; R)$ is the quotient Λ/I, where Λ is the exterior graded R-algebra generated by the vertex set of K_nT, and I is the ideal generated by monomials corresponding to non-faces of K_nT.*

As noted in [7, p. 68], the isomorphism type of a complex K_nT as the one in Theorem 1.1 is well determined. We refer to K_nT as the *n-interaction complex of T*. A description of K_nT as an abstract simplicial complex is given in Definition 1.3. The explicit definition allows us, for instance, to easily deduce a concrete right-angled Artin group presentation for B_nT when T is a linear binary tree (Example 1.6). This complements the inductive method in [3] proving that linearity is a sufficient[1] condition for a tree to have right-angled Artin braid groups.

The definition of K_nT applies for any tree and we show that the resulting combinatorial object encodes much of the ring structure of $H^*(B_nT; R)$, whether T is binary or not. Indeed, we generalize Theorem 1.1 in two directions. On the one hand, the ring-isomorphism assertion $H^*(B_nT; R) \cong \Lambda_R(K_nT)$ holds as long as T is a tree with binary core (Theorem 6.4). Furthermore, we show that, for any tree T, the vertices of K_nT can be thought of as giving an R-basis of $H^1(B_nT; R)$, while the cup-product-based rule $\{v_1, \ldots, v_m\} \mapsto v_1 \cdots v_m$ sets a 1-1 correspondence between the family of $(m-1)$-simplices of K_nT and an R-basis of $H^m(B_nT; R)$. More importantly, while cup squares are known to vanish in $H^*(B_nT; R)$, certain (square-free) products $v_1 \cdots v_m$ are non-zero even when $\{v_1, \ldots, v_m\}$ fails to be a face of K_nT (this can happen only if T is not a tree with binary core). In any such case, we give a closed formula (Theorem 5.1) to write any such product $v_1 \cdots v_m$ as an R-linear combination of basis elements, thus completing a full description of the cup-product structure in the cohomology of B_nT for any tree T. Details are summarized in Theorem 1.7.

The techniques used in this work (discrete Morse theoretic approach to cup products) should be a valuable tool in understanding the algebraic topology properties of discrete models for other spaces, such as non-particle configuration spaces, as well as generalized (e.g., no-k-equal) configuration spaces.

Remark 1.2. Ghrist's pioneering work led to conjecture that any pure braid group $P_n\Gamma$ on a graph Γ would be a right-angled Artin group. In the case of full braid groups $B_n\Gamma$, [13, 14] give two characterizations (one combinatorial and another cohomological) of the right-angled-Artin condition. For instance, for $\Gamma = T$ a tree,

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[1]The condition is known to be necessary and sufficient.

B_nT is a right-angled Artin group if and only if $H^*(B_nT)$ is the exterior face ring of a *flag* complex. Theorem 1.1 and its generalized version in Theorem 6.4 assert that, in the full braid group setting and for trees with binary core, Ghrist's conjecture is true after removal of the flag requirement.

The description of the complex K_nT, as well as an explicit statement of Theorem 1.7, and a couple of explicit illustrations (Examples 1.5 and 1.6) of Theorem 1.1 require a few preparatory constructions. Unless otherwise noted, throughout the rest of the section T stands for an arbitrary tree.

Fix once and for all a planar embedding together with a root (a vertex of degree 1) for T. Order the vertices of T as they are first encountered through the walk along the tree that (a) starts at the root vertex, which is assigned the ordinal 0, and that (b) takes the left-most branch at each intersection given by an essential vertex (turning around when reaching a vertex of degree 1). Vertices of T will be denoted by the assigned non-negative integer. An edge of T, say with endpoints r and s, will be denoted by the ordered pair (r, s), where $r < s$. Furthermore, the ordering of vertices will be transferred to an ordering of edges by declaring that the ordinal of (r, s) is s. The resulting ordering of vertices and edges will be referred to as the T-order.[2]
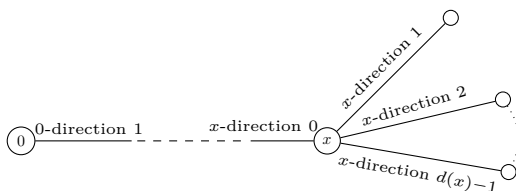

FIGURE 1. The $d(x)$ x-directions from an essential vertex x

Let $d(x)$ stand for the degree of a vertex x of T, so there are $d(x)$ "directions" from x. For a vertex x different from the root, the direction from x that leads to the root is defined to be the x-direction 0; x-directions $1, 2, \ldots, d(x) - 1$ (if any) are then chosen following the positive orientation coming from the planar embedding. See Figure 1. For instance, if x is not the root and the vertex y incident to x in x-direction 0 is not essential (i.e. $d(y) \leq 2$), then $y = x - 1$. Likewise, if $d(x) \geq 2$, then $x + 1$ is the vertex incident to x in x-direction 1. It will be convenient to think of the only direction from the root vertex 0 as 0-direction 1, in particular there is no 0-direction 0.

Fix essential vertices $x_1 < \cdots < x_m$ of T. The complement in T of the set $\{x_1, \ldots, x_m\}$ decomposes into $1 + \sum_{i=1}^{m} (d(x_i) - 1)$ components
$$C_{i,\ell_i} = C_{i,\ell_i}(x_1, \ldots, x_m),$$
where $0 \leq i \leq m$, $\ell_0 = 1$, and $1 \leq \ell_i \leq d(x_i) - 1$ for $i > 0$. The closure of each C_{i,ℓ_i} is a subtree of T. $C_{0,1}$ is the component containing the root 0, while C_{i,ℓ_i} (for $i > 0$) is the component whose closure contains x_i and is located on the x_i-direction ℓ_i. The set $B(C_{i,\ell_i})$ of "bounding" vertices of a component C_{i,ℓ_i} is defined to be the intersection of the closure of C_{i,ℓ_i} with $\{x_1, \ldots, x_m\}$. Note that $x_i \in B(C_{i,\ell_i})$ for $i > 0$, however the root 0 is not considered to be a bounding vertex of $C_{0,1}$, just

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[2]This of course depends on the embedding and root chosen.

as no leave of T (i.e., a vertex of degree 1 other than the root) is considered to be a bounding vertex of any C_{i,ℓ_i}.

Definition 1.3 (The n-interaction complex of T, K_nT)**.**

(a) The vertex set V_nT of K_nT is the collection of all 4-tuples $\nu = \langle k, x, p, q \rangle$, where k is a non-negative integer number, x is an essential vertex of T, and $p = (p_1, \ldots, p_r)$ and $q = (q_1, \ldots, q_s)$ are tuples of non-negative integer numbers satisfying the three conditions
   - $r + s = d(x) - 1$, with $r > 0 < s$;
   - $k + |p| + |q| = n - 1$, where $|p| := \sum_{j=1}^r p_j$ and $|q| := \sum_{j=1}^s q_j$;
   - $p_j > 0$ for at least one $j \in \{1, \ldots, r\}$.

   We stress that r (i.e., the length of p) is one of the parameters determining the 4-tuple ν. For instance, if $d(x) = 6$ and $n = 4$, then $\langle 1, x, (0,1,0), (1,0) \rangle$ and $\langle 1, x, (0,1), (0,1,0) \rangle$ are two different elements in V_nT. The length s of q, on the other hand, is determined by r and $d(x)$.

(b) For $\nu_1, \ldots, \nu_m \in V_nT$ with $\nu_i = \langle k_i, x_i, p_i, q_i \rangle$, $p_i = (p_{i,1}, \ldots, p_{i,r_i})$, $q_i = (q_{i,1}, \ldots, q_{i,s_i})$ and so that $x_1 < \cdots < x_m$, consider the components $C_{0,1}$ and C_{i,ℓ_i} ($1 \le i \le m$ and $1 \le \ell_i \le d(x_i) - 1$) of $T \setminus \{x_1, \ldots, x_m\}$ as defined above. Then, for $C \in \{C_{0,1}, C_{i,\ell_i}\}$, the C-local information of ν_j, denoted by $\ell_C(\nu_j)$, is defined by

$$\ell_{C_{0,1}}(\nu_j) = \begin{cases} k_j, & \text{if } x_j \in B(C_{0,1}); \\ 0, & \text{otherwise,} \end{cases}$$

and, for $i > 0$,

(1)
$$\ell_{C_{i,\ell_i}}(\nu_j) = \begin{cases} p_{i,\ell_i}, & \text{if } j = i \text{ and } \ell_i \le r_i; \\ q_{i,\ell_i - r_i}, & \text{if } j = i \text{ and } \ell_i > r_i; \\ k_j, & \text{if } j \ne i \text{ and } x_j \in B(C_{i,\ell_i}); \\ 0, & \text{in any other case.} \end{cases}$$

   Note that $\ell_C(\nu_j) = 0$ whenever $x_j \notin B(C)$.

(c) The n-interaction complex of T is the abstract simplicial complex K_nT whose vertex set is V_nT and whose $(m-1)$-simplices are given by families of vertices $\nu_1, \ldots, \nu_m$ as in item (b) satisfying

(2)
$$\sum_{j=1}^m \ell_{C_{i,\ell_i}}(\nu_j) \ge n \left(\mathrm{card}(B(C_{i,\ell_i})) - 1 \right),$$

   for all $i \in \{0, 1, \ldots, m\}$ and all relevant ℓ_i, and in such a way that, for every $i > 0$, (2) is a strict inequality for at least one $\ell_i \in \{1, \ldots, r_i\}$.

It is an easy arithmetic exercise (whose verification is left to the reader) to check that K_nT is indeed a simplicial complex.

Definition 1.3 is dictated by discrete Morse theoretic considerations —reviewed in later sections. Our choice for using angle brackets instead of parentheses for 4-tuples in V_nT will be justified later in the paper (Remark 6.2). More important at this point is to explain the role of K_nT as an object measuring "local interactions" between systems of "local informations" around essential vertices of T. For starters, we refer to a vertex $\nu = \langle k, x, (p_1, \ldots, p_r), (q_1, \ldots, q_s) \rangle \in V_nT$ as a system of local
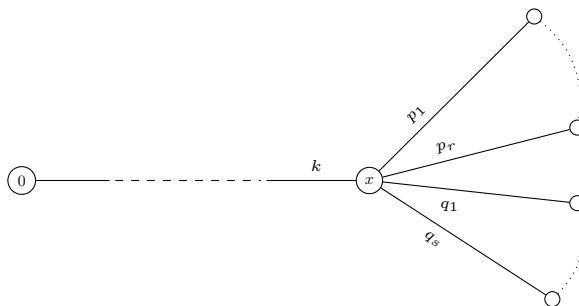
FIGURE 2. The local information given by a vertex $\langle k, x, (p_1, \ldots, p_r), (q_1, \ldots, q_s)\rangle$ of $K_n T$

informations around the essential vertex x of T. Indeed, as illustrated in Figure 2, we think of:

(i) k as the local information of ν in x-direction 0,

(ii) p_j $(1 \leq j \leq r)$ as the local information of ν in x-direction j, and

(iii) q_j $(1 \leq j \leq s)$ as the local information of ν in x-direction $j + r$.

In these terms, (1) gives a systematic way to spell out the information ingredients on a given family of systems of local informations. Likewise, item (c) in Definition 1.3 asserts that a family $\{\nu_1, \ldots, \nu_m\}$ of systems of local informations around essential vertices $x_1 < \cdots < x_m$ of T assembles a simplex of $K_n T$ if, for each component C of $T \setminus \{x_1, \ldots, x_m\}$, the sum of the C-local informations of vertices x_j bounding C is suitably large, depending on n and on the number of bounding vertices of C.

Definition 1.4. Let $\nu_1, \nu_2, \ldots, \nu_m \in V_n T$ be a family of systems of local informations around essential vertices $x_1 < x_2 < \cdots < x_m$ of T. We say that $\nu_1, \ldots, \nu_m$ interact strongly provided $\{\nu_1, \ldots, \nu_m\}$ is a simplex of $K_n T$. We say that $\nu_1, \ldots, \nu_m$ interact weakly provided (2) holds for all relevant i and ℓ_i but $\{\nu_1, \ldots, \nu_m\}$ fails to be a simplex of $K_n T$ —so that, in fact, (2) is an equality for some $i > 0$ and all $\ell_i \in \{1, \ldots, r_i\}$. In all other cases, we say that $\nu_1, \ldots, \nu_m$ do not interact.
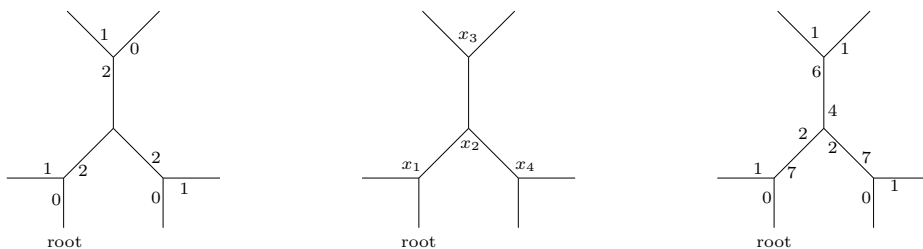

FIGURE 3. Three different aspects of the minimal non-linear tree T_0

Example 1.5. Figure 3 shows three aspects of the smallest possible non-linear tree T_0. The four essential vertices are labelled (following the T_0-order) in the central

picture. The fact that the 4-fold product

$$(3) \qquad \langle 0, x_1, (1), (7) \rangle \langle 2, x_2, (4), (2) \rangle \langle 6, x_3, (1), (1) \rangle \langle 7, x_4, (1), (0) \rangle \in H^4(B_9 T_0; R)$$

is a basis element follows from Theorem 1.1, as inspection in the picture on the right of Figure 3 reveals that the factors in (3) interact strongly. Note that $r = s = 1$ for each factor in (3), and that the cases with a strict inequality in (2) hold as required in the last clause of item (c) of Definition 1.3. Likewise, interaction analysis in the picture on the left exhibits the well known fact that $K_4 T_0$ is not flag (i.e., $B_4 T$ is not a right-angled Artin group): the three basis elements $\langle 0, x_1, (1), (2) \rangle$, $\langle 2, x_3, (1), (0) \rangle$ and $\langle 2, x_4, (1), (0) \rangle$ in $H^1(B_4 T_0; R)$ have pairwise strong interactions (so their three double products are part of a basis of $H^2(B_4 T_0)$), but the three basis elements do not interact (so their triple product vanishes).
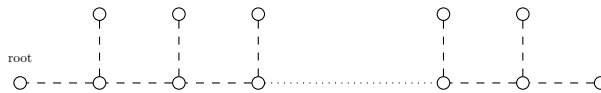

FIGURE 4. A planar embedding of a binary linear tree

Example 1.6. Let T be a binary tree whose essential vertices lie along a single embedded arc. Choosing the planar embedding shown in Figure 4, we see that $B_n T$ has a right-angled Artin group presentation with generators $\langle k, x, p, q \rangle$, where x is an essential vertex of T and k, p, q are non-negative integer numbers[3] satisfying $p > 0$ and $k + p + q = n - 1$. In these terms, $B_n T$ has a commutativity relation $\langle k, x, p, q \rangle \langle k', x', p', q' \rangle = \langle k', x', p', q' \rangle \langle k, x, p, q \rangle$ whenever $x < x'$ and $q + k' \geq n$, where the former inequality refers to the T-order resulting from the embedding. Note that the chosen planar embedding of T rules out weak interactions.

Theorem 1.7. *For any tree T, any non-negative integer n and any commutative ring R with unit 1, there is a set-theoretic inclusion $V_n T \hookrightarrow H^1(B_n T; R)$ so that the faces of $K_n T$ yield, via cup-product of their vertices, a graded basis of $H^*(B_n T; R)$. For instance, the empty face $\varnothing \in K_n T$ corresponds to the unit $1 \in H^0(B_n T; R) = R$. Furthermore, any product $\langle k, x, p, q \rangle \cdot \langle k', x', p', q' \rangle$ with $x = x'$ vanishes (in particular cup-squares vanish), as do cup-products of non-interacting elements in $V_n T$.*

The only piece of multiplicative information missing in Theorem 1.7, namely a description of cup-products of weak-interacting basis elements in $V_n T$, is fully addressed in Section 5 (see Theorem 5.1) through the concept of "interaction parameters" introduced in Section 4 (Definition 4.3).

Remark 1.8. The only obstructions for realizing $H^*(B_n T; R)$ in Theorem 1.7 as the exterior face ring determined by $K_n T$ are the non-vanishing products whose factors interact weakly. For trees with binary core, such weak-interacting non-trivial products are effectively ruled out in the final section of this paper (Theorem 6.4) by means of a suitable change of basis that adjusts the inclusion $V_n T \hookrightarrow H^1(B_n T; R)$ in Theorem 1.7.

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[3] Instead of writing the 1-tuples (p) and (q), we have simply written p and q.

Remark 1.9. The results in this paper allow us to recover and generalize Scheirer's main technical tool [16, Lemma 3.6] for studying Farber's topological complexity of B_nT. Extensions of Scheirer's results will be the topic of a future publication.

In the rest of the paper we shall omit writing the coefficient ring R in cohomology groups and associated (co)chain complexes.

2. Preliminaries

We start by collecting the ingredients and facts we need: cup-products in the cubical setting [11, 12], reviewed in Section 2.1, Forman's discrete Morse theory [8, 9], reviewed in Subsection 2.2, and Farley-Sabalka's gradient field on Abrams' discrete model for (ordered and unordered) graph configuration spaces [1, 2, 6, 13], reviewed in Subsection 2.3. This will set the notation we use in the rest of the paper.

2.1. **Cup products in cubical sets.** An elementary cube in $\mathbb{R}^k$ is a cartesian product $c = I_1 \times \cdots \times I_k$ of intervals $I_i = [m_i, m_i + \epsilon_i]$, where $m_i \in \mathbb{Z}$ and $\epsilon_i \in \{0, 1\}$. For simplicity, we write $[m] := [m, m]$ for a degenerate interval. We say that c is an ℓ-cube if there are ℓ non-degenerate intervals among the cartesian factors I_j of c, say $I_{i_1}, \ldots, I_{i_\ell}$ with $1 \leq i_1 < \cdots < i_\ell \leq k$. In such a case, the product orientation of c is determined by (a) the orientation (from smaller to larger endpoints) of the non-degenerate intervals $I_{i_1}, \ldots, I_{i_\ell}$, and (b) the order $i_1 < \cdots < i_\ell$, i.e., the order of factors in the cartesian product. Under these conditions, and for $1 \leq r \leq \ell$, set

(4)
$$\delta_{2r}(c) = I_1 \times \cdots \times I_{i_r - 1} \times [m_{i_r} + 1] \times I_{i_r + 1} \times \cdots \times I_k,$$
$$\delta_{2r-1}(c) = I_1 \times \cdots \times I_{i_r - 1} \times [m_{i_r}] \times I_{i_r + 1} \times \cdots \times I_k.$$

Then, for a cubical set $X \subset \mathbb{R}^k$, i.e., a union of elementary cubes in $\mathbb{R}^k$, the boundary map $\partial \colon C_\ell(X) \to C_{\ell-1}(X)$ in the oriented cubical chain complex $C_*(X)$ is determined by

(5)
$$\partial(c) = \sum_{r=1}^{\ell} (-1)^{r-1} \left(\delta_{2r}(c) - \delta_{2r-1}(c) \right).$$

For instance, the oriented cubical boundary of the square $[0, 1] \times [0, 1]$ can be depicted as
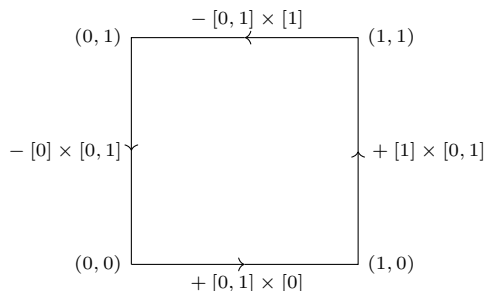
Example 2.1. Let T be a tree whose vertices and edges have been ordered as described in the previous section. Think of T as cubical set. In fact, orient the edges of T from the smaller to the larger endpoints and fix an orientation-preserving embedding $T \subset \mathbb{R}^t$ of cubical sets, where elementary cubes in $\mathbb{R}^t$ have product orientation. Thus, a vertex of T becomes a 0-cube $[k_1] \times \cdots \times [k_t]$ in $\mathbb{R}^t$, while an

oriented edge in T corresponds in $\mathbb{R}^t$ to an oriented 1-cube $I_1 \times \cdots \times I_t$, i.e., an elementary cube whose factors are degenerate, except for one of them.

Cup products in cubical cohomology are fairly similar to their classic simplicial counterparts. At the oriented cubical cochain level, there is a cup product graded map $C^*(X) \times C^*(X) \to C^*(X)$ that is associative, R-bilinear and is described on basis elements as follows. Firstly, for intervals $[a, b]$ and $[a', b']$, let

$$[a, b] \cdot [a', b'] := \begin{cases} [a, b'], & \text{if } b = a' \text{ and either } a = b \text{ or } a' = b' \text{ (or both);} \\ 0, & \text{otherwise.} \end{cases}$$

Then, for elementary cubes $c = I_1 \times \cdots \times I_k$ and $d = J_1 \times \cdots \times J_k$ in X, the cubical cup product $c \cdot d$ of the corresponding basis elements[4] $c, d \in C^*(X)$ vanishes if either $I_i \cdot J_i = 0$ for some $i \in \{1, \ldots, k\}$ or if $(I_1 \cdot J_1) \times \cdots \times (I_k \cdot J_k)$ is not a cube in X; otherwise $c \cdot d$ is up to a sign $\epsilon_{c,d}$, the dual of the cube $(I_1 \cdot J_1) \times \cdots \times (I_k \cdot J_k)$. Given our product-orientation settings, the sign is given by the usual algebraic-topology convention:

$$\epsilon_{c,d} = \sum_{j=1}^{k-1} \left(\dim J_j \sum_{i=j+1}^{k} \dim I_i \right).$$

Remark 2.2. Particularly agreeable is the fact that a finite cartesian product of cubical sets comes equipped for free with the obvious structure of a cubical set. For instance, in the situation of Example 2.1, the cartesian power T^n is a (product-oriented) cubical set in $\mathbb{R}^{nt}$. In such a setting, an oriented cube $c = c_1 \times \cdots \times c_n$ in T^n (where each c_i is either a vertex or an edge of T) corresponds in $\mathbb{R}^{nt}$ to an oriented cube $(I_{1,1} \times \cdots \times I_{1,t}) \times \cdots \times (I_{n,1} \times \cdots \times I_{n,t})$ where, for each $i = 1, \ldots, n$, at most one of the intervals $I_{i,1}, \ldots, I_{i,t}$ is non-degenerate. These considerations, coupled with the fact that cubes of a single factor T are at most one-dimensional, yield the next explicit description of cubical cup-products associated to T and T^n.

Proposition 2.3. *The cup product in $C^*(T)$ of the duals of a pair of (oriented) cubes c and d in T is given by the dual of*

$$c \cdot d = \begin{cases} (x, y), & \text{if } c = (x, y), \text{ an edge of } T, \text{ and } d = y, \text{ a vertex of } T; \\ (x, y), & \text{if } c = x, \text{ a vertex of } T, \text{ and } d = (x, y), \text{ an edge of } T; \\ x, & \text{if } c = d = x, \text{ a vertex of } T; \\ 0, & \text{otherwise.} \end{cases}$$

More generally, let D be a (product-oriented) cubical subset of T^n. The cup product in $C^*(D)$ of the duals of a pair of cubes $c = c_1 \times \cdots \times c_n$ and $d = d_1 \times \cdots \times d_n$ in D vanishes provided $c_i \cdot d_i = 0$ for some $i \in \{1, \ldots, n\}$ or, else, provided the cube $c \cdot d := (c_1 \cdot d_1) \times \cdots \times (c_n \cdot d_n)$ is not contained in D. Otherwise, the cup product is the multiple $(-1)^{\varepsilon_{c,d}}$ of the dual of $c \cdot d$, where

$$\varepsilon_{c,d} = \sum_{j=1}^{n-1} \left(\dim(d_j) \sum_{i=j+1}^{n} \dim(c_i) \right).$$

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[4]We shall omit the use of an asterisk for dual elements. The intended meaning will be clear from the context.

2.2. **Discrete Morse theory.** Let X denote a finite regular cell complex with face poset $(\mathcal{F}, \subset)$, i.e., $\mathcal{F}$ is the set of (closed) cells of X partially ordered by inclusion. For a cell $a \in \mathcal{F}$, we write $a^{(p)}$ to indicate that a is p-dimensional. We think of the Hasse diagram $H_{\mathcal{F}}$ of $\mathcal{F}$ as a directed graph: it has vertex set $\mathcal{F}$, while directed edges (called also "arrows") are given by the family of ordered pairs $(a^{(p+1)}, b^{(p)})$ with $b \subset a$. Such an arrow will be denoted as $a^{(p+1)} \searrow b^{(p)}$. Let W be a partial matching on $H_{\mathcal{F}}$, i.e., a directed subgraph of $H_{\mathcal{F}}$ whose vertices have degree precisely 1. The modified Hasse diagram $H_{\mathcal{F}}(W)$ is the directed graph obtained from $H_{\mathcal{F}}$ by reversing all arrows of W. A reversed edge is denoted as $b^{(p)} \nearrow a^{(p+1)}$, in which case a is said to be W-collapsible and b is said to be W-redundant.

Discrete Morse theory focuses on gradient paths, i.e., directed paths in $H_{\mathcal{F}}(W)$ given by an alternate chain of up-going and down-going arrows,

$$(6) \qquad a_0 \nearrow b_1 \searrow a_1 \nearrow \cdots \nearrow b_k \searrow a_k \quad \text{and} \quad c_0 \searrow d_1 \nearrow c_1 \searrow \cdots \searrow d_k \nearrow c_k.$$

A gradient path as the one on the left (right) hand-side of (6) is called an upper (respectively, lower) path, and the gradient path is called elementary when $k = 1$, or constant when $k = 0$. The sets of upper and lower paths that start on a p-cell a and end on a p-cell b are denoted by $\overline{\Gamma}(a, b)$ and $\underline{\Gamma}(a, b)$, respectively. Concatenation of upper/lower paths $\overline{\Gamma}(a, b) \times \overline{\Gamma}(b, c) \to \overline{\Gamma}(a, c)$ and $\underline{\Gamma}(a, b) \times \underline{\Gamma}(b, c) \to \underline{\Gamma}(a, c)$ is defined in the obvious way; for instance, any upper/lower path is a concatenation of corresponding elementary paths. An upper/lower path is called a cycle if $a_0 = a_k$ in the upper case of (6), or $c_0 = c_k$ in the lower case. (By construction, the cycle condition can only hold with $k > 1$.) The matching W is said to be a gradient field on X if $H_{\mathcal{F}}(W)$ has no cycles. In such a case, cells of X that are neither W-redundant nor W-collapsible are said to be W-critical or, simply, critical when W is clear from the context. We follow Forman's convention to use capital letters to denote critical cells.

It is well known that a gradient field on X carries all the homotopy information of X. For our purposes, we only need to recall how gradient paths recover (co)homological information. In the rest of the section we assume W is a gradient field on X.

Start by fixing an orientation on each cell of X and, for cells $a^{(p)} \subset b^{(p+1)}$, consider the incidence number $\iota_{a,b}$ of a and b, i.e., the coefficient (± 1, since X is regular) of a in the expression of $\partial(b)$. Here ∂ is the boundary operator in the cellular chain complex $C_*(X)$. The Morse cochain complex $\mathcal{M}^*(X)$ is then defined to be the graded R-free[5] module generated in dimension $p \geq 0$ by the duals[6] of the oriented critical cells $A^{(p)}$ of X. The definition of the Morse coboundary map in $\mathcal{M}^*(X)$ requires the concept of multiplicity of upper/lower paths. In the elementary case, multiplicity is given by

$$(7) \quad \mu(a_0 \nearrow b_1 \searrow a_1) = -\iota_{a_0, b_1} \cdot \iota_{a_1, b_1} \quad \text{and} \quad \mu(c_0 \searrow d_1 \nearrow c_1) = -\iota_{d_1, c_0} \cdot \iota_{d_1, c_1},$$

---

[5] Cochain coefficients are taken in a ground ring R, as we are interested in cup-products.

[6] Recall we omit the use of an asterisk for dual elements.

and, in the general case, it is defined to be a multiplicative function with respect to concatenation of elementary paths. The Morse coboundary is then defined by

$$
\tag{8} \partial(A^{(p)}) = \sum_{B^{(p+1)}} \left(\sum_{b^{(p)} \subset B} \left(\iota_{b,B} \sum_{\gamma \in \overline{\Gamma}(b,A)} \mu(\gamma) \right) \right) \cdot B.
$$

In other words, the Morse theoretic incidence number of A and B is given by the number of "mixed" gradient paths $\overline{\gamma}$ from B to A given as the concatenation of an arrow $B \searrow b$ and a path $\gamma \in \overline{\Gamma}(b, A)$, counted with multiplicity $\mu(\overline{\gamma}) := \iota_{b,B} \cdot \mu(\gamma)$.

Gradient paths yield, in addition, a homotopy equivalence between $\mathcal{M}^*(X)$ and the usual cellular cochain complex $C^*(X)$. Indeed, the formulae
$$
\tag{9}
\overline{\Phi}(A^{(p)}) = \sum_{a^{(p)}} \left(\sum_{\gamma \in \overline{\Gamma}(a,A)} \mu(\gamma) \right) a \qquad \text{and} \qquad \underline{\Phi}(a^{(p)}) = \sum_{A^{(p)}} \left(\sum_{\gamma \in \underline{\Gamma}(A,a)} \mu(\gamma) \right) A
$$

define (on generators) cochain maps $\overline{\Phi} \colon \mathcal{M}^*(X) \to C^*(X)$ and $\underline{\Phi} \colon C^*(X) \to \mathcal{M}^*(X)$ inducing cohomology isomorphisms $\overline{\Phi}^*$ and $\underline{\Phi}^*$ with $(\underline{\Phi}^*)^{-1} = \overline{\Phi}^*$.

2.3. **Abrams discrete model and Farley-Sabalka's gradient field.** For a tree T, think of T^n as the cubical set described in Remark 2.2. Abrams discrete model for $\mathrm{Conf}_n T$ is the largest cubical subset $\mathrm{D}_n T$ of T^n inside $\mathrm{Conf}_n T$. In other words, $\mathrm{D}_n T$ is obtained by removing open cubes from T^n whose closures intersect the fat diagonal. As usual, the symmetric group Σ_n acts on the right of $\mathrm{D}_n T$ by permuting factors. The action permutes in fact cubes, and the quotient complex is denoted by $\mathrm{UD}_n T$. Following Farley-Sabalka's lead, from now on we use the notation $(a_1, \ldots, a_n)$, and even (a), for a cube $a_1 \times \cdots \times a_n$ in T^n (so each a_i is either a vertex or an edge of T), and the notation $\{a_1, \ldots, a_n\}$, and even $\{a\}$, for the corresponding Σ_n-orbit. Beware not to confuse the parenthesis notation with a point of T^n, or the braces notation with a set of elements of T —even if all the a_i's are vertices. The "coordinates" a_i in a cube (a) or in its Σ_n-orbit $\{a\}$ are referred to as the ingredients of the cube. Closures of ingredients of cubes in $\mathrm{D}_n T$ and $\mathrm{UD}_n T$ are therefore pairwise disjoint.

In his Ph.D. thesis, Abrams showed that $\mathrm{D}_n T$ is a Σ_n-equivariant strong deformation retract of $\mathrm{Conf}_n T$ provided T is n-sufficiently subdivided in the sense that each path in T between distinct vertices of degree not equal to 2 passes through at least $n-1$ edges. Such a condition will be in force throughout the paper, although it is not a real restriction because T can be subdivided as needed without altering the homeomorphism type of its configuration spaces. The Σ_n-equivariance of the strong deformation retraction above implies that $\mathrm{UD}_n T$ is a strong deformation retract of $\mathrm{UConf}_n T$. Consequently, we will switch attention from $\mathrm{Conf}_n T$ and $\mathrm{UConf}_n T$ to their homotopy equivalent discrete models $\mathrm{D}_n T$ and $\mathrm{UD}_n T$.

For a vertex x of T different from the root 0, let e_x be the unique edge of T of the form (y, x) —recall this requires $y < x$. Let c be a cube either in $\mathrm{D}_n T$ or $\mathrm{UD}_n T$. A vertex-ingredient x of c is said to be blocked in c if $x = 0$ or, else, if replacing in c the ingredient x by the edge e_x fails to render a cube in the corresponding discrete model; x is said to be unblocked in c otherwise. An edge-ingredient e of a cube c is said to be order-disrespectful in c provided e is of the form (x, y) and there is a vertex ingredient z in c with $x < z < y$ and z adjacent to x (in particular

x must be an essential vertex); e is said to be order-respecting in c otherwise. Blocked vertex-ingredients and order-disrespectful edge ingredients in c are said to be critical. Farley-Sabalka's gradient field (on D_nT and UD_nT) then works as follows. Order the ingredients of a cube c by their T-ordering (as described in Section 1), and look for non-critical ingredients:

(i) If the first such ingredient is an unblocked vertex y in c, then c is redundant, and one sets $c \nearrow c'$, where c' is the cube obtained from c by replacing y by e_y. We say that the pairing $c \nearrow c'$ creates the edge e_y. In this case e_y is an order-respecting edge in c', and all ingredients of c' smaller than e_y are critical.

(ii) If the first such ingredient is an order-respecting edge (w, z) in c, then c is collapsible, and one sets $c'' \nearrow c$, where c'' is the cube obtained from c by replacing (w, z) by z. Again, we say that the edge (w, z) is created by the pairing $c'' \nearrow c$. In this case z is an unblocked vertex in c'', and all ingredients of c'' smaller than e_z are critical.

(iii) If all ingredients of c are critical, then c is critical.

Definition 2.4. For a vertex x and a non-negative integer t, let $S_x(t)$ stand for the family of vertices $x, x + 1, \ldots, x + t - 1$. We think of $S_x(t)$ as a size-t stack of vertices supported by x. Whenever we use such a stack of vertices, the n-sufficiently subdivided condition on T will assure the existence of the required t vertices. Furthermore, for $\ell \in \{0, 1, \ldots, d(x) - 1\}$, let $x[\ell]$ denote the vertex adjacent to x that lies in x-direction ℓ. For instance $x[0] = x - 1$ and $x[1] = x + 1$, if x is essential.
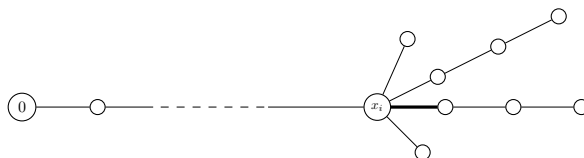

FIGURE 5. Critical ingredients blocked by the root ($k = 2$) and by an order-disrespectful edge $(x_i, x_i[3])$ ($r_i = 2$, $t_{i,1} = 1$, $t_{i,2} = 3$, $t_{i,3} = 2$ and $t_{i,4} = 1$)
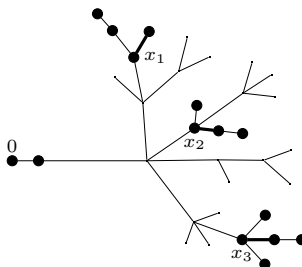

FIGURE 6. A critical 3-cell $\{2 \,|\, x_1, (2), (0) \,|\, x_2, (1, 0), (1) \,|\, x_3, (1), (1, 1)\}$

As illustrated in Figures 5 and 6, ingredients of a critical m-cube are spelled out through

(a) a stack $S_0(k)$ of k vertices supported by the root (here $k \geq 0$, i.e., $S_0(k)$ can be empty);

(b) m pairwise different essential vertices $x_1, \ldots, x_m$ of T and, for each $i = 1, 2, \ldots, m$, an order-disrespectful edge $(x_i, x_i[r_i + 1])$ with $1 \leq r_i < d(x_i) - 1$;

(c) for each $i = 1, 2, \ldots, m$ and each $\ell = 1, 2, \ldots, d(x_i) - 1$, a stack $S_{i,\ell} = S_{y_{i,\ell}}(t_{i,\ell})$ of $t_{i,\ell}$ vertices supported by the vertex

$$y_{i,\ell} := \begin{cases} x_i[\ell], & \text{if } \ell \neq r_i + 1; \\ x_i[\ell] + 1, & \text{if } \ell = r_i + 1, \end{cases}$$

subject to the requirements

(d) some stacks $S_{i,\ell}$ might be empty, i.e., $t_{i,\ell} \geq 0$ for all i and ℓ. Yet, for each i, there must exist an $\ell \in \{1, 2, \ldots, r_i\}$ with $t_{i,\ell} > 0$ (recall that $(x_i, x_i[r_i+1])$ is order-disrespectful);

(e) $k + m + \sum_{i,\ell} t_{i,\ell} = n$, i.e., the total number of ingredients is n.

The critical cube in the unordered discrete model $\mathrm{UD}_n T$ determined by the above information will be denoted as

$$(10) \qquad \left\{ k \,|\, x_1, p_1, q_1 \,|\, \cdots \,|\, x_m, p_m, q_m \right\},$$

where $p_i = (t_{i,1}, \ldots, t_{i,r_i})$ and $q_i = (t_{i,r_i+1}, \ldots, t_{i,d(x_i)-1})$. Vertical bars are meant to stress the fact that each pair of parameters p_i and q_i is ordered and attached to x_i. Other than that, (10) is indeed a set formed by the triples (x_i, p_i, q_i) and the singleton k. Figure 6 illustrates a typical critical cube.

Remark 2.5. In any arrow $d \nearrow c$ of Farley-Sabalka's modified Hasse diagram, d is an even face of c, i.e., in the notation of (4), $d = \delta_{2r}(c)$ for some $r \in \{1, 2, \ldots, \dim(c)\}$.

Remark 2.6. By construction, Farley-Sabalka's gradient field in $\mathrm{D}_n T$ is Σ_n-equivariant and, by passing to the quotient, it yields the corresponding gradient field in $\mathrm{UD}_n T$. Consequently, gradient paths can equivalently be analyzed in either the ordered or unordered settings. Indeed, a gradient path in $\mathrm{UD}_n T$ corresponds to a "Σ_n-orbit" of gradient paths in $\mathrm{D}_n T$. Due to the cup-product descriptions in Subsection 2.1, we find it more convenient to perform the gradient-path analysis at the level of the cubical set $\mathrm{D}_n T$.

3. Gradient-path dynamics

Recall from Subsection 2.1 that the product orientation of a p-dimensional cube $(c_1, \ldots, c_n)$ in $\mathrm{D}_n T$ depends on (the orientation of edges —from the smaller to the larger vertex— in T and on) the coordinate order $c_{i_1}, \ldots, c_{i_p}$, i.e. where $i_1 < \cdots < i_p$, of the edge-ingredients. In particular, the quotient cube $\{c_1, \ldots, c_n\}$ in $\mathrm{UD}_n T$ inherits no well defined orientation. Definition 3.1 avoids the problem and is well suited for the analysis of gradient paths in $\mathrm{D}_n T$.

Definition 3.1 (Gradient orientation, cf. Subsection 2.3 of [5])**.** The listing (x_1, y_1), $\ldots$, (x_p, y_p) of edge-ingredients of a p-cube c in $\mathrm{D}_n T$ or in $\mathrm{UD}_n T$ is said to be in gradient order if $x_1 < \cdots < x_p$, where the latter is the T-ordering of vertices discussed in Section 1. The gradient orientation of c is defined just as the product orientation, except that the gradient order of the edge-ingredients is used (rather than the coordinate order).

In the rest of the paper, and unless explicitly noted otherwise, we use gradient orientations. In doing so, the definitions of the cubes $\delta_{2r}(c)$ and $\delta_{2r-1}(c)$ in (4) require a corresponding adjustment. Namely, if the edge-ingredients of a p-cube c are listed in gradient order as $(x_1, y_1), \ldots, (x_p, y_p)$, then replacing the edge (x_r, y_r) by the vertex y_r or x_r yields $\delta_{2r}(c)$ or $\delta_{2r-1}(c)$, respectively. Remark 2.5 and the expression in (5) for cubical boundaries then remain unaltered. A first advantage of gradient orientations is that the map induced at the cochain level by the projection $\pi \colon \mathrm{D}_n T \to \mathrm{UD}_n T$ involves no signs,

$$(11) \qquad \pi^*(\{c\}) = \sum_{\sigma \in \Sigma_n} (c) \cdot \sigma.$$

(Recall we omit asterisks for duals.) In view of Remark 2.6, the homotopy equivalences in (9) satisfy:

Lemma 3.2. *The following diagram is commutative:*

$$
\begin{array}{ccccc}
\mathcal{M}^*(\mathrm{D}_n T) & \xrightarrow{\overline{\Phi}} & C^*(\mathrm{D}_n T) & \xrightarrow{\Phi} & \mathcal{M}^*(\mathrm{D}_n T) \\
\Big\uparrow{\scriptstyle \pi^*} & & \Big\uparrow{\scriptstyle \pi^*} & & \Big\uparrow{\scriptstyle \pi^*} \\
\mathcal{M}^*(\mathrm{UD}_n T) & \xrightarrow[\overline{\Phi}]{} & C^*(\mathrm{UD}_n T) & \xrightarrow[\underline{\Phi}]{} & \mathcal{M}^*(\mathrm{UD}_n T).
\end{array}
$$

Remark 3.3. The Morse differential in $\mathrm{UD}_n T$ is trivial (see [4] or Proposition 3.10). Therefore, for each $m \geq 0$, a graded basis of $H^m(\mathrm{UD}_n T)$ is given by the cohomology classes of the $\overline{\Phi}$-images of the duals of the critical cubes (10). By abuse of notation,[7] the π^*-image[8] of the cohomology class so determined will also be denoted by the corresponding expression (10). There is no loss of information because vertical maps in the previous diagram are injective and, more importantly, they induce injections in cohomology (the latter assertion follows from a standard transfer argument and the torsion-freeness of $H^*(\mathrm{UD}_n T)$).

This section's goal is the description of a cocycle in $C^*(\mathrm{D}_n T)$ that represents a given cohomology class $\{k \,|\, x, p, q\} \in \mathrm{Im}(\pi^*)$ (Proposition 3.9). This requires the following discussion of dynamics for upper-paths that end at critical cubes.

Definition 3.4. An edge-ingredient (x, y) of a cube c of $\mathrm{D}_n T$ is said to be

- edge order-respecting in c, written as "(x, y) is $\mathrm{eor}(c)$", if there are no edge-ingredients (a, b) in c with $x < a < b < y$.
- strongly order-respecting in c, written as "(x, y) is $\mathrm{sor}(c)$", if (x, y) is $\mathrm{eor}(c)$ and there is no vertex-ingredient v in c with $x < v < y$.

A Farley-Sabalka pairing $\delta_{2i}(c) \nearrow c$ that creates an edge-ingredient that is $\mathrm{sor}(c)$ is said to be of sor type; otherwise, it is said to be of branch type. Likewise, $\delta_{2i}(c) \nearrow c$ is said to be of eor type if the edge-ingredient it creates is $\mathrm{eor}(c)$. An upper elementary path $\delta_{2i}(c) \nearrow c \searrow \delta_j(c)$ is said to be of falling-vertex type (sor type, branch type, respectively) provided $j = 2i - 1$ ($\delta_{2i}(c) \nearrow c$ is of sor type, $\delta_{2i}(c) \nearrow c$ is of branch type, respectively).

---

[7]The context clarifies the meaning.

[8]We prefer to compute products in the ordered setting in view of the explicit descriptions in Subsection 2.1.

Note that if y is the vertex-ingredient in $\delta_{2i}(c)$ that is responsible for a pairing $\delta_{2i}(c) \nearrow c$, say creating the edge-ingredient (x, y) of c, then $\delta_{2i-1}(c)$ is obtained from $\delta_{2i}(c)$ by replacing the vertex y by x. In other words, in the falling-vertex type path $\delta_{2i}(c) \nearrow c \searrow \delta_{2i-1}(c)$, the vertex-ingredient y "falls" to its predecessor x. In particular, elementary paths of falling-vertex type have multiplicity 1.

Example 3.5. Any edge-ingredient $(x, x+1)$ of c is sor(c). On the other hand, for an essential vertex x and a positive direction $\ell \in \{1, 2, \ldots, d(x)-1\}$ from x, an edge-ingredient $(x, x[\ell])$ of c is sor(c) if and only if c has no ingredient, neither vertex nor edge, in any of the components of $T \setminus \{x\}$ lying in x-directions $1, 2, \ldots, \ell - 1$. Furthermore, if (x, y) is an edge-ingredient of a face $\delta_j(c)$ of some cube c of $\mathrm{D}_n T$, then (x, y) is sor($\delta_j(c)$) if and only if (x, y) is sor(c).

The final observation in Example 3.5 is freely used in the proof of:

Proposition 3.6. *Let $(x_1, y_1), \ldots, (x_p, y_p)$ be the gradient-order listing of the edge-ingredients of a p-cube c in $\mathrm{D}_n T$.*

(1) *If an arrow $\delta_{2i}(c) \nearrow c$ in the modified Hasse diagram for $\mathrm{D}_n T$ is of eor type, then (x_i, y_i) is sor(c) and, for any $k > 2i$, $\delta_k(c)$ is collapsible.*

(2) *If the edge (x_i, y_i) is sor(c), then there is no upper path starting at a face $\delta_j(c)$ with $j < 2i - 1$ and ending at a critical cube.*

Proof. (1) By definition, $\delta_{2i}(c) \nearrow c$ creates the edge-ingredient (x_i, y_i), which is assumed to be eor(c). Since ingredients of $\delta_{2i}(c)$ smaller than y_i are critical, (x_i, y_i) is in fact sor(c). Thus, for $k \neq 2i, 2i - 1$, (x_i, y_i) is sor($\delta_k(x)$) and, therefore, order-respecting in $\delta_k(x)$. On the other hand, for $k > 2i$, $\delta_k(c)$ and c have the same ingredients smaller than y_i, so that all ingredients in $\delta_k(c)$ smaller than (x_i, y_i) are critical. Thus, by definition, $\delta_k(x)$ is collapsible for $k > 2i$.

(2) Under the stated hypothesis, assume (for a contradiction) there is a gradient path

$$(12) \qquad c \searrow \delta_j(c) =: c_0 \nearrow d_1 \searrow c_1 \nearrow \cdots \nearrow d_m \searrow c_m$$

with $j < 2i - 1$, $m \geq 0$ and c_m critical. Then (x_i, y_i) is sor(c_0) and, in particular, (x_i, y_i) is order-respecting in c_0, which forces $m > 0$. Recursively, if (x_i, y_i) is an edge-ingredient of both $c_{\ell-1}$ and c_ℓ (and so of d_ℓ), and (x_i, y_i) is sor($c_{\ell-1}$), then (x_i, y_i) is forced to be (sor(d_ℓ) and, thus,) sor(c_ℓ). It is not possible that (x_i, y_i) is an edge-ingredient of all the c_ℓ's, for then (x_i, y_i) would be sor(c_m), which is impossible as c_m is critical. Let k be the first integer ($1 \leq k \leq m$) for which (x_i, y_i) is not an ingredient of c_k —so that (x_i, y_i) is sor(c_ℓ) for $0 \leq \ell < k$. In particular, (x_i, y_i) is order-respecting in c_{k-1}. Thus, the vertex-ingredient v of c_{k-1} responsible for the pairing $c_{k-1} \nearrow d_k$ in (12) satisfies $v < y_i$ and, in fact, $v < x_i$, since (x_i, y_i) is sor(c_{k-1}). On the other hand, since the edge (u, v) created by $c_{k-1} \nearrow d_k$ is order-respecting in d_k, and since c_k is obtained from d_k by replacing the edge (x_i, y_i) by either x_i or y_i, the inequalities $u < v < x_i < y_i$ yield that

$$(13) \qquad\qquad (u, v) \text{ is order-respecting in } c_k \text{ too.}$$

In particular, c_k is not critical, so $k < m$. Let w be the vertex-ingredient of c_k responsible for the pairing $c_k \nearrow d_{k+1}$. By (13), we get the first inequality in $w < v < x_i < y_i$, so

- (w is an ingredient of c_k) $\Rightarrow$ (w is an ingredient of d_k and, therefore, of c_{k-1});

- (w is unblocked in c_k) $\Rightarrow$ (w is unblocked in d_k and, therefore, in c_{k-1}).

But, by definition, v is the minimal unblocked vertex in c_{k-1}, so $v \leq w$, a contradiction. $\square$

Proposition 3.6 implies that upper paths ending at critical cubes have a forced behavior most of the time:

Corollary 3.7. *Let γ be an upper path in $\mathrm{D}_n T$ that ends at a critical cube. Any upper elementary factor of γ of sor type is of falling-vertex type.*

Example 3.8. Let us be specific about the dynamics of an upper path $\gamma \colon c_0 \nearrow d_1 \searrow c_1 \nearrow \cdots \searrow c_m$ that ends at a critical 1-cube c_m. By the Σ_n-equivariance of the gradient field, we can assume $c_0 = (u_1, \ldots, u_i, v_1, \ldots, v_j, (y, y[d]), w_1, \ldots, w_k)$ with $d \in \{1, 2, \ldots, d(y) - 1\}$ and

$$u_1 < \cdots < u_i < y < v_1 < \cdots < v_j < y[d] < w_1 < \cdots < w_k,$$

i.e., c_0 is the Σ_n-orbit representative whose ingredients appear in the T-ordering. By Corollary 3.7, the start of γ is forced to consist of falling-vertex elementary paths, where the vertices $u_1, \ldots, u_i$ fall, each at a time, until they form the stack $S_0(i)$ of i vertices supported (and blocked) by the root. At that point γ arrives at the 1-cube $(S_0(i), v_1, \ldots, v_j, (y, y[d]), w_1, \ldots, w_k)$, and we see that j must be positive, for otherwise γ would have reached a collapsible 1-cube. In particular y must be an essential vertex and $d > 1$. Then, again by Corollary 3.7, it is the turn of vertices $v_1, \ldots, v_j$ that are forced to fall, each at a time, until they form stacks $S_{y[\ell]}(t_\ell)$ of vertices blocked by y in y-directions $\ell = 1, \ldots, d - 1$. At that point γ arrives at a 1-cube of the form

$$(14) \qquad \big(S_0(i), S_{y[1]}(t_1), \ldots, S_{y[d-1]}(t_{d-1}), (y, y[d]), w_1, \ldots, w_k\big).$$

Not all of the stacks $S_{y[\ell]}(t_\ell)$ are empty, so (14) has $(y, y[d])$ as a critical edge-ingredient. The falling-vertex process is also forced by Corollary 3.7 on those vertices $w_1, \ldots, w_k$ that are located in positive y-directions (if any), and this takes γ to a 1-cube of the form

$$(S_0(i), S_{y[1]}(t_1), \ldots, S_{y[d-1]}(t_{d-1}), (y, y[d]), S_{y[d]+1}(t_d), S_{y[d+1]}(t_{d+1}),$$
$$\ldots, S_{y[d(y)-1]}(t_{d(y)-1}), w_\rho, \ldots, w_k),$$

with $w_\rho, \ldots, w_k$ all lying in y-direction 0. Branching starts from this point on, with explicit options discussed in the next paragraph.
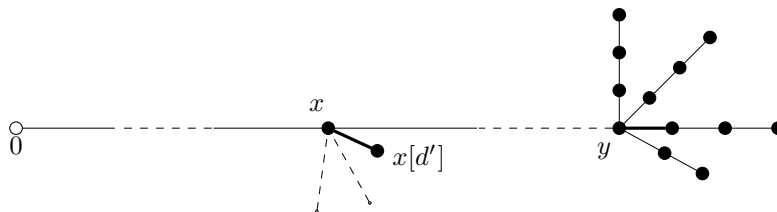

FIGURE 7. A portion of the 1-cube $d_{\lambda+1}$ with its recently created edge $(x, x[d'])$

If no vertices $w_\rho, \ldots, w_k$ are left, then γ would have reached its final critical destination c_m. Otherwise, w_ρ is forced to fall until γ reaches, via some branch

type pairing $c_\lambda \nearrow d_{\lambda+1}$, the 2-cube $d_{\lambda+1}$ depicted in Figure 7. At this point there are two options for $d_{\lambda+1} \searrow c_{\lambda+1}$. In the first option, $c_{\lambda+1}$ is obtained from $d_{\lambda+1}$ by replacing the recently created edge $(x, x[d'])$ by x, i.e., with an upper elementary path $c_\lambda \nearrow d_{\lambda+1} \searrow c_{\lambda+1}$ of falling-vertex type. In such a case, γ is forced to continue with the vertex x falling until it is added to the stack of vertices blocked by the root 0. This leaves us at a situation similar to the one at the start of this paragraph. In the second option, $c_{\lambda+1}$ is obtained from $d_{\lambda+1}$ by replacing the edge $(y, y[d])$ by either of its end points. In such a case, γ is forced to continue:

(1) with the falling of the vertices that are now unblocked in the neighborhood of y (see Figure 7), until they form a stack of vertices blocked by x —thus starting a critical situation around the edge $(x, x[d'])$— and, then,

(2) with the falling of the vertices (if any) in x-directions from d' to $d(x) - 1$, which form (possibly empty) stacks of vertices blocked either by x or $x[d']$ —thus completing the critical situation around the edge $(x, x[d'])$.

Again, this leaves us at a situation similar to the one at the start of this paragraph, but now with the edge $(x, x[d'])$ playing the role of the edge $(y, y[d])$. The branching process in this paragraph then repeats, necessarily a finite number of times, until all vertices $w_\rho, \ldots, w_k$ have been considered, when γ reaches its critical destination c_m.

Proposition 3.9. *A cocycle in $C^*(\mathrm{D}_n T)$ representing a 1-dimensional cohomology class $\{k \,|\, x, p, q\}$ in $\mathrm{Im}(\pi^*)$, with $p = (p_1, \ldots, p_r)$ and $q = (q_1, \ldots, q_s)$, is given by*

$$(15) \qquad \sum \big(u_1, \ldots, u_k, v_1, \ldots, v_{|p|}, (x, x[r+1]), w_1, \ldots, w_{|q|}\big) \cdot \sigma,$$

where the summation runs over

- *all permutations $\sigma \in \Sigma_n$,*
- *all possible vertices $u_1 < \cdots < u_k$ in the component of $T \backslash \{x\}$ in x-direction 0,*
- *all possible vertices $v_1 < \cdots < v_{|p|}$ in the components of $T \backslash \{x\}$ in x-directions from 1 to r so that, for $i \in \{1, \ldots, r\}$, p_i of the vertices $v_1 < \cdots < v_{|p|}$ lie in x-direction i,*
- *all possible vertices $w_1 < \cdots < w_{|q|}$ in the components of $T \backslash \{x\}$ in x-directions greater than r so that, for $j \in \{r+1, \ldots, d(x) - 1\}$, q_{j-r} of the vertices $w_1 < \cdots < w_{|q|}$ lie in x-direction j.*

Proof. By construction, the representing cocycle z we need is obtained by chasing, on the left square of the diagram in Lemma 3.2, the dual of the unordered critical cube $\{c\}$ whose ordered critical representative is

$$(c) := (S_0(k), S_{x[1]}(p_1), \ldots, S_{x[r]}(p_r), (x, x[r+1]), S_{x[r+1]+1}(q_1), S_{x[r+2]}(q_2),$$
$$\ldots, S_{x[d(x)-1]}(q_s)).$$

By (9) and (11),

$$(16) \qquad\qquad z = \overline{\Phi} \circ \pi^* (\{c\}) = \sum_{\gamma \in \mathcal{G}} \mu(\gamma) \cdot \mathcal{S}_\gamma,$$

where $\mathcal{G}$ is the set of upper paths γ that start at a 1-cube $\mathcal{S}_\gamma$ and finish at a 1-cube of the form $c \cdot \sigma$ with $\sigma \in \Sigma_n$. Let $\mathcal{G}'$ be the set of paths $\gamma \in \mathcal{G}$ all of whose upper elementary factors are of falling-vertex type. Since $\mu(\gamma) = 1$ for $\gamma \in \mathcal{G}'$, the analysis

in Example 3.8 shows that the summands in (15) arise from the summands in (16) with $\gamma \in \mathcal{G}'$. It thus suffices to show

$$\sum_{\gamma \in \mathcal{G} \setminus \mathcal{G}'} \mu(\gamma) \cdot \mathcal{S}_\gamma = 0, \tag{17}$$

which will be done by constructing an involution $\iota \colon \mathcal{G} \setminus \mathcal{G}' \to \mathcal{G} \setminus \mathcal{G}'$ such that every pair of paths γ and $\iota(\gamma)$ has the same origin but opposite multiplicities, i.e.,

$$\mathcal{S}_{\iota(\gamma)} = \mathcal{S}_\gamma \quad \text{and} \quad \mu(\iota(\gamma)) = -\mu(\gamma) \tag{18}$$

—thus their contributions to (17) cancel each other out. For a path $\gamma \in \mathcal{G} \setminus \mathcal{G}'$, let $\gamma_{\text{last}} = (c \nearrow d \searrow e)$ denote the last elementary factor of γ that is not of falling-vertex type. In the notation of Example 3.8, e is obtained from d by replacing an edge $(y, y[d])$ by either y or $y[d]$, and both options are possible. Then $\iota(\gamma)$ is defined so to start with the same factorization of γ into elementary paths, except for the elementary factor γ_{last}, for which the other end-point of $(y, y[d])$ is taken, and after which the rest of the elementary factors are of falling-vertex type —just like for γ. Note that the ending 1-cubes of γ and $\iota(\gamma)$ lie in the same Σ_n-orbit, so $\iota(\gamma) \in \mathcal{G} \setminus \mathcal{G}'$. The required properties (18) follow from (the construction and from) the fact that elementary paths of falling-vertex type have multiplicity 1. $\qquad\square$

The cancelation phenomenon in the previous proof allows us to give an easy gradient-path explanation of the main result in [4]: the vanishing of the Morse differential in $\mathrm{UD}_n T$. A variant of the cancellation phenomenon will also play an important role in our evaluation of cup products (Theorem 5.1). Thus, in preparation for that argument, we spell out the gradient proof of:

Proposition 3.10. *The Morse differential in* $\mathrm{UD}_n T$ *vanishes.*

Proof. By Remark 2.6, it suffices to do the gradient path analysis directly at the level of $\mathrm{UD}_n T$. For a pair of unordered critical cubes $c^{(k)}$ and $d^{(k-1)}$, let $\Gamma(c, d)$ be the set of mixed gradient paths $\gamma \colon c \searrow \bullet \nearrow \bullet \searrow \cdots \searrow d$. By (8), we only need to construct an involution $\iota \colon \Gamma(c, d) \to \Gamma(c, d)$ so that, for every $\gamma \in \Gamma(c, d)$, $\mu(\iota(\gamma)) = -\mu(\gamma)$. (Recall that the multiplicity of $\gamma \in \Gamma(c, d)$ is the incidence number for $c \searrow \bullet$ multiplied by the multiplicity of the remaining upper path $\bullet \nearrow \bullet \searrow \cdots \searrow d$.) Let $\Gamma(c, d)_{\text{fall}}$ consist of the paths in $\Gamma(c, d)$ all of whose upper elementary factors are of falling-vertex type. The definition of the restricted $\iota_{\text{fall}} \colon \Gamma(c, d)_{\text{fall}} \to \Gamma(c, d)_{\text{fall}}$ uses the two forms of replacing by a vertex the edge-ingredient at the start of the path. Likewise, for $\Gamma(c, d)_{\text{branch}} := \Gamma(c, d) - \Gamma(c, d)_{\text{fall}}$, the definition of the restricted $\iota_{\text{branch}} \colon \Gamma(c, d)_{\text{branch}} \to \Gamma(c, d)_{\text{branch}}$ uses the two forms of replacing by a vertex the edge ingredient at the last upper elementary factor that is not of falling-vertex type. $\qquad\square$

Propositions 2.3 and 3.9 immediately yield:

Corollary 3.11. *The product of two basis elements* $\{k, x, p, q\}, \{k', x', p', q'\} \in \mathrm{Im}(\pi^*)$ *vanishes provided* $x = x'$. *In particular, squares of 1-dimensional elements in* $\mathrm{Im}(\pi^*)$ *are trivial.*

4. Cup products I: Upper gradient paths

The goal for this section and the next one is to get at a workable description of products

$$(19) \quad \left\{ k_1 \,\middle|\, x_1, (p_{1,1}, \ldots, p_{1,r_1}), (q_{1,1}, \ldots, q_{1,s_1}) \right\} \cdots$$
$$\left\{ k_m \,\middle|\, x_m, (p_{m,1}, \ldots, p_{m,r_m}), (q_{m,1}, \ldots, q_{m,s_m}) \right\}$$

in $\mathrm{Im}(\pi^*)$. Associated to such a product, from now on we set $p_i := (p_{i,1}, \ldots, p_{i,r_i})$, $q_i := (q_{i,1}, \ldots, q_{i,s_i})$, $|p_i| := \sum_{\ell=1}^{r_i} p_{i,\ell}$, $|q_i| := \sum_{\ell=1}^{s_i} q_{i,\ell}$, and make free use of (i) the order-disrespectful edge $(x_i, x_i[r_i+1])$ encoded in the i-th factor of (19), of (ii) the conditions $k_i + \sum_j p_{i,j} + \sum_{j'} q_{i,j'} = n-1$, $r_i + s_i = d(x_i) - 1$ and $r_i, s_i \geq 1$, and of (iii) the fact that, for each i, k_i and all of the $p_{i,\ell}$ and $q_{i,\ell}$ are non-negative, with not all of the $p_{i,\ell}$ being zero. Additionally, in view of Corollary 3.11, we can safely assume $x_1 < \cdots < x_m$. Last, we use the shorthand

$$d_i := d(x_i) - 1 \quad \text{and} \quad \overline{x}_i := x_i[r_i + 1].$$

We start by tuning up the definition in Section 1 of the components C_{i,ℓ_i} of $T \setminus \{x_1, \ldots, x_m\}$.

Definition 4.1 (Leaves and pruned trees). Set $T_{0,1} := C_{0,1}$ and, for $1 \leq i \leq m$ and $1 \leq \ell_i \leq d(x_i) - 1$,

$$T_{i,\ell_i} := \begin{cases} C_{i,\ell_i} \cup \{x_i\}, & \text{if } \ell_i \neq r_i + 1; \\ C_{i,\ell_i} \setminus \mathrm{Int}(x_i, \overline{x}_i), & \text{if } \ell_i = r_i + 1, \end{cases}$$

where $\mathrm{Int}(x_i, \overline{x}_i)$ stands for the interior of the edge $(x_i, \overline{x}_i)$. We think of each T_{i,ℓ_i} $(0 \leq i \leq m)$ as a rooted but possibly pruned tree. Namely, in the notation of Section 1 and setting $x_0 := 0$, the root of T_{i,ℓ_i} is x_i, if $i = 0$ or if $i > 0$ with $\ell_i \neq r_i + 1$, whereas the root of T_{i,r_i+1} is $\overline{x}_i$. Furthermore, the set of pruned leaves of T_{i,ℓ_i} is $L_{i,\ell_i} := B(C_{i,\ell_i}) \setminus \{x_i\}$.

Remark 4.2. Just as the sets L_{i,ℓ_i} give a partition of $\{x_1, \ldots, x_m\}$, the union of the trees T_{i,ℓ_i} agrees with the difference $T \setminus \bigcup_{i=1}^m \mathrm{Int}(x_i, \overline{x}_i)$. Actually, each vertex of T other than x_i for $1 \leq i \leq m$, as well as each semi-open edge $(x,y) \setminus \{y\}$ of T not of the form $(x_i, \overline{x}_i) \setminus \{\overline{x}_i\}$ with $1 \leq i \leq m$, belongs to a tree T_{i,ℓ_i} for a unique ℓ_i.

Definition 1.4 is recast by the second part of:

Definition 4.3.

(1) For a τ-tuple of integers $t = (t_1, \ldots, t_\tau)$, we write $t \geq 0$ to mean that $t_j \geq 0$ for all $j \in \{1, \ldots, \tau\}$, reserving the expression $t > 0$ to mean that $t \geq 0$ with $t_j > 0$ for at least one $j \in \{1, \ldots, \tau\}$. Also, when $t \geq 0$, we write $\underline{t}$ to denote a generic tuple of integers $(t'_1, \ldots, t'_\tau) \geq 0$ satisfying $t'_j \leq t_j$ for all $j \in \{1, \ldots, \tau\}$ with in fact $t'_j < t_j$ for at least one $j \in \{1, \ldots, \tau\}$. We make no distinction between 1-tuples (t_1) and integer numbers t_1 so, accordingly, we use $\underline{t_1}$ instead of $\underline{(t_1)}$.

(2) The interaction parameters $\mathcal{R}_0$, $\mathcal{P}_i := (\mathcal{P}_{i,1}, \ldots, \mathcal{P}_{i,r_i})$ and $\mathcal{Q}_i := (\mathcal{Q}_{i,1}, \ldots, \mathcal{Q}_{i,s_i})$ of the factors in (19) are given by

$$\mathcal{R}_0 := n + \sum_{x_j \in L_{0,1}} (k_j - n),$$

$$\mathcal{P}_{i,\ell_i} := p_{i,\ell_i} + \sum_{x_j \in L_{i,\ell_i}} (k_j - n), \text{ for } i \in \{1, \ldots, m\} \text{ and } \ell_i \in \{1, \ldots, r_i\}, \text{ and}$$

$$\mathcal{Q}_{i,\ell_i} := q_{i,\ell_i} + \sum_{x_j \in L_{i,\ell_i+r_i}} (k_j - n), \text{ for } i \in \{1, \ldots, m\} \text{ and } \ell_i \in \{1, \ldots, s_i\}.$$

If $\mathcal{R}_0 \geq 0$, $\mathcal{P}_i \geq 0$ and $\mathcal{Q}_i \geq 0$ for all $i = 1, \ldots, m$, we say that the factors in (19) interact weakly and, if in addition $\mathcal{P}_i > 0$ for some i, we say that the factors in (19) interact strongly. Otherwise, we say that the factors in (19) do not interact.

Although not reflected in the notation, pruned trees and leaves depend on the essential vertices x_i, while interaction parameters depend on the complete information encoded by the factors in (19). Latter in the paper we will need to use pruned trees, their pruned leaves, as well as interaction parameters of subproducts of (19). In such a case, we will use a notation of the type $T_{i,\ell_i}(x_1, \ldots, x_m)$, $L_{i,\ell_i}(x_1, \ldots, x_m)$, $\mathcal{R}_0(x_1, \ldots, x_m)$, $\mathcal{P}_{i,\ell_i}(x_1, \ldots, x_m)$, $\mathcal{Q}_{i,\ell_i}(x_1, \ldots, x_m)$, as well as $\mathcal{P}_i(x_1, \ldots, x_m)$ and $\mathcal{Q}_i(x_1, \ldots, x_m)$ in order to clarify the factors under consideration.

Next we adapt the expression in (15) for usage within the T_{i,ℓ_i}-notation. In terms of the cocycle representative
(20)
$$\sum \Big(U_i, V_i, (x_i, \overline{x}_i), W_i \Big) \cdot \sigma := \sum \Big(u_1, \ldots, u_{k_i}, v_1, \ldots, v_{|p_i|}, (x_i, \overline{x}_i), w_1, \ldots, w_{|q_i|} \Big) \cdot \sigma$$

in Proposition 3.9 for $\{k_i \,|\, x_i, p_i, q_i\}$, (19) is represented by the sum of all possible products

$$(21) \qquad \cdots \Big((U_i, V_i, (x_i, \overline{x}_i), W_i) \cdot \sigma_i \Big) \cdots \Big((U_j, V_j, (x_j, \overline{x}_j), W_j) \cdot \sigma_j \Big) \cdots .$$

A number of vanishing such products can be ruled out as follows. Fix integers $1 \leq i < j \leq m$. Proposition 2.3 implies that if a product (21) is non-zero, then $(U_i, V_i, (x_i, \overline{x}_i), W_i)$ must have x_j, but cannot have $\overline{x}_j$, as one of its vertex ingredients. Likewise, $(U_j, V_j, (x_j, \overline{x}_j), W_j)$ must have $\overline{x}_i$, but cannot have x_i, as one of its vertex ingredients. Actually, together with Remark 4.2, this shows that non-zero products (21) are best organized (and easily evaluated —see below) by replacing each Σ_n-representative

$$(22) \qquad\qquad (u_1, \ldots, u_{k_i}, v_1, \ldots, v_{|p_i|}, (x_i, \overline{x}_i), w_1, \ldots, w_{|q_i|})$$

in (20) by the one written in a "block" form $(B_0^i, B_1^i, \ldots, B_m^i)$. Here each tuple of ingredients B_j^i starts with the relevant x_j- or $\overline{x}_j$-information (if $j > 0$), and continues with a repacking of the vertex ingredients of (22) that lie in the trees $T_{j,\ell}$ for all relevant ℓ. In detail, for the i-th factor in (19) and each of the corresponding summands in (22), let

(a) $B_0^i := B_{0,1}^i$ be the tuple of vertex ingredients of (22) that lie in $T_{0,1}$, written in T-order;

(b) $B_i^i := \left((x_i, \overline{x}_i), B_{i,1}^i, \ldots, B_{i,d_i}^i\right)$, where $B_{i,\ell}^i$ is the tuple of vertex ingredients of (22) that lie in $T_{i,\ell}$, written in T-order;

(c) If $i < j$, $B_j^i := (x_j, B_{j,1}^i, \ldots, B_{j,d_j}^i)$, where $B_{j,\ell}^i$ is the tuple of vertex ingredients of (22) that lie in $T_{j,\ell}$, written in T-order;

(d) If $j < i$, $B_j^i := (\overline{x}_j, B_{j,1}^i, \ldots, B_{j,d_j}^i)$, where $B_{j,\ell}^i$ is the tuple of vertex ingredients of (22) that lie in $T_{j,\ell}$, written in T-order.

Thus, summands in (20) that have a chance to contribute with non-vanishing products (21) to a cocycle representative of (19) can be written as

$$\left(B_{0,1}^i \middle| \cdots \middle| \overline{x}_{i'}, B_{i',1}^i, \ldots, B_{i',d_{i'}}^i \middle| \cdots \middle| (x_i, \overline{x}_i), B_{i,1}^i, \ldots, B_{i,d_i}^i \middle| \cdots \right.$$
$$\left. \middle| x_{i''}, B_{i'',1}^i, \ldots, B_{i'',d_{i''}}^i \middle| \cdots \right) \cdot \sigma,$$

where vertical bars are used interchangeably by commas, and are intended to make reading easier. Proposition 2.3 then implies that a product (21), written as

$$\left(\left(B_{0,1}^1 \middle| (x_1, \overline{x}_1), B_{1,1}^1, \ldots, B_{1,d_1}^1 \middle| \cdots \middle| x_m, B_{m,1}^1, \ldots, B_{m,d_m}^1\right) \cdot \sigma_1\right)$$
$$\cdots \left(\left(B_{0,1}^m \middle| \overline{x}_1, B_{1,1}^m, \ldots, B_{1,d_1}^m \middle| \cdots \middle| (x_m, \overline{x}_m), B_{m,1}^m, \ldots, B_{m,d_m}^m\right) \cdot \sigma_m\right),$$

is non-zero if and only if $\sigma_i = \sigma_j =: \sigma$ and $B_{t,\ell}^i = B_{t,\ell}^j =: B_{t,\ell}$ for all relevant i, j, t, ℓ, in which case (21) becomes

$$(23) \quad \text{sign}(\widetilde{\sigma}) \left(B_{0,1} \,|\, (x_1, \overline{x}_1), B_{1,1}, \cdots, B_{1,d_1} \,|\, \cdots \,|\, (x_m, \overline{x}_m), B_{m,1}, \cdots, B_{m,d_m}\right) \cdot \sigma,$$

where $\widetilde{\sigma}$ is the permutation determined by the sequence of positions of the edges $(x_1, \overline{x}_1), \ldots, (x_m, \overline{x}_m)$ in the tuple

$$\left(B_{0,1} \,|\, (x_1, \overline{x}_1), B_{1,1}, \ldots, B_{1,d_1} \,|\, \cdots \,|\, (x_m, \overline{x}_m), B_{m,1}, \ldots, B_{m,d_m}\right) \cdot \sigma.$$

Note that the cube in (23) is product-oriented (as required by Proposition 2.3), and that (23) agrees with the gradient-oriented cube

$$\left(B_{0,1} \,|\, (x_1, \overline{x}_1), B_{1,1}, \cdots, B_{1,d_1} \,|\, \cdots \,|\, (x_m, \overline{x}_m), B_{m,1}, \cdots, B_{m,d_m}\right) \cdot \sigma,$$

since $x_1 < \cdots < x_m$. This proves the first half of the next generalization of Proposition 3.9:

Proposition 4.4. *The product* (19) *is represented in* $C^*(D_n T)$ *by the gradient-oriented cocycle*

$$(24) \qquad \sum \left(B_{0,1} \middle| (x_1, \overline{x}_1), B_{1,1}, \ldots, B_{1,d_1} \middle| \cdots \middle| (x_m, \overline{x}_m), B_{m,1}, \ldots, B_{m,d_m}\right) \cdot \sigma,$$

where the summation runs over all permutations $\sigma \in \Sigma_n$ *and all possible tuples* $B_{t,\ell}$ *of vertices written in* T-*order, taken from the corresponding pruned trees* $T_{t,\ell}$, *and having the following lengths: Any block* $B_{0,1}$ *must have* $\mathcal{R}_0$ *ingredients, while any block* $B_{t,\ell}$ *with* $t > 0$ *must have* $\mathcal{P}_{t,\ell}$ *ingredients for* $1 \leq \ell \leq r_t$, *and* $\mathcal{Q}_{t,\ell-r_t}$ *ingredients for* $r_t < \ell \leq d_t$. *In particular,* (19) *vanishes provided its factors do not interact.*

Note that $\mathcal{R}_0 + \sum_{i,\ell} \mathcal{P}_{i,\ell} + \sum_{i,\ell} \mathcal{Q}_{i,\ell} = n - m$ in Definition 4.3. This is compatible with the fact that cubes in (24), if any, have n ingredients. See also Corollary 4.5.
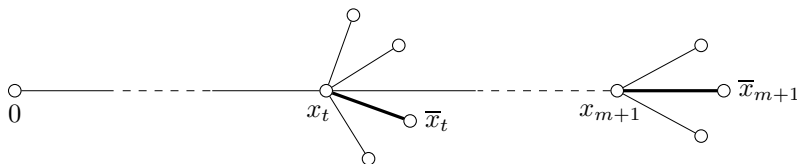
FIGURE 8. The edge $(x_{m+1}, \overline{x}_{m+1})$ belongs to $T_{t,3}$, so the path from x_t to x_{m+1} does not pass through an essential vertex x_j

Proof. It remains to prove the assertions about the sizes of blocks $B_{t,\ell}$, and that all possible such blocks appear in (24). As for the sizes, proceeding by induction on m (with Proposition 3.9 grounding the argument), it suffices to consider a product $\pi_1 \cdot \pi_2$ with

(25)
$$\pi_1 = \left[\left(B_{0,1} \middle| (x_1, \overline{x}_1), B_{1,1}, \ldots, B_{1,d_1} \middle| \cdots \middle| (x_m, \overline{x}_m), B_{m,1}, \ldots, B_{m,d_m} \right) \cdot \sigma \right],$$
$$\pi_2 = \left[\left(U \middle| (x_{m+1}, \overline{x}_{m+1}), V_1, \ldots, V_{d_{m+1}} \right) \cdot \sigma' \right],$$

where $x_1 < \cdots < x_m < x_{m+1}$, and where the structure of the blocks $B_{t,\ell}$ is as specified in the proposition. Here we are assuming (a) that U is a tuple of k_{m+1} vertex ingredients written in T-order and lying in x_{m+1}-direction 0, (b) that any tuple V_ℓ with $1 \leq \ell \leq r_{m+1}$ consists of $p_{m+1,\ell}$ vertex ingredients written in T-order and lying in x_{m+1}-direction ℓ, and (c) that any tuple $V_{\ell+r_{m+1}}$ with $1 \leq \ell \leq s_{m+1}$ consists of $q_{m+1,\ell}$ vertex ingredients written in T-order and lying in x_{m+1}-direction $\ell+r_{m+1}$. In addition, we make the conventions $d_{m+1} := d(x_{m+1})-1$ and $\overline{x}_{m+1} := x_{m+1}[r_{m+1} + 1]$, and assume the relations $d_{m+1} = r_{m+1} + s_{m+1}$, $r_{m+1} \geq 1 \leq s_{m+1}$ and $k_{m+1} + \sum_{\ell=1}^{r_{m+1}} p_{m+1,\ell} + \sum_{\ell=1}^{s_{m+1}} q_{m+1,\ell} = n-1$. Furthermore, signs and orientations will be ignored in the rest of the proof, as they have been carefully addressed in the discussion previous to this proposition. In particular, we can safely work at the unordered-cube level, thus ignoring the permutations σ and σ' in (25) and, instead, thinking of tuples of ingredients as sets of ingredients.

Consider the pruned trees $T_{t,\ell} := T_{t,\ell}(x_1, \ldots, x_m)$ and $T'_{t,\ell} := T_{t,\ell}(x_1, \ldots, x_{m+1})$, as well as the pruned leaves $L_{t,\ell} := L_{t,\ell}(x_1, \ldots, x_m)$ and $L'_{t,\ell} := L_{t,\ell}(x_1, \ldots, x_{m+1})$. There are three cases, depending on whether the edge $(x_{m+1}, \overline{x}_{m+1})$ belongs to $T_{0,1}$, or to $T_{t,\ell}$ with $1 \leq t \leq m$ and $1 \leq \ell \leq r_t$, or to $T_{t,\ell}$ with $1 \leq t \leq m$ and $r_t < \ell \leq d_t$, and the argument is virtually identical in each. We consider only the situation depicted in Figure 8, where the edge $(x_{m+1}, \overline{x}_{m+1})$ belongs to $T_{t,\ell}$ for some $t \in \{1, 2, \ldots, m\}$ and some $\ell \in \{1, 2, \ldots, r_t\}$. In such a case we have

(i) $T_{\tau,\lambda} = T'_{\tau,\lambda}$ and $L_{\tau,\lambda} = L'_{\tau,\lambda}$, for $1 \leq \tau \leq m$ as long as $\tau \neq t$ or $\lambda \neq \ell$;

(ii) $T_{t,\ell} \setminus \text{Int}(x_{m+1}, \overline{x}_{m+1}) = T'_{t,\ell} \bigsqcup \left(\bigcup_{\lambda=1}^{d_{m+1}} T'_{m+1,\lambda} \right)$;

(iii) $L'_{t,\ell} = L_{t,\ell} \cup \{x_{m+1}\}$ and $L'_{m+1,\lambda} = \varnothing$ for $\lambda \in \{1, \ldots, d_{m+1}\}$.

By Proposition 2.3, the product $\pi_1 \cdot \pi_2$ of the elements in (25) vanishes unless

$$\left(\left(\{\overline{x}_1, \ldots, \overline{x}_m\} \sqcup B_{0,1} \sqcup \left(\bigsqcup_{\substack{1 \le \tau \le m \\ 1 \le \lambda \le d_\tau}} B_{\tau,\lambda} \right) \right) \setminus B_{t,\ell} \right) \subseteq U,$$

$$\{x_{m+1}\} \sqcup \left(\bigsqcup_{\lambda=1}^{d_{m+1}} V_\lambda \right) \subseteq B_{t,\ell}$$

and

$$U \setminus \left(\left(\left(\{\overline{x}_1, \ldots, \overline{x}_m\} \sqcup B_{0,1} \sqcup \left(\bigsqcup_{\substack{1 \le \tau \le m \\ 1 \le \lambda \le d_\tau}} B_{\tau,\lambda} \right) \right) \setminus B_{t,\ell} \right) \right)$$

$$= B_{t,\ell} \setminus \left(\{x_{m+1}\} \sqcup \left(\bigsqcup_{\lambda=1}^{d_{m+1}} V_\lambda \right) \right) =: B'_{t,\ell},$$

in which case

$$\pi_1 \cdot \pi_2 = (B_{0,1} \,|\, (x_1, \overline{x}_1), \mathcal{B}_1 \,|\, \cdots \,|\, (x_m, \overline{x}_m), \mathcal{B}_m \,|\, (x_{m+1}, \overline{x}_{m+1}), V_1, \ldots, V_{d_{m+1}}),$$

where $\mathcal{B}_\tau$ is a shorthand for the sequence $B_{\tau,1}, \ldots, B_{\tau,d_\tau}$ provided $\tau \ne t$, whereas $\mathcal{B}_t$ stands for the sequence

$$B_{t,1}, \ldots, B_{t,\ell-1}, B'_{t,\ell}, B_{t,\ell+1}, \ldots, B_{t,d_t}.$$

The induction is complete in view of items (i)–(iii) above and

$$|B'_{t,\ell}| = |B_{t,\ell}| - \left(1 + \sum_{\lambda=1}^{r_{m+1}} p_{m+1,\lambda} + \sum_{\lambda=1}^{s_{m+1}} q_{m+1,\lambda} \right)$$

$$= p_{t,\ell} + \sum_{x_\lambda \in L_{t,\ell}} (k_\lambda - n) - (n - k_{m+1}) = p_{t,\ell} + \sum_{x_\lambda \in L'_{t,\ell}} (k_\lambda - n),$$

which shows that $B'_{t,\ell}$ has the prescribed cardinality. The inductive analysis makes it clear also that all blocks $B_{t,\ell}$ with the structure indicated in the proposition indeed appear in (24). $\qquad\square$

Corollary 4.5. *The product* (19) *agrees with the basis element* $\{\mathcal{R}_0 \,|\, x_1, \mathcal{P}_1, \mathcal{Q}_1 \,|\, \cdots \,|\, x_m, \mathcal{P}_m, \mathcal{Q}_m\}$ *provided the factors of* (19) *interact strongly. Recall* $\mathcal{P}_i = (\mathcal{P}_{i,1}, \mathcal{P}_{i,2}, \ldots, \mathcal{P}_{i,r_i})$ *and* $\mathcal{Q}_i = (\mathcal{Q}_{i,1}, \mathcal{Q}_{i,2}, \ldots, \mathcal{Q}_{i,s_i})$.

Proof. By the strong interaction hypothesis, a summand in (24) that is the target of a lower gradient path γ must actually be critical (and γ must be constant) with ingredients equal to those associated to $\{\mathcal{R}_0 \,|\, x_1, \mathcal{P}_1, \mathcal{Q}_1 \,|\, \cdots \,|\, x_m, \mathcal{P}_m, \mathcal{Q}_m\}$. The conclusion then follows from (9) and (11). $\qquad\square$

Lemma 4.6. *Fix essential vertices* $x_1 < \cdots < x_m$ *and take positive integer numbers* r_i *and* s_i *with* $r_i + s_i = d(x_i) - 1$ *for* $1 \le i \le m$. *Let* R_0, $P_{i,\ell}$, $Q_{i,k}$,

with $1 \leq i \leq m$, $1 \leq \ell \leq r_i$ *and* $1 \leq k \leq s_i$, *be non-negative integers satisfying* $n - m = R_0 + \sum_{i=1}^{m} (\sum_{\ell=1}^{r_i} P_{i,\ell} + \sum_{k=1}^{s_i} Q_{i,k})$. *Then the system*

$$n + \sum_{x_j \in L_{0,1}} (k_j - n) = R_0,$$

$$p_{i,\ell} + \sum_{x_j \in L_{i,\ell}} (k_j - n) = P_{i,\ell} \quad (i = 1, \ldots, m, \ \ell = 1, \ldots, r_i),$$

$$q_{i,k} + \sum_{x_j \in L_{i,k+r_i}} (k_j - n) = Q_{i,k} \quad (i = 1, \ldots, m, \ k = 1, \ldots, s_i)$$

has a unique solution of non-negative integer numbers $\{k_i, p_{i,1}, \ldots, p_{i,r_i}, q_{i,1}, \ldots, q_{i,s_i}\}_{i=1}^{m}$ *satisfying the condition* $n - 1 = k_i + \sum_{\ell=1}^{r_i} p_{i,\ell} + \sum_{k=1}^{s_i} q_{i,k}$ *for each* $i \in \{1, \ldots, m\}$. *If, in addition, for each* $i \in \{1, \ldots, m\}$ *there exists* $\ell \in \{1, \ldots, r_i\}$ *with* $P_{i,\ell} > 0$, *then the unique solution satisfies that, for each* $i \in \{1, \ldots, m\}$, *there exists* $\ell \in \{1, \ldots r_i\}$ *with* $p_{i,\ell} > 0$.

Proof. The two sets of equations with $i = m$ reduce to $p_{m,\ell} = P_{m,\ell}$ ($\ell = 1, 2, \ldots, r_m$) and $q_{m,k} = Q_{m,k}$ ($k = 1, 2, \ldots, s_m$). This also determines

$$(26) \quad k_m := n - \sum_{\ell=1}^{r_m} P_{m,\ell} - \sum_{k=1}^{s_m} Q_{m,k} - 1 = R_0 + \sum_{j=1}^{m-1} \left(\sum_{\ell=1}^{r_j} P_{j,\ell} + \sum_{k=1}^{s_j} Q_{j,k} + 1 \right) \geq 0.$$

The rest of the equations can be written as

$$n + \sum_{x_j \in L_{0,1} \setminus \{x_m\}} (k_j - n) = R_0' := R_0 + \begin{cases} n - k_m, & \text{if } x_m \in L_{0,1} \\ 0, & \text{otherwise} \end{cases},$$

$$p_{i,\ell} + \sum_{x_j \in L_{i,\ell} \setminus \{x_m\}} (k_j - n) = P_{i,\ell}' := P_{i,\ell} + \begin{cases} n - k_m, & \text{if } x_m \in L_{i,\ell} \\ 0, & \text{otherwise} \end{cases},$$

$$q_{i,k} + \sum_{x_j \in L_{i,k+r_i} \setminus \{x_m\}} (k_j - n) = Q_{i,k}' := Q_{i,k} + \begin{cases} n - k_m, & \text{if } x_m \in L_{i,k+r_i} \\ 0, & \text{otherwise} \end{cases},$$

for $i = 1, \ldots, m-1$, $\ell = 1, \ldots, r_i$ and $k = 1, \ldots, s_i$. The result then follows by induction since

$$R_0' + \sum_{j=1}^{m-1} \left(\sum_{\ell=1}^{r_j} P_{j,\ell}' + \sum_{k=1}^{s_j} Q_{j,k}' + 1 \right) = R_0 + \sum_{j=1}^{m-1} \left(\sum_{\ell=1}^{r_j} P_{j,\ell} + \sum_{k=1}^{s_j} Q_{j,k} + 1 \right) + n - k_m$$

$$= R_0 + \sum_{j=1}^{m} \left(\sum_{\ell=1}^{r_j} P_{j,\ell} + \sum_{k=1}^{s_j} Q_{j,k} + 1 \right) = n,$$

where the second equality uses (26). □

Proof of Theorem 1.7. Corollary 4.5 and Lemma 4.6 yield a set theoretic identification $\mathcal{S}_m = \mathcal{B}_m$, where $\mathcal{S}_m$ is the set of products (19) whose factors interact strongly, and $\mathcal{B}_m$ is the m-dimensional basis of $\text{Im}(\pi^*)$ with basis elements

$$\{R_0 \mid x_1, (P_{1,1}, \ldots, P_{1,r_1}), (Q_{1,1}, \ldots, Q_{1,s_1}) \mid \cdots \mid x_m, (P_{m,1}, \ldots, P_{m,r_m}),$$
$$(Q_{m,1}, \ldots, Q_{m,s_m})\}.$$

Together with Corollary 3.11 and Proposition 4.4, this completes the proof, where $\langle k, x, p, q \rangle \in V_n T$ is identified with (the π^*-preimage of) $\{k \mid x, p, q\} \in \mathrm{Im}(\pi^*)$. $\square$

Note that the cohomology ring $H^*(\mathrm{UD}_n T)$ is generated by 1-dimensional classes, a fact already known from [7]. It is not true that a product (19) vanishes when its factors interact but non-strongly. The description of such products relies on the dynamics of lower gradient paths.

5. Cup products II: Lower gradient paths

Let Π_1 stand for a product (19) whose factors interact strongly, so Corollary 4.5 applies. Choose an additional 1-dimensional basis element $\{k_x \mid x, (p_{x,1}, \ldots, p_{x,r_x}), (q_{x,1}, \ldots, q_{x,s_x})\}$ of $\mathrm{Im}(\pi^*)$ with $x < x_1 < \cdots < x_m$ and where the standard conditions and conventions are assumed, namely,

$$(27) \qquad p_x := (p_{x,1}, \ldots, p_{x,r_x}) > 0 \quad \text{and} \quad q_x := (q_{x,1}, \ldots, q_{x,s_x}) \geq 0,$$

where $r_x \geq 1 \leq s_x$, $r_x + s_x = d_x := d(x) - 1$, $|p_x| := \sum_{\ell=1}^{r_x} p_{x,\ell}$, $|q_x| := \sum_{\ell=1}^{s_x} q_{x,\ell}$, $k_x + |p_x| + |q_x| = n - 1$ and $\overline{x} := x[r_x + 1]$. Consider the interaction parameters $P_i := \mathcal{P}_i(x_1, \ldots, x_m)$ and $Q_i := \mathcal{Q}_i(x_1, \ldots, x_m)$ of the factors of Π_1 ($i \in \{1, \ldots, m\}$), as well as the first three interaction parameters $R_0 := \mathcal{R}_0(x, x_1, \ldots, x_m)$, $P_x := \mathcal{P}_1(x, x_1, \ldots, x_m)$ and $Q_x := \mathcal{Q}_1(x, x_1, \ldots, x_m)$ of the factors of $\Pi_2 := \{k_x \mid x, p_x, q_x\} \cdot \Pi_1$. This section is devoted to proving:

Theorem 5.1. *In the situation above, if the factors of Π_2 interact but non-strongly, then*

$$(28)$$

$$\Pi_2 = -\sum_a \left\{ R_0 - |a| \Big| x, a, Q_x \Big| x_1, P_1, Q_1 \Big| \cdots \Big| x_m, P_m, Q_m \right\}$$

$$(29)$$

$$+ \sum_{\ell=1}^{s_x - 1} \sum_{a,b} \left\{ R_0 - |a| - b - 1 \Big| x, Q_x^{(\ell,a,b)}, Q_x^{(\ell,+)} \Big| x_1, P_1, Q_1 \Big| \cdots \Big| x_m, P_m, Q_m \right\}$$

$$(30)$$

$$- \sum_{\ell=1}^{s_x - 1} \sum_{a,b} \left\{ R_0 - |a| - b \Big| x, Q_x^{(\ell,a,b)}, Q_x^{(\ell,-)} \Big| x_1, P_1, Q_1 \Big| \cdots \Big| x_m, P_m, Q_m \right\}.$$

In the above expression we set $a := (a_1, \ldots, a_{r_x})$, $|a| := a_1 + \cdots + a_{r_x}$, $Q_x^{(\ell,+)} := (Q_{x,\ell+1}, Q_{x,\ell+2}, \ldots, Q_{x,s_x})$, $Q_x^{(\ell,-)} := (Q_{x,\ell+1} - 1, Q_{x,\ell+2}, \ldots, Q_{x,s_x})$ and $Q_x^{(\ell,a,b)} := (a_1, \ldots, a_{r_x}, Q_{x,1} + b + 1, Q_{x,2}, \ldots, Q_{x,\ell})$. The summation in (28) runs over all r_x-tuples a of non-negative integer numbers satisfying $1 \leq |a| \leq R_0$. The inner summation in (29) runs over all r_x-tuples a of non-negative integer numbers and all non-negative integer numbers b satisfying $|a| + b < R_0$. The inner summation in (30) is empty if $Q_{x,\ell+1} = 0$, otherwise it runs over all r_x-tuples a of non-negative integer numbers and all non-negative integer numbers b satisfying $|a| + b \leq R_0$.

Since summands in (28)–(30) are basis elements, Theorem 5.1 and the results in the previous section give a recursive method to effectively assess cup-products in $\mathrm{Im}(\pi^*) \cong H^*(B_n T)$.
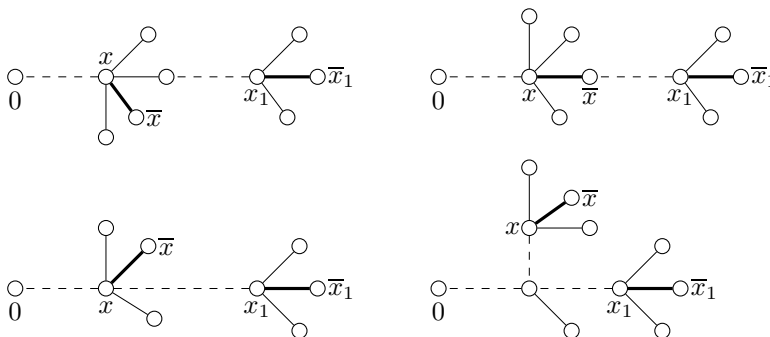
FIGURE 9. The four possible configurations with $x < x_1$

Proof of Theorem 5.1 (Preparation). We have seen that Π_2 is represented in $C^*(D_nT)$ by the gradient-oriented cocycle

$$(31) \quad \sum \Big(B_{0,1}\Big|(x,\overline{x}), B_{x,1}, \ldots, B_{x,d_x}\Big|(x_1,\overline{x}_1), B_{1,1}, \ldots, B_{1,d_1}\Big| \cdots$$
$$\Big|(x_m,\overline{x}_m), B_{m,1}, \ldots, B_{m,d_m}\Big) \cdot \sigma,$$

where the summation runs over all permutations $\sigma \in \Sigma_n$ and over all possible blocks $B_{*,*}$ of vertices written in T-order, taken from the corresponding trees $T_{*,*}$ determined by the factors of Π_2, and having sizes as prescribed in Proposition 4.4 in terms of the relevant interaction parameters. The goal now is to identify the $\underline{\Phi}$-image of (31) which, by (9), is the element in $\mathcal{M}^*(D_nT)$

$$(32) \qquad\qquad\qquad \sum_{\gamma \in \mathcal{G}} \mu(\gamma) \cdot \mathcal{S}_\gamma.$$

Here $\mathcal{G}$ is the set of lower paths γ starting at an $(m+1)$-critical cube $\mathcal{S}_\gamma$ and finishing at a summand of (31). We start by identifying (in the next two paragraphs) key characteristics of ending cubes for paths in $\mathcal{G}$.

Firstly, the condition $x < x_1$ forces one of the four configurations depicted in Figure 9. In any of those configurations, vertices x_i with $i > 1$ lie either on a component of $T \setminus \{x_1\}$ in positive x_1-direction or "below" the horizontal segment joining the root and x_1. As a result, the equalities $P_i = \mathcal{P}_i(x_1,\ldots,x_m) = \mathcal{P}_{i+1}(x,x_1,\ldots,x_m)$ and $Q_i = \mathcal{Q}_i(x_1,\ldots,x_m) = \mathcal{Q}_{i+1}(x,x_1,\ldots,x_m)$ hold for $i = 1,\ldots,m$. The interaction hypotheses then yield

$$(33) \qquad\qquad\qquad\qquad P_x = 0,$$

the r_x-tuple consisting of zeros. This and (27) rule out the two configurations on the right of Figure 9, as well as the one on the bottom left, since the equality $P_x = p_x$ is forced for those configurations. The only possible configuration, i.e., the one on the top left of Figure 9, will be assumed in the rest of the section.

Secondly, redundant summands in (31) can be neglected, as none of those can be the destination of a lower path. Furthermore, (33) shows that no summand in (31) is critical. We thus focus on collapsible summands in (31) which (in addition to their size and distribution properties summarized at the start of the proof) are forced to satisfy the following two properties: For one, ingredients of each $B_{0,1}$ that are smaller than x form a stack of vertices blocked by the root of T. In

addition, for any $i \in \{1, 2, \ldots, m\}$ with $\overline{x}_i$ smaller than $\overline{x}$, all ingredients of each $B_{i,\ell}$ $(1 \leq \ell \leq d_i)$ are blocked (this uses the fact that P_i is not the zero tuple), so the tuple $((x_i, \overline{x}_i), B_{i,1}, \ldots, B_{i,d_i})$ assembles a (unique, by block-size limitations) critical situation around x_i. It follows that each summand $c \cdot \sigma$ in (31) relevant for (32) is collapsible by a branch-type pairing that creates the edge $(x, \overline{x})$, as depicted in (34)
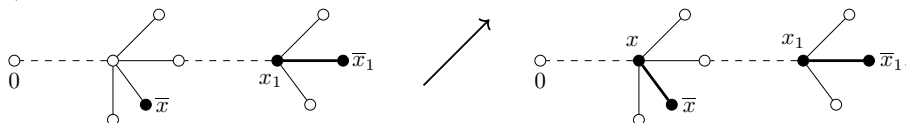
Note that any $(m + 1)$-cube c_0 that has been identified on the right of (34) as a potential destination of a path $\gamma \in \mathcal{G}$ supports a gradient path $\lambda \colon c_0 \searrow c_1 \nearrow \cdots \searrow c_t$ with c_t a critical m-cube. For instance, start by replacing the edge $(x_1, \overline{x}_1)$ in c_0 by x_1, and let the rest of the path consist of falling-vertex elementary factors. It follows that the concatenation of γ and λ and, therefore, γ itself obey the rule in Corollary 3.7: any upper elementary factor of sor type is of falling-vertex type. Such a fact, together with cancellation phenomena similar to the one in the proof of Proposition 3.9, is used in the rest of the argument in order to analyze paths determining (32). As in the proof of Proposition 3.10, the analysis can equivalently be done at the level of $C^*(\mathrm{UD}_n T)$, which means that an ordered cube $c \cdot \sigma$ can be replaced by the corresponding orbit $\{c\}$. Following the lead in Proposition 3.9, we first identify the actual sets of paths whose contributions in (32) give (28)–(30).

The summation in (28) arises from a set $\mathcal{L}_0^- \subset \mathcal{G}$ of paths having a single "lock" dynamics. Explicitly, each r_x-tuple a of non-negative integer numbers satisfying $0 < |a| \leq R_0$ determines a lower gradient path $\lambda_{a,0}^- \in \mathcal{L}_0^-$ that departs from the critical $(m + 1)$-cube

$$\left\{ R_0 - |a| \,\middle|\, x, a, Q_x \,\middle|\, x_1, P_1, Q_1 \,\middle|\, \ldots \,\middle|\, x_m, P_m, Q_m \right\}$$

by replacing the edge $(x, \overline{x})$ by $\overline{x}$ —this opens the lock. Then $\lambda_{a,0}^-$ continues with the falling of the $|a|$ vertices that were blocked by x, after which $\lambda_{a,0}^-$ ends with the pairing that closes the lock by creating the edge $(x, \overline{x})$ required in (34). Since both opening and closing locks are associated to the same face (the gradient-orientated δ_2-face), and since falling-vertex elementary paths have multiplicity 1, we see from (7) that $\mu(\lambda_{a,0}^-) = -1$. Thus, $\mathcal{L}_0^- \subseteq \mathcal{G}$ yields (28).

The set of paths $\mathcal{L}_0^-$ is contained in a slightly larger subset $\mathcal{L}^- \subset \mathcal{G}$ which consists of paths $\lambda_{a,b}^-$, where a runs over r_x-tuples of non-negative integer numbers and b runs over non-negative integers numbers satisfying $|a| > 0$ and $|a| + b \leq R_0$. Explicitly, $\lambda_{a,b}^-$ starts by taking face δ_2 (lock opening) of the critical $(m + 1)$-cube

$$\left\{ R_0 - |a| - b \,\middle|\, x, a, Q_x + (b, 0, \ldots, 0) \,\middle|\, x_1, P_1, Q_1 \,\middle|\, \ldots \,\middle|\, x_m, P_m, Q_m \right\}.$$

Here and below we take the coordinate-wise sum of tuples. Then $\lambda_{a,b}^-$ continues with the falling of the $|a|$ vertices that were blocked by x, followed (if $b > 0$) by the falling of the b vertices $S_b(\overline{x})$, to finish with the falling of $\overline{x} + b$ until it creates the required branch-type pairing (34) —which closes the lock. As in the case of $\mathcal{L}_0^-$, paths in $\mathcal{L}^-$ have multiplicity -1. Likewise, there is the family $\mathcal{L}^+ \subset \mathcal{G}$ consisting of paths $\lambda_{a,b}^+$, with a and b as above, except that the inequality $|a| + b \leq R_0$ is

replaced by the strict inequality $|a| + b < R_0$. Explicitly, $\lambda_{a,b}^+$ starts by taking face δ_1 (inverse lock opening) of the critical $(m+1)$-cube

$$\left\{ R_0 - |a| - b - 1 \,\Big|\, x, a, Q_x + (b+1, 0, \ldots, 0) \,\Big|\, x_1, P_1, Q_1 \,\Big|\, \ldots \,\Big|\, x_m, P_m, Q_m \right\}.$$

Then $\lambda_{a,b}^+$ continues with the falling of x, followed by the falling of the $|a|$ vertices that were blocked by x, followed (if $b > 0$) by the falling of the b vertices $S_{\overline{x}+1}(b)$, to finish with the falling of $\overline{x}+b+1$ until it creates the required branch-type pairing (34) —which closes the lock. Note that paths in $\mathcal{L}^+$ have multiplicity $+1$.

Figure 10 summarizes dynamics of paths in $\mathcal{L}^-$ (top) and paths in $\mathcal{L}^+$ (bottom), with lock opening/closing represented by arrows. Note the shifting on the b vertices falling from x-direction $r_x + 1$, as well as on the vertices that make up B_{x,r_x+1} at the end of the path. The key point is that if $b > 0$, the paths $\lambda_{a,b}^-$ and $\lambda_{a,b-1}^+$ share origin, so their contributions in (32) cancel each other out. The only unmatched paths are those in $\mathcal{L}^-$ with parameter $b = 0$, i.e., paths in $\mathcal{L}_0^-$, whose contribution in (32) has been shown to yield (28).

$$\underbrace{(x,\overrightarrow{x}), \;\; \overline{x}+1, \ldots, \overline{x}+b-1,}_{b \text{ falling vertices}} \;\; \overbrace{\overline{x}+b,}^{\text{closing-lock vertex}} \;\; \underbrace{\overline{x}+b+1, \ldots, \overline{x}+b+Q_{x,1}}_{\text{vertices in } B_{x,r_x+1} \text{ at the end of } \lambda_{a,b}^-}$$

$$\underbrace{(x,\overleftarrow{x}), \;\; \overline{x}+1, \ldots, \overline{x}+b,}_{b \text{ falling vertices}} \;\; \overbrace{\overline{x}+b+1,}^{\text{closing-lock vertex}} \;\; \underbrace{\overline{x}+b+2, \ldots, \overline{x}+b+Q_{x,1}+1}_{\text{vertices in } B_{x,r_x+1} \text{ at the end of } \lambda_{a,b}^+}$$
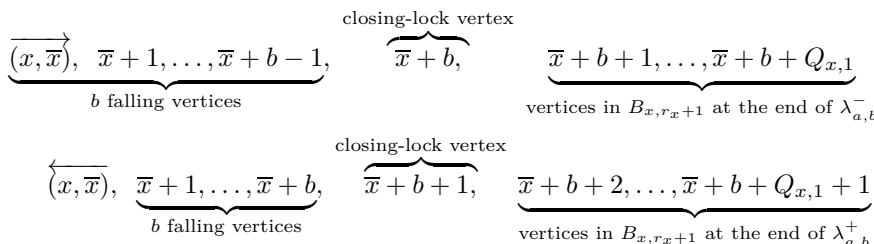
FIGURE 10. Dynamics of paths in $\mathcal{L}^-$ (top) and $\mathcal{L}^+$ (bottom)

By construction, $\mathcal{L}^- \cup \mathcal{L}^+$ consists of those paths in $\mathcal{G}$ that start by taking a face δ_i with $i = 1, 2$ of a critical $(m+1)$-cube with edges

$$(x, \overline{x}), (x_1, \overline{x}_1), \ldots, (x_m, \overline{x}_m),$$

and that evolve exclusively though falling-vertex elementary paths before reaching the required pairing (34). Next we describe similar sets of paths contributing in (32) with (29) and (30). In such sets of paths, an edge

$$(35) \qquad\qquad (x, x[r+1]) \quad \text{with} \;\; r \neq r_x$$

plays the role of the edge $(x, \overline{x}) = (x, x[r_x + 1])$ in $\mathcal{L}^\pm$.

Paths $\mathcal{K}_\ell^-$ with $1 \le \ell \le s_x - 1$ ($r = r_x + \ell$, in the notation of (35)): If $Q_{x,\ell+1} = 0$, set $\mathcal{K}_\ell^- = \varnothing$, otherwise $\mathcal{K}_\ell^-$ consists of paths $\kappa_{\ell,a,b}^- \in \mathcal{G}$, where a runs over r_x-tuples of non-negative integer numbers and b runs over non-negative integer numbers satisfying $|a| + b \le R_0$. Explicitly, if $a = (a_1, \ldots, a_{r_x})$, then $\kappa_{\ell,a,b}^-$ starts by taking face δ_2 of the critical $(m+1)$-cube

$$\left\{ R_0 - |a| - b \,\Big|\, x, (a_1, \ldots, a_{r_x}, Q_{x,1} + b + 1, Q_{x,2}, \ldots, Q_{x,\ell}), \right.$$
$$\left. (Q_{x,\ell+1} - 1, Q_{x,\ell+2}, \ldots, Q_{x,s_x}) \,\Big|\, x_1, P_1, Q_1 \,\Big|\, \ldots \,\Big|\, x_m, P_m, Q_m \right\},$$

and evolves through falling-vertex elementary paths as depicted by the chart[9]

$$\overrightarrow{(x, x[r+1])}, \quad \underbrace{\overline{x}, \dots, \overline{x}+b-1}_{b \text{ falling vertices}}, \quad \overbrace{\overline{x}+b,}^{\text{closing-lock vertex}} \quad \underbrace{\overline{x}+b+1, \dots, \overline{x}+b+Q_{x,1}}_{\text{vertices in } B_{x,r_x+1} \text{ at the end of } \kappa^-_{\ell,a,b}}$$

before reaching the required pairing (34). Both opening and closing locks of $\kappa^-_{\ell,a,b}$ are associated to a (gradient-oriented) δ_2 face, so that $\mu(\kappa^-_{a,b}) = -1$. The contribution in (32) of the paths in $\mathcal{K}^-_1 \cup \cdots \cup \mathcal{K}^-_{s_x-1}$ thus gives raise to (30). Note that no path that starts from the origin of a given $\kappa^-_{a,b}$ by taking face δ_1 —instead of δ_2— and that evolves through falling-vertex elementary paths can arrive at a summand of (31). This is why the contribution to (32) of the set of paths in the next paragraph does not cancel out terms in (30).

Paths $\mathcal{K}^+_\ell$ with $1 \le \ell \le s_x - 1$ ($r = r_x + \ell$, in the notation of (35)): $\mathcal{K}^+_\ell$ consists of paths $\kappa^+_{\ell,a,b} \in \mathcal{G}$, where a runs over r_x-tuples of non-negative integer numbers and b runs over non-negative integer numbers satisfying $|a| + b < R_0$. Explicitly, if $a = (a_1, \dots, a_{r_x})$, then $\kappa^+_{\ell,a,b}$ starts by taking face δ_1 of the critical $(m+1)$-cube

$$\Big\{ R_0 - |a| - b - 1 \Big| x, (a_1, \dots, a_{r_x}, Q_{x,1} + b + 1, Q_{x,2}, \dots, Q_{x,\ell}),$$
$$(Q_{x,\ell+1}, Q_{x,\ell+2}, \dots, Q_{x,s_x}) \Big| x_1, P_1, Q_1 \Big| \dots \Big| x_m, P_m, Q_m \Big\},$$

and evolves through falling-vertex elementary paths as depicted by the chart

$$\underbrace{\overleftarrow{(x, x[r+1])}, \quad \overline{x}, \dots, \overline{x}+b-1}_{b+1 \text{ falling vertices}}, \quad \overbrace{\overline{x}+b,}^{\text{closing-lock vertex}} \quad \underbrace{\overline{x}+b+1, \dots, \overline{x}+b+Q_{x,1}}_{\text{vertices in } B_{x,r_x+1} \text{ at the end of } \kappa^+_{\ell,a,b}}$$

before reaching the required pairing (34). Now $\mu(\kappa^+_{a,b}) = 1$, so the contribution in (32) of the paths in $\mathcal{K}^+_1 \cup \cdots \cup \mathcal{K}^+_{s_x-1}$ gives raise to (29). Again, no path that starts from the origin of a given $\kappa^+_{a,b}$ by taking face δ_2 —instead of δ_1— and that evolves through falling-vertex elementary paths can arrive at a summand of (31). □

Remark 5.2. Since the closing-lock pairing (34) must come from x-direction $r_x + 1$, paths corresponding to cases with $r < r_x$ in (35) have no contribution in (32). Specifically, any path $\gamma \in \mathcal{G}$ that starts from a critical cell with edges $(x, x[r+1]), (x_1, \overline{x}_1), \dots, (x_m, \overline{x}_m)$, where $r < r_x$, by taking a face δ_i with $i = 1, 2$, and that reaches the pairing (34) through falling-vertex elementary paths, has a companion path γ' that starts from the same critical cell by taking the face δ_{3-i}, and that also evolves through falling-vertex elementary paths until it reaches the closing-lock pairing (34) —so that $\mu(\gamma') = -\mu(\gamma)$ and $(\gamma')' = \gamma$. Note that, in the ordered setting, γ and its companion path γ' arrive at summands of (31) whose ingredients differ only by a permutation (so $\gamma' \in \mathcal{G}$ as well). The phenomenon noticed in this remark is in fact the key to finishing the proof of the main result in this section.

Proof of Theorem 5.1 (Conclusion). Let $\mathcal{J}$ stand for the set of paths analyzed up to this point, i.e., the paths in $\mathcal{G}$ that (I) depart from a critical $(m+1)$-cube with gradient-ordered edges $(x, x[\ell]), (x_1, \overline{x}_1), \dots, (x_m, \overline{x}_m)$, (II) start by taking the face

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[9]As in the case of $\mathcal{L}^\pm$, the $|a|$ vertices falling from x-directions 1 through r_x are not shown in the chart.

δ_1 or δ_2 and (III) reach the ending branch-type pairing (34) exclusively through falling-vertex elementary paths. It suffices to construct an involution $\iota\colon \mathcal{G}' \to \mathcal{G}'$, with $\mathcal{G}' := \mathcal{G} \setminus \mathcal{J}$, such that each pair of paths γ and $\iota(\gamma)$ shares origin and has opposite multiplicity. With this in mind, we first note that condition (II) is forced by conditions (I) and (III). Indeed, in any gradient path $e \searrow e' \nearrow \cdots$ all of whose upper elementary factors are of falling-vertex type,

$$(36) \qquad \textit{the edge ingredients of } e' \textit{ are present in all steps of the path.}$$

Therefore $\mathcal{G}'$ is partitioned into two sets, $\mathcal{G}'_{\mathrm{fall}}$ and $\mathcal{G}'_{\mathrm{branch}}$, where the former set consists of the paths in $\mathcal{G}$ that satisfy (III) without satisfying (I), and the latter set consists of the paths in $\mathcal{G}$ that do not satisfy (III). We construct involutions $\iota_{\mathrm{fall}}\colon \mathcal{G}'_{\mathrm{fall}} \to \mathcal{G}'_{\mathrm{fall}}$ and $\iota_{\mathrm{branch}}\colon \mathcal{G}'_{\mathrm{branch}} \to \mathcal{G}'_{\mathrm{branch}}$ with the required properties.

For a path $\gamma = a_0 \searrow b_1 \nearrow a_1 \searrow \cdots \searrow b_k \nearrow a_k$ in $\mathcal{G}'_{\mathrm{fall}}$, the observation in (36) and the form of the closing-lock pairing $b_k \nearrow a_k$ imply that all edges $(x_i, \overline{x}_i)$, $1 \le i \le m$, must be ingredients of a_0. The additional edge of the critical $(m+1)$-cube a_0 must then have the form $(y, y[d])$, with $y \notin \{x, x_1, \ldots, x_m\}$, which is then replaced by either y or $y[d]$ at the beginning of γ. Given the form of $b_k \nearrow a_k$, y must lie in x-direction $r_x + 1$. Then, as in the proof of Proposition 3.10, the definition of ι_{fall} is based on the two options for $a_0 \searrow b_1$, as both lead to summands of (31) —unlike the situation in Remark 5.2, the ending cube of $\iota_{\mathrm{fall}}(\gamma)$ might fail to be in the Σ_n-orbit of the ending cube of γ. Likewise, the definition of ι_{branch} is based on the two forms of replacing by a vertex the edge ingredient at the last upper elementary factor that is not of falling-vertex type.

6. Exterior-face basis for trees with binary core

We have made a careful distinction between $\mathrm{Im}(\pi^*)$ and $H^*(\mathrm{UD}_nT; R)$ in the previous sections so as to provide clear proof arguments. In this section we use the resulting algebro-combinatorial description of cup-products and have no need to make any further distinction between these isomorphic rings. Accordingly, we transfer the notation and descriptions of elements in $\mathrm{Im}(\pi^*)$ back to $H^*(\mathrm{UD}_nT; R)$. In particular, the notation and conventions in the paragraph containing (19) will be carried over this final section, directly in the context of $H^*(\mathrm{UD}_nT; R)$, with the simplifications discussed below.

Definition 6.1. A tree T is said to have binary core provided that, for each essential vertex x of T, at most two of the components of $T \setminus \{x\}$ in x-directions $1, 2, \ldots, d_x$ carry essential vertices (recall $d_x := d(x) - 1$).

Throughout this section, T stands for a tree with binary core (e.g. an actual binary tree). In addition, we assume that the chosen planar embedding of T has been adjusted so that, for any essential vertex x of T,

$$(37) \quad \textit{no component of } T \setminus \{x\} \textit{ in } x\textit{-direction } j \textit{ with } 1 \le j \le d_x - 2$$
$$\textit{carries an essential vertex.}$$

There are two reasons for sticking to such a hypothesis. For one, the existence of non-vanishing products whose factors are given by weak-interacting basis elements

$$\{k_i \,|\, x_i, (p_{i,1}, \ldots, p_{i,r_i}), (q_{i,1}, \ldots, q_{i,s_i})\}$$

with $x_1 < \cdots < x_m$, i.e., the obstructions in Remark 1.8, is somehow restricted (cf. Example 1.6), while our description of the corresponding product is greatly

simplified. Explicitly, in the setting and notation of Theorem 5.1, since the top left configuration in Figure 9 holds, (37) forces $s_x = 1$, i.e., the edge $(x, \overline{x})$ must lie in the largest x-direction, with x_1 then lying in the second largest x-direction $r_x = d_x - 1$. In particular, the product Π_2 takes the simpler form

$$(38) \qquad \Pi_2 = -\sum \left\{ R_0 - |a| \,\Big|\, x, a, Q_x \,\Big|\, x_1, P_1, Q_1 \,\Big|\, \cdots \,\Big|\, x_m, P_m, Q_m \right\},$$

where the sum runs over all r_x-tuples of integer numbers $a = (a_1, \ldots, a_{r_x})$ with $a > 0$ and $|a| \le R_0$.

The second advantage for working under the situation in (37) is that, for $1 \le i \le m$ and $j \le d_i - 2$, any set of pruned leaves $L_{i,j}$ associated to a product (19) is empty. As a result, the corresponding $C_{i,j}$-local interaction is "vacuous" in the sense that the $C_{i,j}$-instance of (2) simplifies to $\ell_{C_{i,j}}(\nu_i) \ge 0$ —a condition which is certainly true. In fact, still in the context of (19), there will be no local interactions in the positive x_i-directions leading to a weak interaction situation as long as $p_{i,j} > 0$ for some $j \le \min\{r_i, d_i - 2\}$ (cf. (33)). In particular, it makes sense to reset the notation for pruned leaves in the presence of (37): we shall set $L_1(x_i) := L_{i,d_i-1}$ and $L_2(x_i) := L_{i,d_i}$ when $i > 0$, and $L_1(x_0) := L_{0,1}$ (recall from Definition 4.1 that x_0 stands for the root of T).

Expression (38) suggests redefining some of the basis elements $\langle k, x, (p_1, \ldots, p_r), (q_1, \ldots, q_s) \rangle \in H^1(B_n T)$ in the proof of Theorem 1.7. Namely, for the purposes of this section, if $p_1 = \cdots = p_{r-1} = 0$ and $s = 1$, we set
$$(39)$$
$$\langle k, x, (p_1, \ldots, p_r), (q_1, \ldots, q_s) \rangle := \sum \left\{ k - |a| \,\Big|\, x, (a_1, \ldots, a_{r-1}, p_r + a_r), (q_1) \right\},$$

where the summation runs over all r-tuples $a = (a_1, \ldots, a_r) \ge 0$ with $|a| \le k$, otherwise we keep

$$\langle k, x, (p_1, \ldots, p_r), (q_1, \ldots, q_s) \rangle := \left\{ k \,\Big|\, x, (p_1, \ldots, p_r), (q_1, \ldots, q_s) \right\}.$$

Remark 6.2. We use the angle-bracket notation $\langle k, x, p, q \rangle$ since we have reserved the parenthesis notation for cubes in $D_n T$ (as tuples of their ingredients). Additionally, the angle-bracket notation is intended to stress the change of basis in (39).

A central task in this section is the analysis of the relationship between ordered[10] products

$$(40) \quad \langle k_1, x_1, p_1, q_1 \rangle \cdots \langle k_m, x_m, p_m, q_m \rangle \quad \text{and} \quad \{k_1 \,|\, x_1, p_1, q_1\} \cdots \{k_m \,|\, x_m, p_m, q_m\}.$$

We say that any of these products is a strong interaction product if the factors of the product on the right hand-side of (40) interact strongly (in the sense of Definition 4.3).

Remark 6.3. Corollary 4.5, Proposition 4.4 and Theorem 5.1 show that both products in (40) are (possibly empty) linear combinations of basis elements

$$\{\,\centerdot \,|\, x_1, \centerdot, \centerdot \,|\, \cdots \,|\, x_m, \centerdot, \centerdot \,\}.$$

Such a linear combination will be written as

$$\sum \centerdot \{\,\centerdot \,|\, x_1, \centerdot, \centerdot \,|\, \cdots \,|\, x_m, \centerdot, \centerdot \,\}.$$

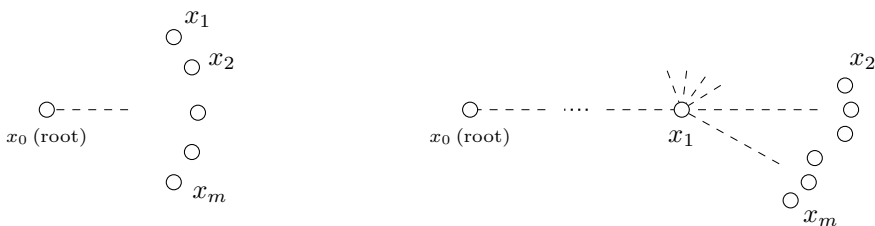[10]In the sense that $x_1 < \cdots < x_m$.

FIGURE 11. Configurations of essential vertices in Lemma 6.5

Here and below, a dot '$\cdot$' stands for either an unspecified ring coefficient or an unspecified tuple[11] of integer numbers, $t = (t_1, t_2, \ldots) \geq 0$, satisfying $t > 0$ when the tuple immediately follows an essential vertex x_i (the context clarifies the option).

Theorem 6.4. *Let T be a tree with binary core, R be a commutative ring with 1, and $n \geq 1$. Then $H^*(B_n T; R) \cong \Lambda_R(K_n T)$. In detail:* (i) *An ordered product $\langle k_1, x_1, p_1, q_1 \rangle \cdots \langle k_m, x_m, p_m, q_m \rangle$ is non-zero if and only if it is a strong interaction product.* (ii) *Two ordered strong interaction products agree if and only if they have the same factors.* (iii) *A graded basis of $H^*(\mathrm{UD}_n T)$ is given by the set of ordered strong interaction products.*

The crux of the matter in the proof of Theorem 6.4 is getting at a precise description of the conditions that have to be satisfied by some of the unspecified dot ingredients in

$$(41) \qquad \langle k_1, x_1, p_1, q_1 \rangle \cdots \langle k_m, x_m, p_m, q_m \rangle = \sum \cdot \{ \cdot \mid x_1, \cdot, \cdot \mid \cdots \mid x_m, \cdot, \cdot \}.$$

With this in mind, the product in (41) will be denoted by ϖ throughout the section, setting

$$R_0 := \mathcal{R}_0(x_1, \ldots, x_m), \ \ P_{i,j} := \mathcal{P}_{i,j}(x_1, \ldots, x_m), \ \ Q_{i,j} := \mathcal{Q}_{i,j}(x_1, \ldots, x_m),$$

$P_i := (P_{i,1}, \ldots, P_{i,r_i})$ and $Q_i := (Q_{i,1}, \ldots, Q_{i,s_i})$, $1 \leq i \leq m$, for the corresponding interaction parameters. Furthermore, we set

$$(42) \qquad\qquad\qquad B_i := (x_i, P_i, Q_i) \ \ \text{and} \ \ \dot{B}_i := (x_i, \cdot, \cdot),$$

where the latter expression stands for any triple with unspecified tuples in the second and third coordinates (subject to the usual restrictions). Additionally, the i-th factor on the left hand-side of (41) will denoted by ϕ_i. For instance, in terms of the notation set forth in Definition 4.3,

$$\phi_i = \{ k_i \mid x_i, p_i, q_i \} + \sum \{ \underline{k_i} \mid x_i, \cdot, q_i \},$$

with a possibly empty summation, whereas Corollary 4.5 asserts that the second product in (40) is trivial or agrees with $\{ R_0 \mid B_1 \mid \ldots \mid B_m \}$ under, respectively, the no-interaction or strong-interaction condition of the factors.

In the following results, some of which are true for general trees, we make free use of the notation and considerations above. Likewise, the use of cup-product descriptions in Sections 4 and 5, with the simplification in (38), we will refer to generically as "interaction reasons".

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[11] As in Definition 4.3, we make no distinction between integer numbers and 1-tuples.

Lemma 6.5.

(1) Assume $L_1(x_0) = \{x_1, x_2, \ldots, x_m\}$ *(left configuration in Figure* 11*), then*

$$\varpi = \begin{cases} \{R_0 \,|\, B_1 \,|\, \cdots \,|\, B_m\} + \sum \cdot \{\underline{R_0} \,|\, \dot{B}_1 \,|\, \cdots \,|\, \dot{B}_m\}, & \text{if } R_0 \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

(2) Assume $L_1(x_1) = \{x_2, x_3, \ldots, x_u\}$ *and* $L_2(x_1) = \{x_{u+1}, \ldots, x_{m-1}, x_m\}$ *with* $1 \leq u \leq m$ *(right configuration in Figure* 11*) with* $u = 1$ *(i.e.* $L_1(x_1) = \varnothing$ *and* $L_2(x_1) = \{x_2, \ldots, x_m\}$*) if* $s_1 = 1$*, then*

$$\varpi = \begin{cases} \{R_0 \,|\, B_1 \,|\, \cdots \,|\, B_m\} + \sum \cdot \{\underline{R_0} \,|\, \dot{B}_1 \,|\, \cdots \,|\, \dot{B}_m\} \\ \qquad\qquad + \sum \cdot \{R_0 \,|\, x_1, P_1, \underline{Q_1} \,|\, \dot{B}_2 \,|\, \cdots \,|\, \dot{B}_m\}, & \text{if } Q_1 \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The first assertion follows by direct inspection of the expression

$$\left(\{k_1 \,|\, x_1, p_1, q_1\} + \sum \{\underline{k_1} \,|\, x_1, \cdot, \cdot\} \right) \cdots \left(\{k_m \,|\, x_m, p_m, q_m\} + \sum \{\underline{k_m} \,|\, x_m, \cdot, \cdot\} \right),$$

noticing that the only non-vacuous interaction occurs in the tree $T_{0,1}$ (so that $P_i = p_i$ and $Q_i = q_i$ for $1 \leq i \leq m$). The second assertion is proved in a similar way, noticing that this time non-vacuous interactions occur only either on T_{1,d_1} or T_{1,d_1-1} (or both). In any case, $R_0 = k_1$, $P_i = p_i$ for $1 \leq i \leq m$, while $Q_i = q_i$ for $2 \leq i \leq m$. $\qquad\square$

A key situation with $L_1(x_1) \cup L_2(x_1) = \{x_2, x_3, \ldots, x_m\}$ not covered by Lemma 6.5(2) is:

Lemma 6.6. *Assume* $L_1(x_1) = \{x_2, x_3, \ldots, x_m\}$. *Then the product of* $\langle k_1, x_1, (p_{1,1}, \ldots, p_{1,d_1-1}), (q_{1,1}) \rangle$ *with* $\{R \,|\, x_2, p_2, q_2 \,|\, \cdots \,|\, x_m, p_m, q_m\}$ *vanishes provided* $p_{1,1} = \cdots = p_{i,d_1-2} = 0$ *and* $p_{1,d_1-1} + R \leq n$.

Proof. We proceed by induction on $p_{1,d_1-1} + R - n = p_{1,d_1-1} + \sum_{j=2}^m (t_j - n) \in \{0, -1, -2, \ldots\}$, where

$$\{R \,|\, x_2, p_2, q_2 \,|\, \cdots \,|\, x_m, p_m, q_m\} = \{t_2 \,|\, x_2, p_2, q_2\} \cdots \{t_m \,|\, x_m, p_m, q_m\}$$

is the unique strong-interaction factorization of $\{R \,|\, x_2, p_2, q_2 \,|\, \cdots \,|\, x_m, p_m, q_m\}$ noted in the proof of Theorem 1.7. Since $p_{1,j} = 0$ for $j = 1, \ldots, d_1 - 2$, the induction is grounded for $p_{1,d_1-1} + R - n = 0$ by

$$\{k_1 \,|\, x_1, (p_{1,1}, \ldots, p_{1,d_1-1}), (q_{1,1})\} \cdot \left(\{t_2 \,|\, x_2, p_2, q_2\} \cdots \{t_m \,|\, x_m, p_m, q_m\} \right)$$

$$= -\sum \{k_1 - |a| \,|\, x_1, a, (q_{1,1}) \,|\, x_2, p_2, q_2 \,|\, \cdots \,|\, x_m, p_m, q_m\}$$

$$= -\sum \{k_1 - |a| \,|\, x_1, (a_1, \ldots, a_{d_1-2}, p_{1,d_1-1} + a_{d_1-1}), (q_{1,1})\} (\{t_2 \,|\, x_2, p_2, q_2\} \cdots \{t_m \,|\, x_m, p_m, q_m\}),$$

where both summations run over tuples $a = (a_1, \ldots, a_{d_1-1}) > 0$ with $|a| \leq k_1$. The inductive step then follows by noticing that, for $p_{1,d_1-1} + R - n < 0$,

$$\langle k_1, x_1, (p_{1,1}, \ldots, p_{1,d_1-1}), (q_{1,1}) \rangle \cdot \left(\{t_2 \,|\, x_2, p_2, q_2\} \cdots \{t_m \,|\, x_m, p_m, q_m\} \right)$$

$$= \langle k_1 - 1, x_1, (p_{1,1}, \ldots, p_{1,d_1-2}, p_{1,d_1-1} + 1), (q_{1,1}) \rangle$$

$$\cdot \left(\{t_2 \,|\, x_2, p_2, q_2\} \cdots \{t_m \,|\, x_m, p_m, q_m\} \right),$$

as

$$\{k_1 - |a| \,|\, x_1, (p_{1,1} + a_1, \dots, p_{1,d_1-2} + a_{d_1-2}, p_{1,d_1-1}), (q_{1,1})\}$$
$$\cdot \Big(\{t_2 \,|\, x_2, p_2, q_2\} \cdots \{t_m \,|\, x_m, p_m, q_m\} \Big)$$

vanishes for $a = (a_1, \dots, a_{d_1-2}, 0) \geq 0$ with $|a| \leq k_1$ by interaction reasons. $\qquad \square$

Corollary 6.7. *If the factors on the left of* (41) *do not yield a strong interaction product, then $\varpi = 0$.*

Proof. By focusing on the factors ϕ_i of ϖ that are involved in a faulty interaction parameter, it suffices to consider three cases: $L_1(x_0) = \{x_1, \dots, x_m\}$, $L_2(x_1) = \{x_2, \dots, x_m\}$ and $L_1(x_1) = \{x_2, \dots, x_m\}$. The first two cases are covered by Lemma 6.5. On the other hand, there are two options for the instances of the third case that are not covered by Lemma 6.5((2)): either $p_{1,j} > 0$ for some $j \in \{1, 2, \dots, d_1 - 2\}$ or $p_{1,j} = 0$ for all $j \in \{1, 2, \dots, d_1 - 2\}$ —in both cases $s_1 = 1$. In the latter option, the result follows from Lemma 6.6; in the former option we have $\langle k_1, x_1, p_1, q_1 \rangle = \{k_1 \,|\, x_1, p_1, q_1\}$ while the condition $p_{1,d_1-1} + \mathcal{R}_0(x_2, \dots, x_m) < n$ is forced by the no-strong-interaction hypothesis, so that the result follows by interaction reasons in view of Lemma 6.5((1)). $\qquad \square$

The proof of Theorem 6.4 will be complete once we set a one-to-one correspondence between the set of ordered strong interaction products ϖ and the graded basis of $H^*(B_n T; R)$ formed by the elements in (10). With this in mind, we start with a two-step approach to the missing case in Lemma 6.5((2)):

Lemma 6.8. *Assume $L_1(x_1) = \{x_2, x_3, \dots, x_m\}$ with $s_1 = 1$. Then*
(43)
$$\varpi = \begin{cases} \{R_0 \,|\, B_1 \,|\, \cdots \,|\, B_m\} + \sum \cdot \{\underline{R_0} \,|\, \dot{B}_1 \,|\, \cdots \,|\, \dot{B}_m\} & \\ \qquad\qquad + \sum \cdot \{R_0 \,|\, x_1, \underline{P_1}, Q_1 \,|\, \dot{B}_2 \,|\, \cdots \,|\, \dot{B}_m\}, & \text{if } P_1 > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Interactions occur only in T_{1,d_1-1}, so $R_0 = k_1$, $Q_i = q_i$ for $1 \leq i \leq m$, and $P_i = p_i$ for $2 \leq i \leq m$. By Corollary 6.7, only the case $P_1 > 0$ needs to be argued. Use Lemma 6.5((1)) to write $\varpi = \phi_1 \cdot (\phi_2 \cdots \phi_m)$ as

$$\Big(\{R_0 \,|\, x_1, p_1, Q_1\} + \sum \{\underline{R_0} \,|\, x_1, \cdot, Q_1\} \Big) \Big(\{R_0' \,|\, B_2 \,|\, \cdots \,|\, B_m\} + \sum \{\underline{R_0'} \,|\, \dot{B}_2 \,|\, \cdots \,|\, \dot{B}_m\} \Big),$$

where $R_0' = \mathcal{R}_0(x_2, \dots, x_m)$ (so $P_{1,d_1-1} = p_{1,d_1-1} + R_0' - n$). The result then follows by direct inspection, though this time (38) needs to be used in the analysis of the products giving rise to the terms in both summations of (43). $\qquad \square$

Proposition 6.9. *Assume $L_1(x_1) = \{x_2, x_3, \dots, x_u\}$ and $L_2(x_1) = \{x_{u+1}, \dots, x_{m-1}, x_m\}$, with $1 < u < m$ and $s_1 = 1$. Then*

$$\varpi = \begin{cases} \{R_0 \,|\, B_1 \,|\, \cdots \,|\, B_m\} + \sum \cdot \{\underline{R_0} \,|\, \dot{B}_1 \,|\, \cdots \,|\, \dot{B}_m\} & \\ \qquad\qquad + \sum \cdot \{R_0 \,|\, x_1, \underline{P_1}, \underline{Q_1} \,|\, \dot{B}_2 \,|\, \cdots \,|\, \dot{B}_m\}, & \text{if } P_1 > 0 \leq Q_1; \\ 0, & \text{otherwise.} \end{cases}$$

Here and below each expression $\underline{P_1}, \underline{Q_1}$ is meant to represent a pair V_1, W_1 of unspecified tuples of integer numbers with $V_1 = (V_{1,1}, \dots, V_{1,d_1-1})$, $W_1 = (W_{1,1})$

and such that $V_1 > 0 \leq W_1$ and $(V_1, W_1) < (P_1, Q_1)$ in the product ordering, i.e., $V_{1,j} \leq P_{1,j}$ for $j = 1, 2, \ldots, d_1 - 1$ and $W_{1,1} \leq Q_{1,1}$, with at least one of the last d_1 inequalities being strict.

Proof. By Corollary 6.7, it suffices to consider the case $P_1 > 0 \leq Q_1$. Lemmas 6.5((1)) and 6.8 allow us to write $\varpi = (\phi_1 \cdots \phi_u) \cdot (\phi_{u+1} \cdots \phi_m)$ as the product of

$$\{R_0 \,|\, x_1, P_1, q_1 \,|\, B_2 \,|\, \ldots \,|\, B_u\} + \sum \cdot \{\underline{R_0} \,|\, \dot{B}_1 \,|\, \dot{B}_2 \,|\, \ldots \,|\, \dot{B}_u\}$$

$$+ \sum \cdot \{R_0 \,|\, x_1, \underline{P_1}, q_1 \,|\, \dot{B}_2 \,|\, \ldots \,|\, \dot{B}_u\}$$

with

$$\{R_0' \,|\, B_{u+1} \,|\, \ldots \,|\, B_m\} + \sum \cdot \{\underline{R_0'} \,|\, \dot{B}_{u+1} \,|\, \ldots \,|\, \dot{B}_m\},$$

where $R_0' = \mathcal{R}_0(x_{u+1}, \ldots, x_m)$ (so $Q_{1,1} = q_{1,1} + R_0' - n$). The result follows by inspection. $\qquad \square$

We are now ready to set up the strategy for completing the proof of Theorem 6.4. By Lemma 4.6, Remark 6.3 and Corollary 6.7, the goal reduces to describing, for fixed essential vertices $x_1 < \cdots < x_m$, a partial ordering $\preceq$ on the set of basis elements $\{t_0 \,|\, x_1, u_1, v_1 \,|\, \cdots \,|\, x_m, u_m, v_m\}$ of $H^m(\mathrm{UD}_n T)$ such that any strong interaction product (41) can be expressed by a congruence

$$(44) \qquad \langle k_1, x_1, p_1, q_1 \rangle \cdots \langle k_m, x_m, p_m, q_m \rangle \equiv \{R_0 \,|\, B_1 \,|\, \cdots \,|\, B_m\}$$

modulo basis elements that are $\preceq$-smaller than $\{R_0 \,|\, B_1 \,|\, \cdots \,|\, B_m\}$. The partial ordering $\preceq$ we need becomes apparent by writing either of the triples $(x_1, \underline{P_1}, Q_1)$, $(x_1, \underline{P_1}, Q_1)$ and $(x_1, P_1, \underline{Q_1})$ in Proposition 6.9 and Lemmas 6.8 and 6.5((2)), respectively, as $\underline{B_1}$. Indeed, in such terms, the $(P_1 > 0 \leq Q_1)$-conclusions in those results can be written as

(45)
$$\varpi = \{R_0 \,|\, B_1 \,|\, \cdots \,|\, B_m\} + \sum \cdot \{\underline{R_0} \,|\, \dot{B}_1 \,|\, \cdots \,|\, \dot{B}_m\} + \sum \cdot \{R_0 \,|\, \underline{B_1} \,|\, \dot{B}_2 \,|\, \cdots \,|\, \dot{B}_m\}.$$

Definition 6.10. The ℓ-th level of pruned leaves $\mathcal{L}_\ell$ of the essential vertices $x_1 < \cdots < x_m$ is

$$\mathcal{L}_\ell = \mathcal{L}_\ell(x_1, \ldots, x_m) := \begin{cases} L_1(x_0), & \text{if } \ell = 1; \\ \bigcup_{x_i \in \mathcal{L}_{\ell-1}} \left(L_1(x_i) \cup L_2(x_i) \right), & \text{if } \ell > 1. \end{cases}$$

The interaction level of the vertices $x_1 < \cdots < x_m$ is the largest ℓ such that $\mathcal{L}_\ell \neq \varnothing$. Furthermore, extending the notation introduced in (42) and (45), let $B^{(\ell)}$ denote the collection of blocks B_i with $x_i \in \mathcal{L}_\ell$, and let $\dot{B}^{(\ell)}$ stand for any collection of blocks $\dot{B}_i$ with $x_i \in \mathcal{L}_\ell$. On the other hand, $\underline{B}^{(\ell)}$ stands for any collection of blocks (x_i, V_i, W_i), with $x_i \in \mathcal{L}_\ell$, satisfying:

- $V_i > 0 \leq W_i$ and $(V_i, W_i) \leq (P_i, Q_i)$ (the latter in the product ordering) for all $x_i \in \mathcal{L}_\ell$, and
- $(V_i, W_i) \neq (P_i, Q_i)$ for at least one $x_i \in \mathcal{L}_\ell$.

Note that the definition of $\underline{B}^{(\ell)}$ is less restrictive than actually requiring $\underline{B}^{(\ell)}$ to be a collection of blocks $\underline{B_i}$ with $x_i \in \mathcal{L}_\ell$. As in Proposition 6.9, the condition we want for $\underline{B}^{(\ell)}$ is based on a *strict* product-order inequality. The reason for this becomes apparent in the proof of Proposition 6.12.

Example 6.11. Lemma 6.5((1)) gives $\varpi = \{R_0, B^{(1)}\} + \sum \cdot \{\underline{R_0}, \dot{B}^{(1)}\}$ in interaction level 1 (under a strong condition hypothesis). Likewise, (45) becomes

$$(46) \qquad \varpi = \{R_0 \,|\, B^{(1)} \,|\, B^{(2)}\} + \sum \cdot \{\underline{R_0} \,|\, \dot{B}^{(1)} \,|\, \dot{B}^{(2)}\} + \sum \cdot \{R_0 \,|\, \underline{B^{(1)}} \,|\, \dot{B}^{(2)}\}$$

in interaction level 2 (with $\mathcal{L}_1 = \{x_1\}$, so $B^{(1)}$ consist of B_1 alone). In full generality:

Proposition 6.12. *Let $x_1 < \cdots < x_m$ be essential vertices having interaction level ℓ. If ϖ is a strong interaction product, then*

$$\varpi = \{R_0 \,|\, B^{(1)} \,|\, \cdots \,|\, B^{(\ell)}\} + \sum \cdot \{\underline{R_0} \,|\, \dot{B}^{(1)} \,|\, \cdots \,|\, \dot{B}^{(\ell)}\}$$

$$(47) \qquad\qquad + \sum \cdot \{R_0 \,|\, \underline{B^{(1)}} \,|\, \dot{B}^{(2)} \,|\, \cdots \,|\, \dot{B}^{(\ell)}\} + \cdots$$

$$\qquad\qquad + \sum \cdot \{R_0 \,|\, B^{(1)} \,|\, \cdots \,|\, B^{(\ell-2)} \,|\, \underline{B^{(\ell-1)}} \,|\, \dot{B}^{(\ell)}\}.$$

Proof of Theorem 6.4 (Conclusion). Partially order the set of basis elements

$$\{v_0 \,|\, x_1, v_1, w_1 \,|\, \cdots \,|\, x_m, v_m, w_m\}$$

by means of a level-wise lexicographical comparison of their v- and w-ingredients. Then (47) yields the required congruence (44). $\qquad\qquad\square$

Proof of Proposition 6.12. The argument is by direct computation, proceeding by induction on ℓ and with Example 6.11 grounding the induction. The real challenge consists of setting a suitable notation so arguments can be seen clearly. With this in mind, we start by checking the situation in the special case $\mathcal{L}_1 = \{x_1\}$ (so $R_0 = k_1$), i.e., the generalization of (46) to higher interaction levels. In such a situation

$$(48) \qquad\qquad \mathcal{L}_\lambda(x_2, \ldots, x_m) = \mathcal{L}_{\lambda+1}(x_1, \ldots, x_m), \text{ for } \lambda \geq 2.$$

Accordingly, we reset notation and start level-number counting at 2 (rather than at 1) for $x_2 < \cdots < x_m$, so as to make it compatible with that for $x_1 < \cdots < x_m$. Thus, (48) gets replaced by

$$(49) \qquad\qquad \mathcal{L}_\lambda(x_2, \ldots, x_m) = \mathcal{L}_\lambda(x_1, \ldots, x_m), \text{ for } \lambda \geq 3.$$

Let $x_2, x_3, \ldots, x_t$ be the essential vertices lying on the component of $T \setminus \{x_1\}$ in x_1-direction $d_1 - 1$, while $x_{t+1}, x_{t+2}, \ldots, x_m$ be the vertices lying on the component of $T \setminus \{x_1\}$ in x_1-direction d_1 ($1 \leq t \leq m$). Then, if $B^{(\lambda)}$, $\dot{B}^{(\lambda)}$ and $\underline{B^{(\lambda)}}$ stand for collections defined by all the vertices $x_1, \ldots, x_m$, we write

$$(50) \qquad\qquad B^{(\lambda)}_{[\epsilon]}, \quad \dot{B}^{(\lambda)}_{[\epsilon]} \text{ or } \underline{B^{(\lambda)}_{[\epsilon]}},$$

with $\varepsilon = 1$, to denote the corresponding parts coming only from the vertices $x_2, \ldots, x_t$. Likewise, the case $\epsilon = 2$ in (50) stands for the parts that come from the vertices $x_{t+1}, \ldots, x_m$. For instance, $B^{(\lambda)} = B^{(\lambda)}_{[1]} \cup B^{(\lambda)}_{[2]}$. In these terms, we use induction to write $\varpi = \phi_1 \cdot (\phi_2 \cdots \phi_t) \cdot (\phi_{t+1} \cdots \phi_m)$ as the product of the three expressions

$$\{R_0 \,|\, x_1, p_1, q_1\} + \sum \{\underline{R_0} \,|\, x_1, \centerdot, q_1\},$$

$$\left\{R_0' \,\middle|\, B^{(2)}_{[1]} \,\middle|\, \cdots \,\middle|\, B^{(\ell)}_{[1]}\right\} + \sum \cdot \left\{\underline{R_0'} \,\middle|\, \dot{B}^{(2)}_{[1]} \,\middle|\, \cdots \,\middle|\, \dot{B}^{(\ell)}_{[1]}\right\}$$

$$+ \sum_{3 \leq j \leq \ell} \cdot \left\{R_0' \,\middle|\, B^{(2)}_{[1]} \,\middle|\, \cdots \,\middle|\, B^{(j-2)}_{[1]} \,\middle|\, \underline{B^{(j-1)}_{[1]}} \,\middle|\, \dot{B}^{(j)}_{[1]} \,\middle|\, \cdots \,\middle|\, \dot{B}^{(\ell)}_{[1]}\right\},$$

and

$$\left\{ R_0'' \,\Big|\, B_{[2]}^{(2)} \,\Big|\, \cdots \,\Big|\, B_{[2]}^{(\ell)} \right\} + \sum \cdot \left\{ \underline{R_0''} \,\Big|\, \dot{B}_{[2]}^{(2)} \,\Big|\, \cdots \,\Big|\, \dot{B}_{[2]}^{(\ell)} \right\}$$
$$+ \sum_{3 \le j \le \ell} \cdot \left\{ R_0'' \,\Big|\, B_{[2]}^{(2)} \,\Big|\, \cdots \,\Big|\, B_{[2]}^{(j-2)} \,\Big|\, \underline{B_{[2]}^{(j-1)}} \,\Big|\, \dot{B}_{[2]}^{(j)} \,\Big|\, \cdots \,\Big|\, \dot{B}_{[2]}^{(\ell)} \right\},$$

where $R_0' = \mathcal{R}_0(x_2, \ldots, x_t)$ and $R_0'' = \mathcal{R}_0(x_{t+1}, \ldots, x_m)$. Note the compactified notation for the two summations running over j, each of which really stands for sums of summations as in (47). Note also that the interaction level of the vertices $x_2, \ldots, x_t$ (or $x_{t+1}, \ldots, x_m$) could be smaller than ℓ, in which case some of the corresponding collections of blocks are empty. Then, by direct inspection and interaction reasons (using (38) when $s_1 = 1$ and the interaction parameter under consideration lies in x_1-direction $d_1 - 1$), the product of the three expressions above takes the form (47). This completes the proof when $\mathcal{L}_1$ is a singleton.

In general, $\mathcal{L}_1$ consists of, say, vertices $x_1 = x_{i_1} < \cdots < x_{i_k}$, and we evaluate ϖ as the length-k product

$$(51) \qquad\qquad (\phi_1 \cdots \phi_{i_2 - 1})(\phi_{i_2} \cdots \phi_{i_3 - 1}) \cdots (\phi_{i_k} \cdots \phi_m).$$

(This time there is no need to reset notation so as to get the analogue of (49) to hold.) We have just seen that the w-th factor in (51) takes the form

$$\left\{ r_{i_w} \,\Big|\, B_{[w]}^{(1)} \,\Big|\, \cdots \,\Big|\, B_{[w]}^{(\ell)} \right\} + \sum \cdot \left\{ \underline{r_w} \,\Big|\, \dot{B}_{[w]}^{(1)} \,\Big|\, \cdots \,\Big|\, \dot{B}_{[w]}^{(\ell)} \right\}$$
$$+ \sum_{2 \le j \le \ell} \cdot \left\{ r_{i_w} \,\Big|\, B_{[w]}^{(1)} \,\Big|\, \cdots \,\Big|\, B_{[w]}^{(j-2)} \,\Big|\, \underline{B_{[w]}^{(j-1)}} \,\Big|\, \dot{B}_{[w]}^{(j)} \,\Big|\, \cdots \,\Big|\, \dot{B}_{[w]}^{(\ell)} \right\},$$

where

$$B_{[w]}^{(1)} := B_{i_w}, \quad \dot{B}_{[w]}^{(1)} := \dot{B}_{i_w}, \quad \underline{B_{[w]}^{(1)}} := \underline{B_{i_w}}$$

and, for interaction levels larger than 1, a subindex '$[w]$' in a collection of blocks indicates that only blocks in positive x_{i_w}-directions are to be taken. The required form (47) for the product of all these expressions follows again from direct inspection —this time without requiring the use of (38). $\qquad\square$

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