

## BOUNDS ON COHOMOLOGICAL SUPPORT VARIETIES

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ABSTRACT. Over a local ring  $R$ , the theory of cohomological support varieties attaches to any bounded complex  $M$  of finitely generated  $R$ -modules an algebraic variety  $V_R(M)$  that encodes homological properties of  $M$ . We give lower bounds for the dimension of  $V_R(M)$  in terms of classical invariants of  $R$ . In particular, when  $R$  is Cohen–Macaulay and not complete intersection we find that there are always varieties that cannot be realized as the cohomological support of any complex. When  $M$  has finite projective dimension, we also give an upper bound for  $\dim V_R(M)$  in terms of the dimension of the radical of the homotopy Lie algebra of  $R$ . This leads to an improvement of a bound due to Avramov, Buchweitz, Iyengar, and Miller on the Loewy lengths of finite free complexes, and it recovers a result of Avramov and Halperin on the homotopy Lie algebra of  $R$ . Finally, we completely classify the varieties that can occur as the cohomological support of a complex over a Golod ring.

### INTRODUCTION

In local algebra, the theory of cohomological support varieties attaches to any complex  $M$  of modules over a local ring  $R$  an algebraic variety  $V_R(M)$ , defined over the residue field of  $R$ . These were first introduced by Avramov [Avr89b] for local complete intersection rings, taking inspiration from modular representation theory [Qui71]. They were used by Avramov and Buchweitz to establish striking symmetry results in the homological behavior of modules over complete intersection rings [AB00b]. The theory has been extended in [AB00b, AI18, BW15, Jor02, Pol19, Pol21], and now encompasses all local rings. These varieties encode important homological information about  $M$ , as well as ring theoretic properties of  $R$ . For example, the third author showed [Pol19] that  $R$  is complete intersection if and only if  $V_R(R)$  is a point, and thus nontrivial varieties act as obstructions to the complete intersection property.

Throughout,  $R$  will be a noetherian local ring with residue field  $k$ . By Cohen’s structure theorem, the completion of  $R$  at its maximal ideal has the form  $Q/I$  where  $(Q, \mathfrak{m})$  is a regular local ring and  $I$  is an ideal minimally generated by  $n$  elements

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$f_1, \dots, f_n$  of  $\mathfrak{m}^2$ . The cohomological support variety of a complex  $M$  of  $R$ -modules is a conical, algebraic subvariety

$$V_R(M) \subseteq \mathbb{A}_k^n$$

(equivalently, a projective subvariety of  $\mathbb{P}_k^{n-1}$  of dimension one less). In this paper, we give both upper and lower bounds for  $\dim V_R(M)$ , in terms of other invariants of  $R$  and  $M$ , for any bounded complex  $M$  of finitely generated  $R$ -modules.

As a consequence, we give a negative answer to the *realizability question*:

**Question 1.** Given a fixed  $R$ , can every conical subvariety of  $\mathbb{A}_k^n$  be realized as the cohomological support of some bounded complex of finitely generated  $R$ -modules?

When  $R$  is complete intersection, Bergh [Ber07] showed that any conical variety is the cohomological support of some  $R$ -module; see also [AI07]. In general, the third author established a partial negative answer: *if  $R$  is not complete intersection then  $\{0\}$  is not the cohomological support of any finite  $R$ -module* [Pol21, Theorem B]; but the proof there says nothing about the cohomological supports of *complexes*.

One motivation for Question 1 is that constraints on the cohomological support impose constraints on *the lattice of thick subcategories of the bounded derived category  $R$*  (see Section 1.5). In particular, if  $\{0\}$  cannot be realized as the support of any complex, then cohomological supports can be used to detect when thick subcategories are zero. For complete intersection rings, the thick subcategories were classified by Stevenson via cohomological support (phrased in triangulated terms) [Ste14], and the lattice of all such categories has good geometric properties. The situation for general local rings seems to be markedly different; see Proposition 2.14 for the case of Cohen–Macaulay rings, and Corollary 4.4 for the case of Golod rings.

Our nonrealizability result follows from a lower bound on the dimension of cohomological support varieties in terms of the following invariants of  $R$ : the deviations  $\varepsilon_1(R) = \dim Q$  and  $\varepsilon_2(R) = n$ , as well as  $\text{depth}(R)$  and, if  $R$  is artinian, the Loewy length  $\ell_R(R)$ .

**Theorem A.** *For any local ring  $R$  and any bounded complex  $M$  of finitely generated modules with  $H(M) \neq 0$ , we have*

$$\dim V_R(M) \geq \varepsilon_2(R) - \varepsilon_1(R) + \text{depth}(R),$$

*with strict inequality if  $R$  is not complete intersection. Moreover, if  $R$  is artinian then*

$$\dim V_R(M) \geq \varepsilon_2(R) - \ell_R(R) + 1.$$

Both inequalities in the theorem follow from a stronger bound in terms of the *Lusternik–Schnirelmann category* of  $R$ ; see Corollary 2.8.

For Cohen–Macaulay rings, the lower bound  $\varepsilon_2(R) - \varepsilon_1(R) + \text{depth}(R)$  is known as the **complete intersection defect** of  $R$ . This integer is always nonnegative, and vanishes exactly when  $R$  is complete intersection [KK65, Avr77]. Thus Theorem A yields a strong negative answer to the realizability question: *if  $R$  is Cohen–Macaulay but not complete intersection then*

$$\dim V_R(M) \geq 2$$

*for any bounded complex  $M$  of finitely generated modules with  $H(M) \neq 0$ .*

Our upper bounds on the dimension of  $V_R(M)$  exploit a new connection between the cohomological support varieties and the *homotopy Lie algebra* of  $R$ . This is a

graded Lie algebra  $\pi^*(R)$  attached to  $R$  that has seen a number of important applications in local algebra (cf. [Avr10, Section 10]); the  $k$ -vector space ranks of its graded pieces are the deviations of  $R$ . The next result bounds  $\dim V_R(M)$  in terms of the subspace  $\text{rad}(\pi^*(R))$  of radical elements, first studied in [FHJ<sup>+</sup>88]. See Section 1.7 for definitions and discussion.

**Theorem B.** *For any local ring  $R$  and any bounded complex  $M$  of finitely generated free modules*

$$\dim V_R(M) \leq \varepsilon_2(R) - \text{rank}_k \text{rad}^2(\pi^*(R)).$$

This is really a result about  $V_R(R)$  since the complexes considered satisfy  $V_R(M) = V_R(R)$ . Theorem B follows from the more precise Theorem 3.1, which describes how each radical element of  $\pi^*(R)$  gives rise to a hyperplane containing  $V_R(R)$ . In particular, we obtain an upper bound on the rank of the smallest vector subspace of  $\mathbb{A}_k^n$  containing  $V_R(M)$ , which is often larger than  $\dim V_R(M)$ .

Combined with the characterization [Pol19] of local complete intersection rings in terms of  $V_R(R)$ , Theorem B immediately implies a result of Avramov and Halperin [AH87], that if every element of  $\pi^2(R)$  is radical then  $R$  is complete intersection; see Corollary 3.4. A related open problem of Avramov [Avr89a, Problem 4.3] connects the existence of embedded deformations to central elements in  $\pi^2(R)$ . In Remark 3.6 we discuss how our results shed some new light on this question.

As another consequence of Theorem B, as well as the proof of Theorem A, is the following improvement to a central result from [ABIM10] that bounds the Loewy length of finite free complexes in terms of the *conormal free rank*  $\text{cf-rank}(R)$ .

**Theorem C.** *For any local ring  $R$  and any bounded complex  $M$  of finitely generated free modules with  $H(M) \neq 0$ , we have*

$$\sum \ell_R H_n(M) \geq \varepsilon_2(R) - \dim V_R(R) + 1 \geq \text{cf-rank}(R) + 1.$$

The inequality between the outer two terms is [ABIM10, Theorem 10.1]. Equality can hold throughout, for example if  $R$  is complete intersection and  $M$  is the Koszul complex on a minimal set of generators for the maximal ideal of  $R$ . The inequalities in the theorem above are established using Theorem 3.2 and Remark 3.3, and the right hand inequality is often strict; therefore our result above provides a stronger lower bound on  $\sum \ell_R H_n(M)$  than the previous one from [ABIM10]. More specifically, a consequence of Theorem 3.1 is that

$$\varepsilon_2(R) - \text{rank}_k(\text{span}_k V_R(R)) + 1 \geq \text{cf-rank}(R) + 1,$$

and hence when  $V_R(R)$  is not a linear space, Theorem C gives a better lower bound than the one in [ABIM10]. Taking  $R$  to be the group algebra of an elementary abelian  $p$ -group also recovers a theorem from homotopy theory [Car83, AP93] (see Remark 3.3), although in this case our bound coincides with that of [ABIM10].

The Golod property is reflected in the cohomological support varieties, and for these rings we can completely solve the realizability problem stated in Question 1. The following is Theorem 4.1:

**Theorem D.** *Let  $R$  be a Golod ring. For any bounded complex  $M$  of finitely generated modules with  $H(M) \neq 0$ , the cohomological support variety  $V_R(M)$  is either all of  $\mathbb{A}_k^n$  or a (conical) hypersurface, and every hypersurface is indeed a cohomological support variety of some complex. Moreover, if  $R$  is a nonhypersurface ring, then  $R$  has full support  $V_R(R) = \mathbb{A}_k^n$ .*

This theorem also recovers the fact that if  $R$  is Golod but not a hypersurface, then the conormal free rank of  $R$  is zero; see Remark 4.3 for details.

Section 1 contains the necessary background on cohomological support varieties, semifree dg algebras, levels, and the homotopy Lie algebra. We prove our lower bound on the dimension of cohomological support varieties in Section 2, which leads to our nonrealizability result. In Section 3 we establish a relationship between the homotopy Lie algebra of  $R$  and  $V_R(R)$ , and give some applications. We also discuss how this appears to point to a deeper connection between the ring-theoretic properties of  $R$ , the geometry of  $V_R(R)$ , and the structure of the homotopy Lie algebra. Finally, in Section 4 we focus on the case of Golod rings.

## 1. NOTATION AND BACKGROUND

**1.1. Fixed notation.** We begin by introducing a few objects whose notation will remain fixed for the remainder of the paper. Throughout,  $R$  is a local ring with chosen minimal Cohen presentation  $\widehat{R} = Q/I$ . That is,  $(Q, \mathfrak{m}, k)$  is a regular local ring and  $I \subseteq \mathfrak{m}^2$ . We fix a list of minimal generators  $\mathbf{f} = f_1, \dots, f_n$  for  $I$ ; the number  $n$  of generators will be important throughout.

The Koszul complex on  $\mathbf{f}$  over  $Q$  will be denoted

$$E := Q[e_1, \dots, e_n \mid \partial e_i = f_i].$$

As an algebra,  $E$  is the exterior algebra over  $Q$  on variables  $e_1, \dots, e_n$  in bijection with  $f_1, \dots, f_n$ , given homological degree 1. As a differential graded (henceforth dg) algebra,  $E$  is also given the differential uniquely determined by  $\partial(e_i) := f_i$ .

We will also write

$$\mathcal{S} := k[\chi_1, \dots, \chi_n],$$

the polynomial algebra over  $k$  on variables  $\chi_1, \dots, \chi_n$  dual to  $e_1, \dots, e_n$ , with each  $\chi_i$  given cohomological degree 2 (or homological degree  $-2$ ).

**1.2. Graded support.** Throughout, let  $\mathrm{Spec}^* \mathcal{S}$  denote the set of homogeneous primes  $\mathfrak{p}$  in  $\mathcal{S}$  equipped with the Zariski topology, with closed subsets of the form  $\mathcal{V}(\mathcal{I}) := \{\mathfrak{p} \in \mathrm{Spec}^* \mathcal{S} : \mathcal{I} \subseteq \mathfrak{p}\}$  for a homogeneous ideal  $\mathcal{I}$  of  $\mathcal{S}$ . We think of  $\mathrm{Spec}^* \mathcal{S}$  as “homogeneous affine space”; its points correspond to irreducible conical subvarieties of  $\mathbb{A}_k^n$ . We denote by  $\mathbf{0}$  the irrelevant maximal ideal  $(\chi_1, \dots, \chi_n)$  of  $\mathcal{S}$ , thought of as the origin of  $\mathrm{Spec}^* \mathcal{S}$ .

The support of a graded  $\mathcal{S}$ -module  $X$  is

$$\mathrm{Supp}_{\mathcal{S}} X := \{\mathfrak{p} \in \mathrm{Spec}^* \mathcal{S} : X_{\mathfrak{p}} \neq 0\}.$$

When  $X$  is finitely generated over  $\mathcal{S}$ , then  $\mathrm{Supp}_{\mathcal{S}} X$  is a closed subset of  $\mathrm{Spec}^* \mathcal{S}$ :

$$\mathrm{Supp}_{\mathcal{S}} X = \mathcal{V}(\mathrm{ann}_{\mathcal{S}} X).$$

Any such support set is conical (the closure of a set of lines through the origin), and thus, if it is not zero, may be thought of as a projective variety embedded in  $\mathbb{P}_k^{n-1} = \mathrm{Proj} \mathcal{S}$ , with dimension one less. We work with affine varieties in order to exploit the functoriality of  $\mathrm{Spec}^* \mathcal{S}$  with respect to local homomorphisms (as was done in [BGP22]), and as well because it is sometimes useful to view  $\mathrm{Supp}_{\mathcal{S}} X$  as corresponding to a subset of a *vector space* over  $k$ .

We will say a closed set  $V \subseteq \mathrm{Spec}^* \mathcal{S}$  is **linear** if  $V = \mathcal{V}(\mathcal{I})$  where  $\mathcal{I}$  is an ideal generated by linear forms, that is, elements of  $\mathcal{S}^2$ .

**1.3. Dg algebras and dg modules.** Let  $A = \{A_i\}_{i \in \mathbb{Z}}$  be a dg algebra. We write  $D(A)$  for its derived category of dg  $A$ -modules. This is regarded as a triangulated category in the standard way, with  $\Sigma$  denoting the suspension functor. We write  $D^f(A)$  for the full subcategory of  $D(A)$  consisting of those objects  $M$  such that  $H(M)$  is finitely generated over  $H(A)$ . The reader is directed to [ABIM10] for more details.

**1.3.1.** We let  $(-)^{\natural}$  denote the functor that forgets the differential of a graded object. That is to say, if  $M$  is a dg module over a dg algebra  $A$ , then  $M^{\natural}$  is the underlying graded module over the graded algebra  $A^{\natural}$ .

A dg  $A$ -module  $M$  yields a pair of exact functors

$$- \otimes_A^L M: D(A) \rightarrow D(\mathbb{Z}) \quad \text{and} \quad \text{RHom}_A(M, -): D(A) \rightarrow D(\mathbb{Z}),$$

and we write

$$\text{Ext}_A(M, -) := H(\text{RHom}_A(M, -)) \quad \text{and} \quad \text{Tor}^A(M, -) := H(M \otimes_A^L -).$$

As a special case, a map of dg algebras  $\varphi: A \rightarrow B$  yields an adjoint pair

$$(1) \quad - \otimes_A^L B: D(A) \rightarrow D(B) \quad \text{and} \quad \varphi_*: D(B) \rightarrow D(A),$$

where the second functor is restriction of scalars along  $\varphi$ . For  $N$  in  $D(B)$ , we typically write  $N$  for  $\varphi_*(N)$ .

**1.3.2.** When  $A$  is nonnegatively graded, there is a map of dg algebras  $A \rightarrow H_0(A)$ , and restriction of scalars makes each complex of  $H_0(A)$ -module into a dg  $A$ -module. This will be used without further mention, especially applied to the map  $E \rightarrow \widehat{R}$ .

Assume that  $A$  is a nonnegatively graded dg algebra and  $H_0(A)$  is a commutative noetherian ring. By [AINSW19, Appendix B.2], for each  $M$  in  $D^f(A)$  there exists a semifree resolution  $F \xrightarrow{\simeq} M$  over  $A$  with

$$F^{\natural} \cong \bigoplus_{j=i}^{\infty} \Sigma^j (A^{\natural})^{\beta_j}$$

for some nonnegative integers  $\beta_j$ . Let  $s = \sup\{j : H_j(M) \neq 0\}$  and note that as  $A$  is nonnegatively graded, the soft truncation

$$F' = \cdots \rightarrow 0 \rightarrow \text{coker } \partial_{s+1}^F \rightarrow F_{s-1} \rightarrow F_{s-2} \rightarrow \cdots$$

is quasi-isomorphic to  $M$  as a dg  $A$ -module. Moreover,  $(F')^{\natural}$  is finitely generated as a graded  $A^{\natural}$ -module.

**1.4. Cohomological support.** In this subsection, we recall the definition of our main objects of study. The theory of support varieties over a local complete intersection ring was defined by Avramov in [Avr89b]; see also [AB00b] for notable applications of these geometric objects. D. Jorgensen extended this theory in [Jor02] to arbitrary local rings using intermediate hypersurface rings; see also [AI18]. These theories of support are recovered by the cohomological supports in 1.4.1; see [Pol21, Section 5.2].

**1.4.1.** By [AB00a, Section 2],  $\mathcal{S}$  can be identified with a graded  $k$ -subalgebra of  $\text{Ext}_E(k, k)$ . Hence for any dg  $E$ -module  $N$ , through the composition pairing

$$\text{Ext}_E(k, N) \otimes_k \text{Ext}_E(k, k) \rightarrow \text{Ext}_E(k, N) \quad \text{given by} \quad \alpha \otimes \beta \mapsto \alpha\beta,$$

$\text{Ext}_E(k, N)$  is a graded  $\mathcal{S}$ -module. Moreover, for any  $M$  in  $D^f(R)$  the graded  $k$ -space  $\text{Ext}_E(k, \widehat{M})$  is a finitely generated graded  $\mathcal{S}$ -module; see [Pol19, Proposition 3.2.5].

**1.4.2.** The **cohomological support variety** of an object  $M$  in  $D^f(R)$  is

$$V_R(M) := \text{Supp}_{\mathcal{S}} \text{Ext}_E(k, \widehat{M}) \subseteq \text{Spec}^* \mathcal{S}.$$

**1.4.3.** According to [Pol21, Theorem 6.1.6], for any  $M$  in  $D^f(R)$  we have

$$V_R(M) = \text{Supp}_{\mathcal{S}} \text{Ext}_E(k, \widehat{M}) = \text{Supp}_{\mathcal{S}} \text{Ext}_E(\widehat{M}, k),$$

this time using the natural left action of  $\text{Ext}_E(k, k)$  on  $\text{Ext}_E(\widehat{M}, k)$  by Yoneda composition. In other words, we may compute cohomological support varieties using the  $\text{Ext}$ -modules with  $\widehat{M}$  in either argument.

It follows from Nakayama’s lemma that  $\text{Ext}_E(\widehat{M}, k) = 0$  only when  $M \simeq 0$ . This in turn implies that

$$V_R(M) = \emptyset \text{ if and only if } M \simeq 0.$$

*Remark 1.4.4.* The fact that  $V_R(M)$  is independent of the choice of Cohen presentation, and of the minimal generators for the defining ideal of  $\widehat{R}$  in such a presentation, is dealt with by [Pol21, Theorem 6.1.2].

**Example 1.4.5.** By [Pol19, Theorem 3.3.2],  $R$  is complete intersection if and only if  $V_R(R) = \{\mathbf{0}\}$ , the origin. In fact, in [Pol21, Theorem 6.1.6], this equivalence was strengthened to say that  $R$  is complete intersection if and only if  $V_R(M) = \{\mathbf{0}\}$  for some finitely generated  $R$ -module  $M$ .

**Example 1.4.6.** One always has  $V_R(k) = \text{Spec}^* \mathcal{S}$ . Indeed,  $\text{Ext}_E(k, k)$  is a finite rank free graded  $\mathcal{S}$ -module; cf. [Pol21, Remark 3.2.6].

**Example 1.4.7.** Given a ring of central cohomology operators, one has control on the support of the *Koszul objects* introduced in [AI07] (see also [BIK08]). In the setting of the present article,  $\mathcal{S}$  is a ring of central cohomology operators on  $D(E)$  but not  $D(R)$ , unless  $R$  is complete intersection. Regardless, [Pol19, Theorem 3.3.4] established that for any nonzero  $\zeta \in \mathcal{S}^d$ , with  $d$  any nonnegative even integer, there exists an object  $L_\zeta$  in  $D^f(R)$  which plays the role of a Koszul object in the sense that

$$V_R(L_\zeta) = \mathcal{V}(\zeta).$$

That is, each conical hypersurface in  $\text{Spec}^* \mathcal{S}$  is realizable as the cohomological support of an object in  $D^f(R)$ . The objects  $L_\zeta$  are defined as

$$L_\zeta := \text{cone}(k \xrightarrow{\tilde{\zeta}} \Sigma^d k)$$

in  $D^f(R)$ , where  $\tilde{\zeta}$  is a lift of  $\zeta$  along the map

$$\text{Ext}_R(k, k) \cong \text{Ext}_{\widehat{R}}(k, k) \rightarrow \text{Ext}_E(k, k);$$

the fact this map is surjective is [Pol19, Theorem 2.3.2]. Consequently, if  $d > 0$ , then

$$H_i(L_\zeta) = \begin{cases} k & i = 1, d \\ 0 & \text{otherwise.} \end{cases}$$

In Theorem 4.1, we will see that these hypersurfaces are the only proper subsets of  $\text{Spec}^* \mathcal{S}$  that are *always* realizable regardless of  $R$ ; see also Example 1.4.9. The complexes  $L_\zeta$  are analogous to the Carlson modules defined over group algebras; cf. [Ben17, Section 1.10].

**Example 1.4.8.** Recall  $R$  has an **embedded deformation** if there exists a local ring  $S$  and an element  $f$  in the square of the maximal ideal of  $S$  such that  $R \cong S/(f)$ . By [Pol21, Theorem 6.3.5], whenever  $\text{codepth}(R) := \text{depth}(Q) - \text{depth}(R) \leq 3$  we have

$$V_R(R) = \begin{cases} \{0\} & \text{if } R \text{ is complete intersection} \\ \text{Spec}^* \mathcal{S} & \text{if } R \text{ does not admit an embedded deformation} \\ \mathcal{V}(\zeta) & \text{for some } \zeta \in \mathcal{S}^2, \text{ otherwise.} \end{cases}$$

So when the codepth of  $R$  is small,  $V_R(R)$  is always a linear space. In [BGP22, Example 5.4] an example was provided of a nonlinear variety. Namely,

$$V_R(R) = \mathcal{V}(\chi_1\chi_5) = \mathcal{V}(\chi_1) \cup \mathcal{V}(\chi_5) \subseteq \text{Spec}^* k[\chi_1, \dots, \chi_5]$$

where  $R = k[[x, y, z, w]]/(x^2, xy, yz, zw, w^2)$ .

**Example 1.4.9.** One can always construct a finitely generated  $R$ -module  $M$  whose support is contained in a hyperplane. To do this, fix a minimal generator  $f$  of  $I$  of minimal  $\mathfrak{m}$ -adic order, and find a complete intersection ideal  $J \supseteq I$  such that  $f$  is also a minimal generator of  $J$ . By [BGP22, Lemma 4.2], such a  $J$  always exists, and by [BGP22, Lemma 3.6],  $V_R(Q/J)$  is a linear space contained in the hyperplane determined by  $f$ .

**1.5. Thick subcategories and levels.** Here we recall a notion from [ABIM10] that counts the number of mapping cones needed to build one object from another in a triangulated category; see also [BvdB03, Section 2] or [Rou08, Section 3].

**1.5.1.** Let  $\mathbb{T}$  be a triangulated category with suspension functor  $\Sigma$ , and fix an object  $N$  in  $\mathbb{T}$ . We write  $\text{thick}_{\mathbb{T}}^0 N = \{0\}$  and  $\text{thick}_{\mathbb{T}}^1 N$  for the smallest full additive subcategory of  $\mathbb{T}$  containing  $\Sigma^i N$  for all  $i$  and closed under direct summands. Then inductively,  $\text{thick}_{\mathbb{T}}^n N$  is the smallest full subcategory of  $\mathbb{T}$  closed under direct summands that contains all objects  $M$  for which there is an exact triangle

$$M' \rightarrow M \rightarrow M'' \rightarrow$$

with  $M'$  in  $\text{thick}_{\mathbb{T}}^1 N$  and  $M''$  in  $\text{thick}_{\mathbb{T}}^{n-1} N$ . We also set

$$\text{thick}_{\mathbb{T}} N := \bigcup_{n=0}^{\infty} \text{thick}_{\mathbb{T}}^n N,$$

and note that  $\text{thick}_{\mathbb{T}} N$  is exactly the thick closure of  $N$  in  $\mathbb{T}$ ; that is,  $\text{thick}_{\mathbb{T}} N$  is the smallest *triangulated* subcategory of  $\mathbb{T}$  that is closed under direct summands. For  $M$  in  $\mathbb{T}$ , we define  $\text{level}_{\mathbb{T}}^N M$  to be the smallest integer  $n$  such that  $M$  belongs to  $\text{thick}_{\mathbb{T}}^n N$ ; if no such integer exists then  $\text{level}_{\mathbb{T}}^N M = \infty$ .

*Notation 1.5.2.* Let  $A$  be a dg algebra. When calculating level in  $D(A)$  we will write  $\text{level}_A$  instead of  $\text{level}_{D(A)}$ .

**Example 1.5.3.** Recall an  $R$ -complex  $M$  is **perfect** if it is in  $\text{thick}_R R$ . That is,  $M$  is quasi-isomorphic to a finite free  $R$ -complex; the latter is a bounded complex of finite rank free  $R$ -modules.

**Example 1.5.4.** In the notation of Example 1.4.7, for each  $\zeta \in \mathcal{S}^d$  with  $d > 0$ , satisfies  $\text{level}_R^k(L_{\zeta}) = 2$ . Indeed, by definition  $\text{level}_R^k(L_{\zeta}) \leq 2$ . Since  $V_R(N) = \text{Spec}^* \mathcal{S}$  for all nonzero  $N$  in  $\text{thick}_R^1 k$ , the desired equality follows the already noted strict containment  $V_R(L_{\zeta}) \subsetneq \text{Spec}^* \mathcal{S}$ ; cf. Example 1.4.7.

**1.5.5.** Let  $R$  be a local ring and let  $M, N$  be in  $D^f(R)$ . By [Pol19, 3.3.2] if  $M$  is in  $\text{thick}_R N$  then  $V_R(M) \subseteq V_R(N)$ . It follows that the cohomological support varieties may be thought of as invariants of thick subcategories of  $D^f(R)$ ; see Proposition 2.14 for an application of this idea.

**1.5.6.** Given an exact functor of triangulated categories  $\mathfrak{t}: \mathbb{T} \rightarrow \mathbb{S}$ , it follows directly from the definition of level that

$$\text{level}_{\mathbb{S}}^{\mathfrak{t}^N} \mathfrak{t}M \leq \text{level}_{\mathbb{T}}^N M$$

for any pair of objects  $M, N$  in  $\mathbb{T}$ . If  $\mathfrak{t}$  admits a left inverse then equality holds.

In particular, for any map  $\varphi: A \rightarrow B$  of dg algebras, and for any  $M, M'$  in  $D(A)$  and  $N, N'$  in  $D(B)$ , the exact functors in 1.3.1(1) satisfy inequalities

$$\text{level}_B^{M \otimes_A^L B} (M' \otimes_A^L B) \leq \text{level}_A^M M' \quad \text{and} \quad \text{level}_A^N N' \leq \text{level}_B^N N'.$$

**1.6. Semifree dg algebras and the LS category.** We recall some necessary facts on semifree dg algebras in the section. Much of the background needed is contained in [Avr10, Bri18].

**1.6.1.** A dg algebra over a commutative ring  $T$  is called **semifree** if it is concentrated in nonnegative homological degrees and, as a graded  $T$ -algebra, it is isomorphic to the free strictly graded commutative algebra  $T[X]$  on a graded set of variables  $X = X_1 \cup X_2 \cup \dots$  (the elements of  $X_n$  having degree  $n$ ). Explicitly,  $T[X]$  is the tensor product of the exterior algebra over  $T$  on  $X_{\text{odd}}$  with the symmetric algebra over  $T$  on  $X_{\text{even}}$ .

**Example 1.6.2.** The Koszul complex  $E$  on  $\mathfrak{f}$  over  $Q$  is semifree with variables  $X_1 = \{e_1, \dots, e_n\}$  and  $X_i = \emptyset$  for  $i \geq 2$ .

**1.6.3.** A **semifree resolution** of  $\widehat{R}$  over  $Q$  is a quasi-isomorphism  $Q[X] \rightarrow \widehat{R}$  of dg  $Q$ -algebras where  $Q[X]$  is semifree over  $Q$ . Such resolutions may be constructed inductively using the method of Tate, successively adjoining variables to kill homology classes (but using polynomial variables instead of divided power variables); see [Tat57] or [Avr10] for details. By [Avr10, Lemma 7.2.2], when this resolution is constructed by adjoining the minimal possible number of variables at each stage, the obtained dg-algebra  $Q[X]$  has a **decomposable** differential, meaning that  $\partial(x) \in (\mathfrak{m}, X^2)$  for any  $x \in X$ . Such a resolution is called a **minimal model** for  $R$ ; by [Avr10, Proposition 7.2.4] minimal models are unique up to isomorphism.

**1.6.4.** Since  $Q[X]$  is well-defined up to isomorphism, the numbers  $\varepsilon_i(R) = |X_{i-1}|$  for  $i \geq 2$  are well-defined invariants of  $R$ , known as the **deviations** of  $R$ ; by convention  $\varepsilon_1(R) = \dim(Q)$  is the embedding dimension of  $R$ . The Koszul complex  $E$  is exactly the first step in the construction of the minimal model of  $R$ , and therefore  $\varepsilon_2(R) = n$ , the minimal number of generators of  $I$ .

**1.6.5.** Lusternik–Schnirelmann category is a numerical invariant having its origins in topology that extends Loewy length to the realm of dg algebras.

Let  $A = Q[X]$  be a minimal semifree dg algebra as above, with maximal ideal  $\mathfrak{m}_A = (\mathfrak{m}, X)$ . By definition,  $\text{LScat}(A)$  is the smallest integer  $m$  such that there is



a dg algebra  $B$  and a diagram

$$(2) \quad \begin{array}{ccccc} A & \xrightarrow{\iota} & B & \xrightarrow{\rho} & A \\ & & \tau \downarrow \simeq & & \\ & & A/\mathfrak{m}_A^{m+1} & & \end{array}$$

such that  $\rho\iota = \text{id}_A$  and  $\tau\iota = \pi$ , the natural projection from  $A$  to  $A/\mathfrak{m}_A^{m+1}$ . If there is no such integer, then  $\text{LScat}(A) = \infty$ . This definition first appears in [FH80], where it is given as a characterization of the rational LS category of a simply connected space  $Y$ , taking  $A$  to be the minimal Sullivan model associated to  $Y$ .

We remark that (2) implies that the inclusion  $\mathfrak{m}_A^{m+1} \hookrightarrow \mathfrak{m}_A$  is a nullhomotopic chain map; we will use this particular consequence of the definition to prove Theorem 3.1.

**1.6.6.** Adopting the notation from 1.6.5, restriction along the maps in (2) yields exact functors of triangulated categories

$$(3) \quad D(A) \xrightleftharpoons[\pi_*]{F=(\tau_*)^{-1}\rho_*} D(A/\mathfrak{m}_A^{\text{LScat}(A)+1}) \quad \text{with} \quad \pi_*F = \text{id}.$$

This observation will be used in the proof of Lemma 2.6.

**1.6.7.** We associate an important numerical invariant to any local ring  $R$  as in [FHJ<sup>+</sup>88] by passing to a semifree resolution and using the Lusternik–Schnirelmann category for dg algebras.

To be precise, let  $Q[X]$  be a minimal model for  $R$  as in 1.6.3, and set

$$A := k[X] = k \otimes_Q Q[X].$$

Then the **LS category** of  $R$  is by definition the integer

$$\text{LScat}(R) := \text{LScat}(A).$$

A priori the LS category may be infinite, but by [BH86, Lemma 2.3] we have

$$(4) \quad \text{LScat}(R) \leq \text{codepth}(R),$$

and in particular  $\text{LScat}(R)$  is always finite. For artinian rings LS category is, by [BH86, Proposition 2.4], also bounded by the Loewy length of  $R$ :

$$\text{LScat}(R) \leq \ell\ell(R).$$

This inequality is a special case of the *mapping theorem*; see [FH80, Theorem I] and [Bri18, Theorem 20]. Moreover, the inequality in (4) is strict when  $R$  is not complete intersection.

**1.7. The homotopy Lie algebra.** To each local ring  $R$ , one can associate a graded Lie algebra  $\pi^*(R)$  over  $k$ , known as the **homotopy Lie algebra of  $R$** , and its radical subspace

$$\text{rad}(\pi^*(R)) \subseteq \pi^*(R).$$

**1.7.1.** Fix a minimal model  $Q[X]$  for  $R$  as defined in 1.6.3. Following [AH87] we set

$$\pi^i(R) := (kX_{i-1})^\vee \quad \text{for} \quad i \geq 2,$$

where  $(-)^\vee$  denotes  $k$ -space duality. Often we will think of elements of  $\pi^i(R)$  as functionals on  $kX$ , taking the value zero on  $X_j$  when  $j \neq i - 1$ . We note that

$\text{rank}_k \pi^i(R) = \varepsilon_i(R)$  by definition (see 1.6.4), and therefore one can think of  $\pi^*(R)$  as an algebraic object that enriches the deviations of  $R$ .

A Lie bracket can be defined on this graded vector space using the fact that the differential of  $Q[X]$  is decomposable using the following recipe. Given  $\alpha \in \pi^i(R)$  and  $\beta \in \pi^j(R)$ , we need to define  $[\alpha, \beta] \in \pi^{i+j}(R) = (X_{i+j-1})^\vee$ . To do that, let  $x \in X_{i+j-1}$  and suppose that

$$\partial(x) = \sum_{pq} a_{pq} y_p z_q \pmod{\mathfrak{m} + (X)^3},$$

with  $a_{pq} \in Q$  and  $y_p, z_q \in X$ . The bracket  $[\alpha, \beta]$  is then determined by

$$[\alpha, \beta](x) := \sum_{pq} \bar{a}_{pq} ((-1)^{i+1+ij} \alpha(y_p) \beta(z_q) + (-1)^j \beta(y_p) \alpha(z_q))$$

with  $\bar{a}_{pq}$  denoting the class of  $a_{pq}$  in  $k$ ; cf. [AH87, page 175]. This makes  $\pi^{\geq 2}(R)$  into a graded Lie algebra over  $k$ .

We have only defined the homotopy Lie algebra in degrees 2 and above. The full homotopy Lie algebra, including  $\pi^1(R)$ , can be defined along similar lines, cf. [Avr10] or [Bri18].

When  $k$  has characteristic 2 a graded Lie algebra should possess a squaring operation on odd degree elements, in addition to its bracket. Moreover in characteristic 2 or 3 care has to be taken to state the Jacobi identity in full. These technical points do not affect the constructions of this paper, and so we refer to [Avr10, Remark 10.1.2] for details.

**1.7.2.** The homotopy Lie algebra in degree two admits a simple description. Keeping in mind that the Koszul complex  $E = Q[X_1]$ , see Example 1.6.2, is the first step in the construction of  $Q[X]$ , we have a canonical isomorphism

$$\pi^2(R) = (kX_1)^\vee = \text{span}_k \{e_1^\vee, \dots, e_n^\vee\} \xrightarrow{\cong} (I/\mathfrak{m}I)^\vee, \quad e_i^\vee \mapsto f_i^\vee.$$

**1.7.3.** For any element  $\alpha \in \pi^i(R)$  we consider the adjoint action

$$\text{ad}(\alpha) := [\alpha, -]: \pi^*(R) \rightarrow \pi^{*+i}(R).$$

We say that  $\alpha$  is **central** if  $\text{ad}(\alpha) = 0$ , and **radical** if  $\text{ad}(\alpha)^p = 0$  for some  $p$ . The radical of  $\pi^*(R)$  is the set of all radical elements

$$\text{rad}^*(\pi^*(R)) := \{a \in \pi^*(R) \mid \text{ad}(\alpha)^p = 0 \text{ for some } p\}.$$

It follows from the results of [FHJ<sup>+</sup>88] that  $\text{rad}^*(\pi^*(R))$  is a graded Lie ideal in  $\pi^*(R)$ ; in fact it is the maximal solvable ideal. We will be especially interested in radical elements living in degree 2, which we will denote by  $\text{rad}^2(\pi^*(R))$ .

## 2. LOWER BOUNDS ON COHOMOLOGICAL SUPPORT

The goal here is to show there is a uniform lower bound on  $\dim V_R(M)$ , for all nonzero  $M$  in  $D^f(R)$ , based on invariants of  $R$ ; see Theorem 2.7.

First we use a well-traversed bridge from homological algebra over  $R$  to homological algebra over an exterior algebra, exploited in [ABIM10, AI10, IPS22, LP21] to name a few. This is a dg version of the celebrated BGG correspondence [BGG78]; see [ABIM10, Section 7].

**Construction 2.1.** We use the notation from 1.1, and in particular  $E$  is the Koszul complex on  $\mathbf{f}$  over  $Q$ . We also write  $\Lambda := E \otimes_Q k$ , the exterior algebra over  $k$  on  $n$  variables of homological degree 1. Consider the functor

$$\mathbf{t}: D(E) \longrightarrow D(\Lambda)$$

given by  $\mathbf{t} := - \otimes_Q^{\mathbf{L}} k$ . That is, for an object  $N$  in  $D(E)$  we have

$$\mathbf{t}(N) \simeq F \otimes_Q k,$$

where  $F \xrightarrow{\simeq} N$  is a quasi-isomorphism of dg  $E$ -modules, with  $F^{\natural}$  free as a  $Q$ -module. Note  $\mathbf{t}$  restricts to a functor  $D^f(E) \rightarrow D^f(\Lambda)$ , also denoted  $\mathbf{t}$ , since each object of  $D^f(E)$  is perfect over  $Q$ . The second functor of interest is the equivalence from [ABIM10]:

$$\mathbf{h}: D^f(\Lambda) \xrightarrow{\cong} D^f(\mathcal{S})$$

given by  $\mathbf{h} = \mathrm{RHom}_{\Lambda}(k, -)$ . In  $D(\mathcal{S})$  there is an isomorphism  $\mathbf{h}(k) \simeq \mathcal{S}$ , and so, using 1.5.6, for any dg  $\Lambda$ -module  $N$ , the following equality is satisfied

$$\mathrm{level}_{\Lambda}^k(N) = \mathrm{level}_{\mathcal{S}}^{\mathcal{S}}(\mathbf{h}N).$$

**Lemma 2.2.** *In the notation from Construction 2.1,  $\mathbf{t}(k) \simeq \bigwedge_{\Sigma} k^{\varepsilon_1(R)}$  in  $D(\Lambda)$ .*

*Proof.* Since  $Q$  is regular, there is a quasi-isomorphism of dg  $Q$ -algebras  $K^Q \xrightarrow{\simeq} k$ . The dg algebra  $K^Q$  obtains a dg  $E$ -algebra structure by choosing a lift of the canonical map  $E \rightarrow k$  along this quasi-isomorphism. Since  $\mathbf{f} \subseteq \mathfrak{m}^2$ , any such lift  $E \rightarrow K^Q$  will factor in degree 1 as

$$E_1 \rightarrow \mathfrak{m}K_1^Q \rightarrow K_1^Q;$$

see, for example, [Pol19, Section 2.2]. By the definition of  $\mathbf{t}$ , the action of  $\Lambda_1 = E_1 \otimes_Q k$  on  $\mathbf{t}(k) \simeq k \otimes_Q K^Q$  factors through the zero map

$$k \otimes_Q \mathfrak{m}K_1^Q \rightarrow k \otimes_Q K_1^Q,$$

and therefore  $\mathbf{t}(k)$  is isomorphic to a direct sum of shifted copies of  $k$  in  $D(\Lambda)$ .  $\square$

**Proposition 2.3.** *For each  $M$  in  $D^f(R)$ , the following equalities are satisfied:*

$$V_R(K^M) = V_R(M) = \mathrm{Supp}_{\mathcal{S}} \mathbf{H}(\mathbf{h}tM).$$

*Proof.* First note

$$\mathrm{Ext}_E(k, \widehat{K^M}) \cong \mathrm{Ext}_E(k, \widehat{M}) \otimes_k \bigwedge_{\Sigma} k^{\varepsilon_1(R)}$$

as graded  $\mathcal{S}$ -modules. Indeed,  $K^M$  is an iterated mapping cone on a minimal generating set for  $\mathfrak{m}$  starting from  $M$ , and so the isomorphism holds by induction; see also [Pol19, Lemma 3.2.4]. From this isomorphism, the first equality holds. The second equality is from [IPS22].  $\square$

**2.4.** Fix a closed subset  $\mathcal{U}$  of  $\mathrm{Spec}^* \mathcal{S}$ . The dimension of  $\mathcal{U}$  is

$$\dim \mathcal{U} := \dim \mathcal{S}/\mathcal{I},$$

where  $\mathcal{U} = \mathcal{V}(\mathcal{I})$  for a homogeneous ideal  $\mathcal{I}$  of  $\mathcal{S}$ , and the right-hand side is the Krull dimension of  $\mathcal{S}/\mathcal{I}$ . The codimension of  $\mathcal{U}$  is

$$\mathrm{codim} \mathcal{U} := \dim \mathcal{S} - \dim \mathcal{U},$$

and the height of  $\mathcal{I}$  is

$$\mathrm{height} \mathcal{I} := \inf\{n \geq 0 : \mathfrak{p}_0 \subseteq \dots \subseteq \mathfrak{p}_n \text{ in } \mathrm{Spec}^* \mathcal{S} \text{ with } \mathfrak{p}_n \supseteq \mathcal{I}\}.$$

Since  $\mathcal{S}$  is Cohen–Macaulay, by [BH98, Corollary 2.1.4] we have

$$\text{codim } \mathcal{U} = \text{height } \mathcal{I}.$$

**2.5.** For any dg  $\mathcal{S}$ -module  $X$ , the *dg new intersection theorem* states:

$$\text{level}_{\mathcal{S}}^{\mathcal{S}} X \geq \text{height}(\text{ann}_{\mathcal{S}} H(X)) + 1.$$

This is stated and proved in [ABIM10, Theorem 5.1] when  $\mathcal{S}$  is a graded algebra over a field, as it is in this article. However, we record that in light of the work of André [And18], see also [Bha18], the assumption that this graded ring contain a field is no longer necessary.

**Lemma 2.6.** *Let  $(R, \mathfrak{m}, k)$  be a local ring and let  $K^R$  be the Koszul complex on a minimal generating set for  $\mathfrak{m}$ . For  $N$  in  $\text{D}^f(K^R)$ , we have*

$$\text{level}_{K^R}^k N \leq \text{LScat}(R) + 1.$$

*Proof.* First, note that the level and LS-category in the statement are unchanged when we pass to the completion, so we can reduce to the case when  $R$  is complete. Fix a minimal Cohen presentation  $Q \rightarrow R$  and a minimal model  $Q[X] \xrightarrow{\sim} R$ . The quasi-isomorphisms of dg algebras

$$K^R \xleftarrow{\sim} K^Q \otimes_Q Q[X] \xrightarrow{\sim} A := k \otimes_Q Q[X]$$

define an equivalence of categories  $\text{D}(K^R) \cong \text{D}(A)$ ; under this equivalence  $k$  is sent to  $k$  as the quasi-isomorphisms above respect the augmentation maps to  $k$ . Hence, by 1.5.6 it suffices to show  $\text{level}_A^k M \leq \text{LScat}(R) + 1$  for each  $M$  in  $\text{D}^f(A)$ .

Set  $C = A/(X)^{\text{LScat}(R)+1}$  and consider the functor

$$\text{D}(A) \rightarrow \text{D}(C)$$

described in 1.6.6. By 1.6.6(3), this functor admits a left inverse, and so we have further reduced the problem to showing that for each  $M$  in  $\text{D}^f(C)$  there is an inequality

$$\text{level}_C^k M \leq \text{LScat}(R) + 1.$$

To this end, as  $M$  is in  $\text{D}^f(C)$  we can assume the underlying module  $M^\natural$  obtained from  $M$  by forgetting the differential is finitely generated over the graded algebra  $C^\natural$ ; see 1.3.2. Consider the sequence of dg  $C$ -modules

$$0 = (X)^{\text{LScat}(R)+1} M \subseteq \dots \subseteq (X)^2 M \subseteq (X) M \subseteq M.$$

The fact that these are dg  $C$ -submodules of  $M$  uses that  $Q[X]$  was chosen as a minimal model, and so  $C$  satisfies  $\partial(X) \subseteq (X)^2$ . Since  $M^\natural$  is finitely generated over  $C^\natural$ , each quotient

$$(X)^i M / (X)^{i+1} M$$

is a finite dimensional graded  $k$ -space and consequentially the desired inequality follows. □

We now arrive at the main result of the section. The lower bound in it takes inspiration from [ABIM10, Theorem 7]; see also [AI10, Theorem 9.1].

**Theorem 2.7.** *Let  $R$  be a local ring with residue field  $k$ . If  $M$  is a nonzero object in  $\text{D}^f(R)$ , then the following inequalities are satisfied:*

$$\text{LScat}(R) + 1 \geq \text{level}_{K^R}^k(K^M) \geq \varepsilon_2(R) - \dim V_R(M) + 1.$$

*Proof.* The first inequality is Lemma 2.6. For the second inequality, set  $N = K^M$ , and consider the inequalities

$$\begin{aligned} \text{level}_{K^R}^k(N) &\geq \text{level}_R^k(N) && \text{by 1.5.6, restricting along } R \rightarrow K^R \\ &\geq \text{level}_\Lambda^k(\mathfrak{t}N) && \text{by applying } \mathfrak{t} \text{ and 1.5.6} \\ &= \text{level}_\Lambda^k(\mathfrak{t}N) && \text{using Lemma 2.2} \\ &= \text{level}_\mathcal{S}^\mathcal{S}(\text{ht}N) && \text{using Construction 2.1} \\ &\geq \text{height}(\text{ann}_\mathcal{S} \text{H}(\text{ht}N)) + 1 && \text{by 2.5.} \end{aligned}$$

Also, the following equalities are satisfied

$$\begin{aligned} \text{height}(\text{ann}_\mathcal{S} \text{H}(\text{ht}N)) &= \text{codim } \mathcal{V}(\text{ann}_\mathcal{S} \text{H}(\text{ht}N)) && \text{by 2.4} \\ &= \dim \mathcal{S} - \dim V_R(M) \\ &= \varepsilon_2(R) - \dim V_R(M) && \text{by 1.6.4} \end{aligned}$$

where the second equality comes from Section 1.2 and Proposition 2.3 since  $\text{H}(\text{ht}N)$  is a finitely generated  $\mathcal{S}$ -module.

Combining the two strings of inequalities completes the proof. □

The **complete intersection defect** of  $R$ , introduced in [Avr77] and implicitly appearing in [KK65], is defined as

$$\text{cid}(R) := \varepsilon_2(R) - \varepsilon_1(R) + \dim R,$$

or alternatively,  $\text{cid}(R) = \text{rank}_k(I/\mathfrak{m}I) - \text{height}(I)$  [Avr10, Lemma 7.4.1]. This is a nonnegative integer that measures how close  $R$  is to being complete intersection, in particular,  $R$  is complete intersection if and only if  $\text{cid}(R) = 0$ ; see [KK65, Section 1] and [Avr77, (3.2.3)]. Showing this invariant is additive along the base, source and fiber of a flat map was fundamental in Avramov’s solution to the localization problem for complete intersection rings [Avr75, Avr77].

**Corollary 2.8.** *Every nonzero object  $M$  in  $D^f(R)$  satisfies*

$$\dim V_R(M) \geq \varepsilon_2(R) - \text{LScat}(R).$$

*In particular, if  $R$  is Cohen–Macaulay but not complete intersection, then*

$$\dim V_R(M) > \text{cid}(R) > 0$$

*for each nonzero object  $M$  in  $D^f(R)$ .*

*Proof.* The first inequality is immediate from Theorem 2.7. The second inequality now follows from the first and (4) in 1.6.7. □

There is no uniform bound on the difference between  $\dim V_R(R)$  and  $\text{cid}(R)$  when  $R$  is not complete intersection.

**Example 2.9.** Let  $R = k[x_1, \dots, x_i]/(x_1, \dots, x_i)^2$  for  $i \geq 2$ . From the definition of complete intersection defect,

$$\text{cid}(R) = \binom{i+1}{2} - i.$$

On the other hand,  $R$  is Golod, and therefore by Theorem 4.1 it follows that for any nonzero  $M$  in  $D^f(R)$ :

$$\dim V_R(R) = \binom{i+1}{2} \quad \text{and} \quad \dim V_R(M) \geq \binom{i+1}{2} - 1,$$

and in particular  $\dim V_R(R) - \text{cid}(R) = i$ .

**2.10.** The **realizability question** asks: *Can every subset of  $\text{Spec}^* \mathcal{S}$  be realized as  $V_R(M)$  for some  $M$  in  $D^f(R)$ ?* This has been studied in several contexts, see for example [Avr89b, Ber07, BW15, Car84, EHT<sup>+</sup>04, SFB97]. It was shown in [Ber07] that the realizability problem has a positive answer when  $R$  is complete intersection; see also [AI07]. In [Pol21], it was shown that if  $R$  is not complete intersection, then  $\{\mathbf{0}\}$  is never realizable as  $V_R(M)$  when  $M$  is a finitely generated  $R$ -module. However, it was asked in [Pol21] whether the solution to the realizability problem is positive regardless of whether  $R$  is complete intersection—in particular, whether  $\{\mathbf{0}\}$  can be realized by some object in  $D^f(R)$ .

Example 2.9 provides a class of rings for which the realizability problem has a negative answer in a rather drastic fashion. We also have the following for Cohen–Macaulay rings; it is an immediate consequence of Corollary 2.8.

**Corollary 2.11.** *If  $R$  is a Cohen–Macaulay ring, there exists  $M$  in  $D^f(R)$  with  $V_R(M) = \{\mathbf{0}\}$  if and only if  $R$  is complete intersection.*

*Remark 2.12.* In light of Corollary 2.8, a more sensible version of the realizability question is the following problem: *What subsets of  $\text{Spec}^* \mathcal{S}$  of codimension at most  $\text{LScat}(R)$  are realizable as  $V_R(M)$  for some  $M$  in  $D^f(R)$ ?*

*Remark 2.13.* When  $R$  is complete intersection the Koszul complex  $K^R$  belongs to every thick subcategory of  $D^f(R)$  by the work of Dwyer, Greenlees, and Iyengar [DGI06], and so there is a unique minimal thick subcategory (see also [Pol19, Theorem 5.2] for the converse). In contrast, when  $\{\mathbf{0}\}$  is not realizable (as in Corollary 2.11), the cohomological support varieties take values in  $\text{Spec}^* \mathcal{S} \setminus \{\mathbf{0}\} = \text{Proj} \mathcal{S}$ , and we note next that this implies that the lattice of thick subcategories of  $D^f(R)$  exhibits “nonlocal” behaviour.

**Proposition 2.14.** *If  $R$  is a local ring such that  $\{\mathbf{0}\}$  is not the cohomological support of any object in  $D^f(R)$  then there are infinitely many nonzero thick subcategories of  $D^f(R)$  having pairwise zero intersection.*

*Proof.* Given  $X \subseteq \text{Spec}^* \mathcal{S}$  we write  $\mathbb{T}_X$  for the full subcategory of  $D^f(R)$  containing those  $M$  with  $V_R(M) \subseteq X$ ; by 1.5.5 this is a thick subcategory of  $D^f(R)$ .

Pick any nonconstant homogeneous element  $\zeta_1 \in \mathcal{S}$  and let  $X_1$  be a minimal closed subset of  $\mathcal{V}(\zeta_1)$  such that  $\mathbb{T}_{X_1} \neq (0)$ . Such an  $X_1$  exists because  $L_{\zeta_1} \in \mathbb{T}_{\mathcal{V}(\zeta_1)}$  by Example 1.4.7, and  $X_1 \neq \{\mathbf{0}\}$  by hypothesis.

We continue to construct  $X_1, X_2, \dots$  inductively. Assume that  $X_1, \dots, X_n$  are all minimal with the property that  $\mathbb{T}_{X_i}$  are nonzero, and that  $\mathbb{T}_{X_i} \cap \mathbb{T}_{X_j} = (0)$  when  $i \neq j$ . It follows again that each  $X_i \neq \{\mathbf{0}\}$ , and therefore we may choose a nonconstant homogeneous  $\zeta_{n+1} \in \mathcal{S}$  such that  $X_i \not\subseteq \mathcal{V}(\zeta_{n+1})$  for all  $i \leq n$ , and we may choose a minimal  $X_{n+1} \subseteq \mathcal{V}(\zeta_{n+1})$  for which  $\mathbb{T}_{X_{n+1}} \neq (0)$ . To conclude, we note that  $\mathbb{T}_{X_{n+1}} \cap \mathbb{T}_{X_i} \subseteq \mathbb{T}_{X_i \cap X_{n+1}} = (0)$  for all  $i \leq n$ , since  $X_i \cap X_{n+1}$  is strictly contained in  $X_i$  by construction.  $\square$

We expect that the Cohen–Macaulay hypothesis in Corollary 2.11 is unnecessary, and therefore Proposition 2.14 should apply to all rings that are not complete intersection. In the case of Golod rings we can be more precise; see Corollary 4.4.

3. UPPER BOUNDS ON COHOMOLOGICAL SUPPORT

In this section we prove Theorem 3.1, connecting the cohomological support varieties with the homotopy Lie algebra. This result, paired with Theorem 2.7, has several applications that we provide after the proof of Theorem 3.1.

As in Section 1.1 and throughout,  $R$  is a local ring with a Cohen presentation  $\widehat{R} = Q/I$ , where  $I$  is minimally generated by  $\mathbf{f} = f_1, \dots, f_n$ , with the corresponding Koszul complex  $E$  and ring of cohomology operators  $\mathcal{S}$ .

We also use background material from Sections 1.4 and 1.7. In particular, in the notation of 1.7.2, there is a canonical isomorphism between the homotopy Lie algebra in degree 2 and the space of cohomology operators:

$$\pi^2(R) = \text{span}_k\{e_1^\vee, \dots, e_n^\vee\} \xrightarrow{\cong} \mathcal{S}^2 = \text{span}_k\{\chi_1, \dots, \chi_n\}, \quad e_i^\vee \mapsto \chi_i,$$

where, recall,  $(-)^\vee$  denotes  $k$ -space duality. In what follows, for an element

$$\alpha = a_1 e_1^\vee + \dots + a_n e_n^\vee \in \pi^2(R),$$

we will write  $\chi_\alpha := a_1 \chi_1 + \dots + a_n \chi_n$  for the corresponding element of  $\mathcal{S}^2$ .

In the next theorem, for a conical subset  $\mathcal{U}$  of  $\text{Spec}^* \mathcal{S}$ , we write  $\text{span}_k \mathcal{U}$  for the smallest closed subset containing  $\mathcal{U}$  defined by linear forms in  $\mathcal{S}$ . When  $k$  is algebraically closed, upon regarding  $\mathcal{U}$  as a cone in the affine  $n$ -space  $\mathbb{A}_k^n$ , then  $\text{span}_k \mathcal{U}$  is the  $k$ -subspace spanned by  $\mathcal{U}$ .

**Theorem 3.1.** *Under the isomorphism  $\pi^2(R) \xrightarrow{\cong} \mathcal{S}^2$ , radical elements of  $\pi^2(R)$  are sent to linear polynomials vanishing on the subvariety  $V_R(R) \subseteq \text{Spec}^* \mathcal{S}$ . In other words, the isomorphism restricts to an embedding*

$$\text{rad}^2(\pi^*(R)) \hookrightarrow \{\chi \in \mathcal{S}^2 \mid V_R(R) \subseteq \mathcal{V}(\chi)\}.$$

*In particular*

$$\text{rank}_k \text{rad}^2(\pi^*(R)) \leq \varepsilon_2(R) - \text{rank}_k(\text{span}_k V_R(R)).$$

*Proof.* Consider a radical element  $\alpha \in \pi^2(R)$ . By definition, we have an inclusion  $V_R(R) \subseteq \mathcal{V}(\chi_\alpha)$  exactly when  $\chi_\alpha$  acts nilpotently on  $\text{Ext}_E(\widehat{R}, k)$ ; cf. also 1.4.3.

We use the theory of semifree dg algebra resolutions from 1.6.3, taking  $Q[X]$  to be a minimal model for  $\widehat{R}$ , with  $E = Q[X_1]$ . We recall that  $\text{Ext}_E(\widehat{R}, k) \cong \text{Tor}^E(\widehat{R}, k)^\vee$ , see [Pol21, 4.3], and in what follows we will describe the action of  $\chi_\alpha$  in terms of this duality. As  $Q[X]$  is semifree as a dg  $E$ -module we have

$$\text{Ext}_E(\widehat{R}, k) \cong \text{H}(\text{Hom}_E(Q[X], k)) \cong \text{H}(\text{Hom}_k(Q[X] \otimes_E k, k)) \cong (\text{H}(k[X_{\geq 2}]))^\vee,$$

where  $k[X_{\geq 2}] := Q[X] \otimes_E k$ . It is therefore equivalent to show that  $\chi_\alpha^q \text{H}(k[X_{\geq 2}]) = 0$  for some  $q$ .

At this point we need to recall how the cohomology operator  $\chi_\alpha$  may be computed using divided powers; cf. [Avr10, Section 6] and [GL69, Chapter 1] for background on the latter. Starting from  $k[X] = Q[X] \otimes_Q k$  construct the extension  $k[X]\langle y_1, \dots, y_n \rangle$ , where the  $y_i$  are free divided power variables with  $\partial(y_i) = e_i$ . Using [Avr10, Proposition 6.1.7] factoring by the  $e_i$  and  $y_i$  yields a quasi-isomorphism

$$k[X]\langle y_1, \dots, y_n \rangle \xrightarrow{\cong} k[X_{\geq 2}].$$

For each  $i$  there is a unique derivation  $\frac{d}{dy_i}$  on  $k[X]\langle y_1, \dots, y_n \rangle$ , respecting the divided power structure, such that  $\frac{d}{dy_i}(y_i) = 1$ , while  $\frac{d}{dy_i}(y_j) = 0$  for  $i \neq j$  and

$\frac{d}{dy_i}(X) = 0$ ; see [Avr10, Section 6.2]. Writing

$$\frac{d}{dy_\alpha} = a_1 \frac{d}{dy_1} + \cdots + a_n \frac{d}{dy_n}$$

there is by [AB00a, Proposition 2.6] a commutative diagram

$$\begin{CD} \mathbf{H}_*(k[X]\langle y_1, \dots, y_n \rangle) @>\cong>> \mathbf{H}_*(k[X_{\geq 2}]) \\ @V\mathbf{H}_*(\frac{d}{dy_\alpha})VV @VV\chi_\alpha V \\ \mathbf{H}_{*-2}(k[X]\langle y_1, \dots, y_n \rangle) @>\cong>> \mathbf{H}_{*-2}(k[X_{\geq 2}]). \end{CD}$$

In other words,  $\frac{d}{dy_\alpha}$  induces the cohomology operator  $\chi_\alpha$  on  $\text{Tor}^E(\widehat{R}, k)$ .

We can go further and identify  $\chi_\alpha$  as a chain map on  $k[X_{\geq 2}]$ . To do this we use a construction from [Bri22, Construction 2.3]. This time on  $k[X]$  we consider for each  $i$  the derivation  $\frac{d}{de_i}$  for which  $\frac{d}{de_i}(e_i) = 1$ ,  $\frac{d}{de_i}(e_j) = 0$  when  $i \neq j$ , and  $\frac{d}{de_i}(x) = 0$  if  $x \in X_{\geq 2}$ , and we take the linear combination

$$\frac{d}{de_\alpha} = a_1 \frac{d}{de_1} + \cdots + a_n \frac{d}{de_n}.$$

The derivation  $\frac{d}{de_\alpha}$  may not be a chain map, and we take its boundary  $[\partial, \frac{d}{de_\alpha}] = \partial \frac{d}{de_\alpha} + \frac{d}{de_\alpha} \partial$ . For degree reasons  $[\partial, \frac{d}{de_\alpha}](X_1) = 0$ , and it follows that  $[\partial, \frac{d}{de_\alpha}]$  induces a derivation on  $k[X]/(X_1) = k[X_{\geq 2}]$ , that we denote  $\theta_\alpha: k[X_{\geq 2}] \rightarrow k[X_{\geq 2}]$ .

We can extend  $\frac{d}{de_\alpha}$  to a derivation  $\frac{d}{de_\alpha}'$  on  $k[X]\langle y_1, \dots, y_n \rangle$  by setting

$$\frac{d}{de_\alpha}'(y_i^{(p)}) = 0$$

for each  $i$  and  $p$ . With all of this notation in place, we consider the diagram

$$\begin{CD} k[X]\langle y_1, \dots, y_n \rangle @>\cong>> k[X_{\geq 2}] \\ @V[\partial, \frac{d}{de_\alpha}'] + \frac{d}{dy_\alpha}VV @VV\theta_\alpha V \\ k[X]\langle y_1, \dots, y_n \rangle @>\cong>> k[X_{\geq 2}]. \end{CD}$$

A direct computation, by checking on the generators, shows that the diagram is commutative. It follows that on  $\mathbf{H}(k[X_{\geq 2}])$  we have

$$\chi_\alpha = \mathbf{H}\left(\frac{d}{dy_\alpha}\right) = \mathbf{H}\left([\partial, \frac{d}{de_\alpha}'] + \frac{d}{dy_\alpha}\right) = \mathbf{H}(\theta_\alpha).$$

Returning to the homotopy Lie algebra, the definition of  $\pi^*(R)$  implies that, for  $i \geq 3$ , there is a canonical isomorphism

$$\pi^i(R) \cong ((X_{\geq 2})/(X_{\geq 2})^2)_{i-1}^\vee.$$

At the same time, the derivation  $\theta_\alpha$  induces a map

$$\bar{\theta}_\alpha: ((X_{\geq 2})/(X_{\geq 2})^2)_{i+1} \longrightarrow ((X_{\geq 2})/(X_{\geq 2})^2)_{i-1}.$$



And the statement of [Bri22, Proposition 2.4] is that the following diagram is commutative

$$\begin{CD} \pi^{i+2}(R) @<-\text{ad}(\alpha)<< \pi^i(R) \\ @V \cong VV @VV \cong V \\ ((X_{\geq 2})/(X_{\geq 2})^2)_{i+1}^\vee @<-\bar{\theta}_\alpha^\vee<< ((X_{\geq 2})/(X_{\geq 2})^2)_{i-1}^\vee \end{CD}$$

compare also the proof of [AH87, Proposition 4.2]. Therefore, if  $\text{ad}(\alpha)^p = 0$  then as well  $\bar{\theta}_\alpha^p = 0$ , or in other words  $\theta_\alpha^p(X_{\geq 2}) \subseteq (X_{\geq 2})^2$ . Iterating this, there is for any  $m$  some number  $q$  such that  $\theta_\alpha^q(X_{\geq 2}) \subseteq (X_{\geq 2})^m$ .

Now by 1.6.5 and 1.6.7 there is an  $m$  such that the inclusion  $(X_{\geq 2})^m \hookrightarrow (X_{\geq 2})$  is a nullhomotopic chain map. Taking the corresponding  $q$ , the chain map

$$\theta_\alpha^q: k[X_{\geq 2}] \rightarrow k[X_{\geq 2}]$$

factors through this inclusion, and is therefore also nullhomotopic. In particular,  $\chi_\alpha^q = H(\theta_\alpha^q) = 0$  on  $H(k[X_{\geq 2}])$ , and this concludes the proof.  $\square$

We now provide several applications of Theorems 2.7 and 3.1. The first is Theorem C from the introduction.

**Theorem 3.2.** *Let  $M$  be a nonzero perfect complex over  $R$ . Then*

$$\sum \ell_R(H_n(M)) \geq \text{level}_R^k(M) \geq \text{codim } V_R(R) + 1 \geq \text{rank}_k \text{rad}^2(\pi^*(R)) + 1.$$

*Proof.* The first inequality is standard; see [ABIM10, Theorem 6.2]. For the remaining inequalities, applying  $-\otimes_R K^R$  yields the inequality

$$(5) \quad \text{level}_R^k(M) \geq \text{level}_{K^R}^k(K^M);$$

cf. 1.5.6. Also, from Theorem 2.7 we have the first inequality

$$(6) \quad \text{level}_{K^R}^k(K^M) \geq \varepsilon_2(R) - \dim V_R(M) + 1 = \varepsilon_2(R) - \dim V_R(R) + 1;$$

for the equality note that  $M$  is a nonzero perfect complex and so from [Pol19, 3.3.2] we have that  $V_R(M) = V_R(R)$ . Now applying Theorem 3.1 and combining the inequalities from (5) and (6) we obtain the desired result.  $\square$

*Remark 3.3.* Theorem 3.2 recovers and improves the bound from the main theorem in [ABIM10] which replaces the right-hand side in the inequality above with the conormal free rank of  $R$ . The **conormal free rank** of  $R$ , denoted  $\text{cf-rank}(R)$ , is the largest free rank of a conormal module  $I/I^2$  corresponding to a minimal Cohen presentation  $Q \rightarrow \hat{R} \cong Q/I$ . To see that Theorem 3.2 recovers [ABIM10, Theorem 3], we only need to recall the fact that  $\text{rank}_k \text{rad}^2(\pi^*(R)) \geq \text{cf-rank}(R)$  from [Iye01]. Furthermore, whenever  $V_R(R)$  is not a linear space, the bound in Theorem 3.2 is strictly stronger than the one in [ABIM10, Theorem 3] as

$$\text{codim } V_R(R) > \text{codim span}_k V_R(R) \geq \text{cf-rank}(R).$$

Finally, as noted in [ABIM10], this also recovers a theorem from homotopy theory [Car83, AP93]: *For a prime  $p > 0$ , if  $E$  is an elementary abelian  $p$ -group of rank  $r$  that acts freely and cellularly on a finite CW-complex  $X$ , then*

$$\sum \ell_{\mathbb{F}_p E}(H_n(X; \mathbb{F}_p)) \geq r + 1.$$

Together with the characterization of complete intersection rings in terms of  $V_R(R)$  [Pol19], Theorem 3.1 gives a conceptual generalization of the following result of Avramov and Halperin [AH87, Theorem C], which originally was used to prove that a conjecture of Quillen on the cotangent complex holds in characteristic zero [Qui70]. The same result is also an ingredient in the recent proof of Vasconcelos' Conjecture on the conormal module [Bri22], as well as a recent new proof of Quillen's Conjecture in all characteristics [BI23], that had originally been settled by Avramov [Avr99].

**Corollary 3.4.** *The ring  $R$  is complete intersection if and only if every element of  $\pi^2(R)$  is radical.*

*Proof.* If  $R$  is complete intersection then  $\pi^i(R) = 0$  for  $i \geq 3$ , so every element of  $\pi^2(R)$  is central, and in particular radical. If  $\text{rad}^2(\pi^*(R)) = \pi^2(R)$  then by Theorem 3.1 we have  $V_R(R) = \{\mathbf{0}\}$ . By [Pol19, Theorem 3.3.2] this implies that  $R$  is complete intersection.  $\square$

*Remark 3.5.* For accessibility, we have treated here the *absolute* case, concerning the homotopy Lie algebra and cohomological supports defined over a local ring. The discussion generalizes almost verbatim to the *relative* case, beginning with a surjective local map  $\varphi: Q \rightarrow R$  (with  $Q$  not necessarily regular). As long as  $\varphi$  has finite projective dimension, the same proof shows that Theorem 3.1 holds for the relative homotopy Lie algebra  $\pi^*(\varphi)$  (cf. [Bri18]) and the relative support variety  $V_\varphi(R)$  (cf. [AI07, Pol21]). We note that Avramov and Halperin's result Corollary 3.4 was stated in this greater generality in [AH87], and since [Pol19, Theorem 3.3.2] applies in this more general setting we also recover the general statement in [AH87].

*Remark 3.6.* Given an embedded deformation  $R = S/(f)$  (see Example 1.4.8 for a definition), the element  $f$  gives rise to a free summand of the conormal module of  $R$ , and also to a central element in  $\pi^2(R)$  and a hyperplane containing  $V_R(R)$ ; cf. [Iye01] and [Pol21, Proposition 5.3.7], respectively. In [BGP22], the ring  $R$  is said to have **spanning support** if  $V_R(R)$  generates the whole affine space, as a vector space. It follows that if  $R$  has spanning support then  $\text{cf-rank}(R) = 0$  and  $R$  does not have an embedded deformation. Determining whether a given ring has an embedded deformation is in general a difficult question.

Avramov [Avr89a, Problem 4.3] asked if every central element in  $\pi^2(R)$  arises from an embedded deformation of  $R$ . This has been answered affirmatively in some specific cases in [Avr89a] and [Löf94]. Iyengar had previously shown that free summands of  $I/I^2$  give rise to central elements of  $\pi^2(R)$  [Iye01], and Johnson has recently shown that if  $I$  is a licci ideal, then  $R$  has an embedded deformation if and only if  $I/I^2$  has a free summand [Joh22]. However, Dupont has shown that there is a (nonstandard) graded ring  $R$  with a central element in  $\pi^2(R)$  that does *not* admit any embedded deformation compatible with the grading [Dup97]; the ungraded analogue remains open.

*Remark 3.7.* Theorem 3.1 says that each radical element in  $\pi^2(R)$  defines a hyperplane containing  $V_R(R)$ , and therefore to solve [Avr89a, Problem 4.3] it is sufficient to show that every hyperplane containing  $V_R(R)$  gives rise to an embedded deformation of  $R$ ; whether this holds was asked by the third author in [Pol21, Question 6.3.8]. Answering Avramov's problem using this strategy would also show that elements of  $\text{rad}^2(\pi^*(R))$  are in fact central.

Moreover, from a computational and experimental aspect,  $V_R(R)$  is easier to compute than  $\text{rad}^2(\pi^*(R))$ . Indeed, when  $R$  is not complete intersection  $\pi^*(R)$  is an infinite dimensional graded  $k$ -space having exponential growth, and testing whether an element is radical is not realistic in examples. On the other hand, to calculate  $V_R(R)$  one needs only a system of higher homotopies on a minimal resolution of  $\widehat{R}$  over  $Q$  [Eis80, Section 7], and this amounts to computing a finite number of matrices.

*Remark 3.8.* We expect the embedding in Theorem 3.1 to be an isomorphism. That is, when  $k$  is algebraically closed, we expect an equality

$$(7) \quad \text{rank}_k(\text{rad}^2(\pi^*(R))) = \varepsilon_2(R) - \text{rank}_k(\text{span}_k V_R(R)).$$

Such a result would make the connection between Avramov’s question [Avr89a, Problem 4.3] and the third author’s question [Pol21, Question 6.3.8] even stronger: Together, (7) and [Pol21, Question 6.3.8] amount to the statement that  $R$  has an embedded deformation if and only if  $\text{rad}^2(\pi^*(R))$  is nonzero. This would strengthen [Avr89a, Problem 4.3] from central elements to radical elements.

*Remark 3.9.* Notice that in the proof of Theorem 3.2 the following inequality was established:

$$\text{level}_{KR}^k(K^M) \geq \text{rank}_k(\text{rad}^2(\pi^*(R))).$$

Combining this with Lemma 2.6 yields

$$\text{LScat}(R) \geq \text{rank}_k \text{rad}^2(\pi^*(R)).$$

Finally, we note that the methods above allow us to recover, by a new argument, one of the results of the famous *Five Author Paper* [FHJ+88, Theorem A]:

**Corollary 3.10.** *For any local ring  $R$ ,*

$$\text{codepth}(R) \geq \text{rank}_k(\text{rad}^2(\pi^*(R))).$$

*Proof.* Apply Theorem 3.2 when  $M$  is the Koszul complex on a minimal generating set for the maximal ideal of  $R$ ; the inequality follows from the fact that in that case  $\text{level}_R^k(M) = \text{codepth}(R)$ . This is simply a consequence of the rigidity of the Koszul complex [ABIM10, Example 6.7].  $\square$

#### 4. THE REALIZABILITY PROBLEM FOR GOLOD RINGS

For a local ring  $R$  with residue field  $k$ , there is the following coefficient-wise inequality on the Poincaré series  $P_k^R(t)$  of  $k$  due to Serre:

$$P_k^R(t) \preccurlyeq \frac{(1+t)^{\varepsilon_1(R)}}{1-t \sum_{i=1}^{\text{codepth } R} \text{rank}_k H_i(KR)t^i};$$

see for example [Avr10, Proposition 4.1.4]. We say  $R$  is **Golod** if equality holds. By definition these rings possess extremal homological behavior; see [Avr86] or [Avr10, Section 5]. There are two distinct cases: first, when the codepth of  $R$  is at most one (and hence  $R$  is a hypersurface ring), then  $R$  is always Golod and has easy to understand asymptotic properties; second, when  $R$  is Golod with codepth at least two, resolutions of modules typically have exponential growth.

In this section we show that the extremal behavior of Golod rings with codepth at least two is also reflected in the limitations on the cohomological supports that are realizable, that in turn, also puts restrictions on the lattice of thick subcategories over such rings.

**Theorem 4.1.** *If a local ring  $R$  is Golod with  $\text{codepth } R \geq 2$ , then  $R$  has full support  $V_R(R) = \text{Spec}^* \mathcal{S}$ . Moreover, the cohomological support of any nonzero object  $M$  of  $D^f(R)$  satisfies the inequality  $\text{codim } V_R(M) \leq 1$ , and every closed cone having codimension at most one is the cohomological support variety of some object in  $D^f(R)$ .*

*Proof.* By [Bri18, Proposition 9], when  $R$  is Golod we have  $\text{LScat}(R) = 1$ . Hence, by Theorem 2.7 we conclude that  $\text{codim } V_R(M) \leq 1$  whenever  $M$  is a nonzero object in  $D^f(R)$ . The realizability assertion follows from Example 1.4.7.

So it remains to show  $V_R(R) = \text{Spec}^* \mathcal{S}$ . To this end, using the notation from Construction 2.1, since  $R$  is Golod

$$t(K^R) \simeq k \rtimes V$$

as a  $\Lambda$ -module, where  $V = \{V_i\}_{i \geq 1}$  is a graded  $k$ -vector space and where the  $\Lambda$ -action on  $k \rtimes V$  is trivial:

$$\Lambda_1 \cdot k = V_1 \quad \text{and} \quad \Lambda_1 \cdot V = 0;$$

see [Avr86, Theorem 4.6]. Since  $R$  is not a hypersurface ring  $V_2 \neq 0$  and so  $k$  is a summand of  $k \rtimes V$  over  $\Lambda$ . Therefore,

$$\text{Supp}_{\mathcal{S}}(\text{ht}K^R) = \text{Supp}_{\mathcal{S}}(\text{hk}).$$

Finally, since  $\text{hk} \simeq \mathcal{S}$  the desired results follow from Proposition 2.3. □

*Remark 4.2.* The fact that  $V_R(R) = \text{Spec}^* \mathcal{S}$  when  $R$  is a nonhypersurface Golod ring can also be established using the structure of a system of higher homotopies for  $\widehat{R}$  over  $Q$ . This makes use of the  $A_\infty$ -structure on a minimal  $Q$ -free resolution of  $\widehat{R}$  and will be explained in future work.

*Remark 4.3.* As described in Remark 3.6, Theorem 4.1 implies that if  $R$  is a nonhypersurface Golod ring, then the conormal free rank of  $R$  is zero, and in particular  $R$  does not have an embedded deformation. Notice that the fact that  $R$  has no embedded deformation also follows from the fact that  $\pi^{\geq 2}(R)$  is free when  $R$  is Golod, using [Avr86, Theorem 4.6].

Our results show that the bounded derived category of a Golod local ring looks markedly different to that of a complete intersection ring; compare with Remark 2.13 and Proposition 2.14, as well as [EL22, Theorems A & B].

**Corollary 4.4.** *Let  $R$  be a Golod local ring with  $\text{codepth } R \geq 2$ .*

- (1) *For each  $\zeta \in \mathcal{S}$  of positive degree,  $L_\zeta$  is not proxy small.*
- (2) *If  $\zeta, \zeta'$  are in  $\mathcal{S}^{>0}$  with  $\text{gcd}(\zeta, \zeta') = 1$ , then*

$$\text{thick}_R L_\zeta \cap \text{thick}_R L_{\zeta'} = 0.$$

*Proof.* By [Pol19, 3.3.2], if  $L_\zeta$  is proxy small then

$$V_R(R) \subseteq V_R(L_\zeta) = \mathcal{V}(\zeta);$$

the second equality uses Example 1.4.7. However this contradicts Theorem 4.1, and justifies (1).

For (2), if  $M$  is in  $\text{thick}_R L_\zeta \cap \text{thick}_R L_{\zeta'}$ , then

$$V_R(M) \subseteq V_R(\zeta) \cap V_R(\zeta') = \mathcal{V}(\zeta) \cap \mathcal{V}(\zeta') = \mathcal{V}(\zeta, \zeta');$$

the first containment is again [Pol19, 3.3.2], while the next equality follows from Example 1.4.7. On the other hand, since  $\zeta, \zeta'$  form a regular sequence,  $\mathcal{V}(\zeta, \zeta')$

must have codimension exactly 2. Therefore,  $V_R(M)$  would also have codimension at least 2, which contradicts the first part of the theorem.  $\square$

We end with an example of a class of Golod rings where one can explicitly realize all codimension one cones using modules, generalizing [Pol21, Example 6.4.3].

**Example 4.5.** Let  $R$  be a codepth two noncomplete intersection ring, and assume  $R$  that is complete and  $k$  is algebraically closed. In this case,  $R \cong Q/I$  where  $(Q, \mathfrak{m})$  is a regular local ring, and  $I = (xy, yz)$ , with  $x, z$  a regular sequence and  $y$  a nonzero element of  $\mathfrak{m}$ .

For each  $(p, q) \in \mathbb{A}_k^2 \setminus \{(0, 0)\}$  we consider the  $R$ -module

$$M_{(p,q)} := R/(px + qz).$$

If  $p \neq 0$  then  $M_{(p,q)} \cong Q/(px + qz, yz)$ , and if  $p = 0$  then  $M_{(0,b)} \cong Q/(xy, z)$ . In either case  $M_{(p,q)}$  is a complete intersection ring, so setting  $J_{(p,q)} = (xy, yz, px + qy)$ , by Example 1.4.9, we obtain

$$V_R(M_{(p,q)}) = \ker(I/\mathfrak{m}I \rightarrow J_{(p,q)}/\mathfrak{m}J_{(p,q)}) = \{(a, b) \in \mathbb{A}_k^2 \mid qa = pb\}.$$

Note that any codimension one cone in  $\mathbb{A}_k^2$  is a finite union of lines through the origin. Given such a variety

$$V = \bigcup_{i=1}^n \{(a, b) \in \mathbb{A}_k^2 \mid q_i a = p b_i\},$$

we obtain  $V_R(M) = V$  by setting  $M = \bigoplus_{i=1}^n M_{(p_i, q_i)}$  and making use of [Pol21, Proposition 5.1.2].

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