With this brief and imperfect account we must now regretfully leave the subject, consoling ourselves with the reflection that Dr. Fricke's book contains in itself that which will most certainly attract deserved attention to this most beautiful of Klein's creations.

F. N. COLE.

ANN ARBOR, *December* 30, 1891.

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## PERTURBATIONS OF THE FOUR INNER PLANETS.

*Periodic Perturbations of the Longitudes and Radii Vectores of the Four Inner Planets of the First Order as to the Masses.* Computed under the direction of SIMON NEWCOMB. Washington, Navy Department, 1891; 4to, pp. 180.

THIS work forms the concluding part of volume III. of a series of astronomical researches, published under the general title, "*Astronomical Papers, prepared for the use of the American Ephemeris and Nautical Almanac.*"

During the past twelve years, one of the principal works which has been in progress at the office of the Nautical Almanac is that of collecting and discussing data for new tables of the planets. The most recent existing tables, which are now used in all European Ephemerides, are those of Leverrier, the construction of which was the greatest work ever undertaken by that celebrated astronomer. The first tables published, those of the Sun, were issued in 1858; those of Uranus and Neptune appeared about 18 years later. The whole work probably took about 25 years in preparation and publication. Yet the number of observations on which the tables were actually based was only a few hundred in the case of each planet, about 500 being used for Venus, 800 for Mars, and probably yet fewer in the cases of the other planets. The results were not completely discussed, and, in consequence, different data were employed in different tables, making it extremely difficult for future astronomers to derive the results of comparing them with future observations. None except those of the Sun and Mercury, which were the first issued, have shown a satisfactory agreement with subsequent observations. The error in the geocentric place of Venus at the time of the recent transit was surprisingly great, amounting to no less than nine seconds in longitude.

The actual number of observations now available for each of the principal planets is several thousand. The recent ones

are of course better than those available thirty years ago. It therefore seemed desirable to undertake the construction of tables founded on all these observations which could be of value, and on uniform values of the masses of the planets and other elements.

As it was necessary to determine the masses from the periodic perturbations, the first requisite was a determination of the coefficients of these perturbations which should be beyond doubt. Although Leverrier's computations of these coefficients were carried out more fully than those of any of his predecessors, some doubts of their entire accuracy had been expressed. In such intricate computations, which necessarily proceed by successive approximations, and can never pretend to mathematical rigor, the possibility of sensible quantities being omitted can be avoided only by independent computations by different investigators using different methods. The present paper is entirely devoted to the computation of these coefficients. The adopted developments are so radically different from those of Leverrier that there can be no source of error common to the two. The agreement throughout may be called perfect, when compared with the probable error of the best observations. Rarely does a discrepancy amount to the hundredth of a second of arc.

The principal point in which the development differed from that of Leverrier is, that the eccentric anomaly is used, in the beginning, as the independent variable. In this way the series are made, in the first place, more rapidly convergent, and it is thus more easy to be sure of including all sensible terms. The use of this method requires, however, that the eccentric anomalies be changed to mean anomalies by the Besselian transformation. It was supposed that this transformation was one which could be effected with ease and rapidity. But in practice it proved so laborious that it is now doubtful whether the terms saved in the development will compensate for the labor of applying it.

A more radical change from Leverrier's method is, that the perturbations are computed by direct integration of the differential equations of motion, instead of employing the method of variation of elements. Notwithstanding the theoretical elegance of the latter method as developed by Lagrange, it becomes excessively prolix when we attempt to compute the periodic perturbations by it. But when the equations are directly integrated, the coefficients admit of being found with great facility, when once the development of two derivatives of the perturbative function in terms of the mean anomalies is effected. Altogether the method is a combination of the purely numerical process of development employed by Hansen, and the purely analytic one employed by Leverrier.

It is still a question whether the adopted method was actually the shortest, and whether much labor would not have been saved by employing the purely numerical development from the beginning.

The volume of which the above paper forms a part is wholly devoted to the developments of celestial mechanics. The opening paper is the development of the perturbative function in sines and co-sines of multiples of the eccentric anomaly which was employed in computing the perturbations.

This is followed by a determination of the inequalities of the Moon's motion due to the figure of the Earth, prepared by G. W. Hill. This is the most elaborate determination of these difficult inequalities that has ever been made, no less than 165 terms in the Moon's longitude, and yet more in the latitude, being computed. Nearly half the computed terms are, however, entirely insensible, even in the fourth place of decimals of seconds.

The third paper is on the motion of Hyperion, the seventh satellite of Saturn. In it is developed the theory of the curious relation between the mean motions of Hyperion and Titan, which H. Struve has since extended to one or more of the inner satellites.

This is followed by another paper by Mr. Hill, being a computation of certain lunar inequalities due to the action of Jupiter. The inequality in question was first discovered empirically from observations, and was traced by Mr. Neison to a sort of evection due to the action of Jupiter. Mr. Hill's coefficient is, however, only $0''.90$, while observations gave $1''.50$. Probably the theory is more nearly correct, as the uncertainty of observations of the Moon is much greater than in the case of other heavenly bodies, and it is difficult to separate the effects of an inequality of this kind from those of numerous other causes affecting the observations.

It is now expected that the tables of the four inner planets which are founded on the theories developed in the Astronomical Papers, and on the great mass of observations made since 1750, will be ready for the press in a little more than two years.

S. N.