ON THE DEFINITION OF LOGARITHMS.

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AFTER reading Professor Stringham's interesting paper* on "A Classification of Logarithmic Systems," and at his suggestion, I undertook to examine his definitions from the point of view of the theory of functions. The results of my investigation are embodied in the following paper. If I have taken the liberty of deriving some well-known formulæ, I hope that at least the point of view will prove interesting.

Starting from Riemann's definition of the logarithm as that function $\phi(z)$ which satisfies the equation

$$\phi(uz) = \phi(u) + \phi(z), \tag{1}$$

we can easily find an expression for the derivative of the logarithm in terms of the derivative of the independent variable as follows: † Differentiating equation (1) on the assumption that z is constant, we have

$$z\phi'(uz) = \phi'(u);$$

whence, if we write $u = 1$, it follows that

$$z\phi'(z) = \phi'(1).$$

The value of $\phi'(1)$ can be chosen at will and is characteristic for the system of logarithms under consideration. It is called the *modulus* of the system, and we shall denote it by $M = m \text{ cis } \beta$.

Let us write $\phi(z) = w$, and denote time-derivates by dots. The last equation obtained can then be written

$$\dot{w} = M\frac{\dot{z}}{z}. \tag{2}$$

Writing $z = r \text{ cis } \theta$, we find by differentiation

$$\dot{z} = \dot{r} \text{ cis } \theta + r(-\sin\theta + i\cos\theta)\dot{\theta} = (\dot{r} + ir\dot{\theta}) \text{ cis } \theta$$
$$= \left(\frac{\dot{r}}{r} + i\dot{\theta}\right)z, \tag{3}$$

whence follows immediately

$$\dot{w} = M\frac{\dot{z}}{z} = M\left(\frac{\dot{r}}{r} + i\dot{\theta}\right). \tag{4}$$

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* *American Journal of Mathematics*, vol. 14, p. 187.

† See, for example, Durège, "Theorie der Funktionen," 3. Aufl.

Representing values of z in the usual way by points in a plane with polar coördinates r and θ, and the corresponding values of w by points in a second plane, if a point P in the z-plane move along any given curve, the corresponding point Q in the w-plane will likewise move along a curve, and the relative velocities of the two moving points can be computed from equations (3) and (4). The character of the two curves will evidently depend entirely upon the quantities $\dot{r}/r$ and $\dot{\theta}$, to which can be assigned any values, constant or variable, that we choose. The simplest case is obviously that in which $\dot{r}/r$ and $\dot{\theta}$ are regarded as constant, and we shall find that this case is exactly the one of most importance for us.

Let $\dot{r}/r = \lambda$ and $\dot{\theta} = \omega$, to bring our notation into conformity with that of Professor Stringham. Then formulæ (3) and (4) become respectively

$$\dot{z} = (\lambda + i\omega)z, \tag{5}$$

$$\dot{w} = M(\lambda + i\omega), \tag{6}$$

and the movement of the points in the two planes may be described as follows: The velocity of P (representing z) is made up of two components, one along the straight line OP joining the origin to P, the other perpendicular to OP, each of them being in magnitude proportional to the length OP. In other words, P moves along OP with velocity proportional to its distance from the origin O, while at the same time OP revolves about O with constant angular velocity ω; that is to say, P traverses a logarithmic spiral with velocity proportional to the radius vector. The velocity of Q (representing w), on the other hand, is constant in amount and direction, and Q will therefore describe a straight line.

It remains for us to determine more closely the relation between the spiral and the corresponding straight line. The quantity $(\lambda + i\omega)$ is absolutely arbitrary, but occurs as a factor in both $\dot{z}$ and $\dot{w}$ and therefore does not affect the *relative* velocity of P and Q. Let us put

$$\lambda + i\omega = \mu \text{ cis } \psi,$$

then

$$\dot{w} = m\mu \text{ cis } (\beta + \psi) = m\mu \text{ cis } \phi.$$

The point Q then moves always upon a line making an angle ϕ with the axis of real quantities, while, as for the logarithmic spiral, the constant angle which it makes with any radius vector is the angle ψ,—the arbitrary angle $\phi - \beta = \tan^{-1} \lambda/\omega$ of Professor Stringham's paper. The position of the spiral will be completely determined as soon as we fix any one point upon it,—for example, the point at which it crosses the unit circle.

Let us first see what straight line corresponds to the spiral through the point where the axis of real quantities cuts the unit circle. For this point $z = 1$. If in the equation of definition (1) we put $z = 1$, we find

$$\phi(u) = \phi(u) + \phi(1).$$

One value of the logarithm of 1 is therefore 0, and the corresponding straight line in the w-plane is the line *through the origin* which makes an angle ϕ with the axis of real quantities.

In Professor Stringham's paper he obtains a spiral through any given point of the plane by assigning suitable values to the ratio $\lambda : \omega$, letting all the spirals pass through the same point ($z = 1$) of the unit circle. I prefer, however, to retain the same value of this ratio $\lambda : \omega$ and allow the point of intersection with the unit circle to vary. One spiral of this kind will pass through any given point of the plane, and spirals chosen in this way have the obvious advantage that they intersect only at the points 0 and ∞.

We can determine the value of w corresponding to any point of the unit circle by treating the unit circle as a special case of the logarithmic spiral, viz., when $\lambda = 0$. The equations for the velocity of P and Q then become

$$\dot{z} = \omega i z, \qquad \dot{w} = m\omega \operatorname{cis}\left(\frac{\pi}{2} + \beta\right)$$

P then describes the unit circle with uniform angular velocity ω, while Q moves with constant velocity $m\omega$ along the straight line through the origin which makes an angle $\left(\frac{\pi}{2} + \beta\right)$ with the axis of real quantities. The relation between z and w is then evidently this: when $z = \operatorname{cis}\theta$,

$$w = m\theta \operatorname{cis}\left(\frac{\pi}{2} + \beta\right) = \mu i\theta.$$

Now, as P continues to revolve and z passes over and over again through the same values, Q moves continually forward and we obtain an infinite number of values of w corresponding to each value of z. The complete correspondence is given by writing

$$z = \text{cis}\,\theta,$$

$$w = m(\theta + 2n\pi)\,\text{cis}\left(\frac{\pi}{2} + \beta\right)$$

$$= Mi(\theta + 2n\pi),$$

where n is any integer, positive, negative, or zero.

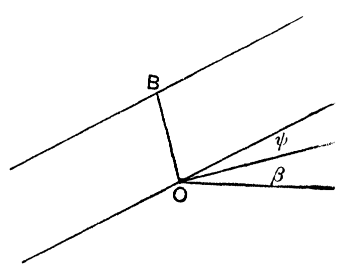
Let us now return to the family of spirals with the same angle ψ, and determine the corresponding lines of the w-plane. To the spiral through the point for which $z = \text{cis}\,\theta$ corresponds the line through the corresponding w. But as this w has an infinite number of values, so to each spiral correspond an infinite number of *parallel* straight lines through the points for which

$$w = Mi(\theta + 2n\pi).$$

In other words, to every z correspond an infinite number of values of w, which differ from each other only by multiples of $2\pi Mi$. This quantity is then the *modulus of periodicity* of the logarithmic function. It reduces to the familiar form $2\pi i$, when $M = 1$.

A word with regard to the conformable representation. It is evident that we shall obtain a complete representation of the z-plane upon a stripe of the w-plane bounded by any two consecutive lines corresponding to any one of the spirals. Furthermore, as the slope of these lines depends on the angle of the spirals and that is arbitrary, the slope may be anything we like, but the distance between corresponding points of the two boundaries is fixed, since it depends on the modulus alone.

Fig. 3.



Finally, it is worth noticing that if we put $\omega = 0$, the spirals all become straight lines through the origin, the angle $\psi = 0$, the slope of the straight lines in the w-plane reduces to β, and Napier's definition of the logarithm only requires to be modified by fixing the slope of the lines and the distance of one of them from the origin.

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