NOTE ON THE DEFINITIONS OF LOGARITHM AND EXPONENTIAL.

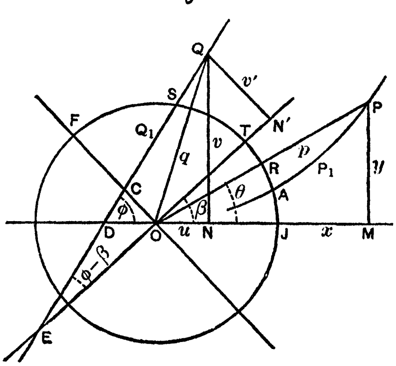
BY PROF. IRVING STRINGHAM.

PROFESSOR HASKELL'S interesting and important investigation calls for a modification of my former definitions of the logarithmic and exponential functions.* I propose that the new specifications shall be as follows:

In a circle whose radius is unity OT is fixed and makes an angle β with the real axis, OR turns about O with a constant speed, Q moves along any line ES in the plane with a constant speed, P along OR with a speed proportional to its distance from O.

Following the notation of my former paper, let the velocities of P, Q, R be:

Fig. 4.



$$
\begin{aligned}
P \text{ in } OR \text{ at } A &= \lambda, \\
Q \text{ in } ES &= \mu, \\
R \text{ in } JRS &= \omega;
\end{aligned}
$$

and let arc $JR = \theta$,

circular measure of $ODS = \phi$,

$$
\begin{aligned}
OP, OQ &= p, q, \\
ON, NQ &= u, v, \\
ON', N'Q &= u', v', \\
OM, MP &= x, y.
\end{aligned}
$$

In all logarithmic systems the relation

$$\frac{\omega}{\lambda} = \tan(\phi - \beta)$$

is assumed to exist. This fixes the ratio of the radial to the transversal velocity of P and determines the *form* of the curve upon which P moves. Its *position* may then be determined by fixing two points upon it, and for this purpose we may assume that P crosses the real axis OJ at the instant Q crosses the line ET, and that P crosses the circumference of the unit circle at the instant Q crosses the line OF which is drawn through the origin perpendicular to ET. As a consequence of these two assumptions, when ES passes through the origin, A coincides with J, and the figure becomes identical with that of my former paper (*loc. cit.* p. 187), which

* *American Journal Mathematics*, vol. 14, p. 187.

thus appears as a special case of the one here presented. Here also the property that $OP = 1$, when $OQ = 0$, is retained.

Referred to this more general figure my former definitions of modulus, base, logarithm, and exponential (*loc. cit.* p. 188) may now stand without any modification whatever. In somewhat changed phraseology they are:

The modulus is

$$\kappa \equiv \frac{\mu}{\sqrt{\lambda^2 + \omega^2}} (\cos \beta + i \sin \beta)$$

$$= \frac{\mu}{\lambda} \cos (\phi - \beta) (\cos \beta + i \sin \beta)$$

and fixes the system as soon as β has been assigned.

The base is the value that OP, or $x + iy$, assumes at the instant when OQ becomes 1, that is, when Q passes through the point J as it moves along some line that intersects the unit circle at J. Let this base be represented by B.

The ratio OP/OP_1 is the exponential, with respect to base B, of the difference $OQ - OQ_1$:

$$(x + iy)/(x_1 + iy_1) = B^{(u + iv) - (u_1 + iv_1)},$$

or with respect to modulus κ:

$$(x + iy)/(x_1 + iy_1) = \exp_\kappa [(u + iv) - (u_1 + iv_1)].$$

In particular, since $x + iy = 1$ when $u + iv = 0$,

$$x + iy = \exp_\kappa (u + iv).$$

Inversely, the difference $OQ - OQ_1$ is the logarithm, with respect to the same modulus or base, of the ratio OP/OP_1:

$$(u + iv) - (u_1 + iv_1) = \log_\kappa (x + iy)/(x_1 + iy_1),$$

or

$$(u + iv) - (u_1 + iv_1) = {}^B\log(x + iy)/(x_1 + iy_1),\text{*}$$

and in particular

$$u + iv = \log_\kappa(x + iy)$$

$$= {}^B\log(x + iy).$$

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* It is convenient to have the means of expressing a logarithm both with respect to its base and its modulus. There are good reasons for preferring the German notation, as here indicated, in the former case, reserving the subscript notation to indicate the modulus; and there is now good usage in English for the German practice, in Cathcart's translation of Harnack's *Differential u. Integral Rechnung*.

From these definitions the laws of operation for logarithms and exponentials and the generalized Eulerian formula,

$$B^{u+iv} = |B|^{u\cos\beta + v\sin\beta}\, e^{\frac{i}{m}(v\cos\beta - u\sin\beta)}, \qquad (\textit{loc. cit.}, \text{ p. } 192)$$

are easily deduced, only slight modifications of my former proofs being necessary.

The position of the point A is determined as follows: Since by definition $\omega = \lambda \tan(\phi - \beta)$

and $$m = \mu/\lambda \,.\, \cos(\phi - \beta),$$

$$\therefore \quad m\omega = \mu \sin(\phi - \beta).$$

But ω and $\mu \sin(\phi - \beta)$ are the rates of change of θ and v' respectively, and $\theta = 0$, $v' = 0$ are simultaneous values;

$$\therefore \quad m\theta = v';$$

and since $\theta = JA$, $v' = OC$ are also simultaneous values;

$$\therefore \quad mJA = OC.$$

From these data the morphosis (*Abbildung*) in the plane as Professor Haskell has constructed it, is readily obtained.

In view of the fact, first pointed out by Professor Haskell, that, whatever be the base or modulus, we may make P move upon a straight line by making Q move parallel to OT, my proposed distinction between linear and non-linear systems (*loc. cit.*, p. 192) is a false classification. Any system has both characteristics. The true classification is into gonic systems with complex modulus and agonic or ordinary systems with real modulus. In order to determine a gonic system completely, we have only to assign a fixed value to the modulus—its versor part by fixing β, its tensor part by fixing the ratio $\mu/\sqrt{\lambda^2 + \omega^2}$. The elements of this ratio may then vary at pleasure, subject only to the condition here imposed, and to the permanent condition $\omega = \lambda \tan(\phi - \beta)$.

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