the surface S and the μ-plane, x behaves in every region of the μ-plane like a rational function; (3) that y is likewise a rational function of μ; for the Riemann surface that represents x as a function of y is conformally related to S and hence to the μ-plane, and y behaves therefore in every region of the μ-plane like a rational function; (4) that μ is a rational function of λ; this can also be inferred easily from the conformal representation, but as it follows immediately from equations (1) and (4), we will not insist on this method.

Thus all of the statements of § 2 have been proven.

Example.
$$x=\frac{(\lambda^4+\lambda^{-4}+14)^3}{108\ (\lambda^4+\lambda^{-4}-2)^2}=\frac{4\ (\mu^2-\mu+1)^3}{27\ (\mu-1)^2}$$

$$y=-\left(\frac{\lambda^2-1}{2\lambda}\right)^2=\mu$$

$$\mu=-\tfrac{1}{4}\,(\lambda-\lambda^{-1})^2$$

See Klein-Fricke, *Modulfunctionen*, vol. I. p. 75, for the division of the λ-plane. The $\rho=4$ regions N, N', N'', N''' appear in the figure on p. 80. The notation λ, μ is there just the reverse of that in this paper.

HARVARD UNIVERSITY,
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NOTES ON THE EXPRESSION FOR A VELOCITY-POTENTIAL IN TERMS OF FUNCTIONS OF LAPLACE AND BESSEL.

BY PROFESSOR JAMES MCMAHON.

1. *Differential equation for ψ.* The partial differential equation to be satisfied by a velocity-potential in an elastic fluid is, in rectangular coördinates,*

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{a^2}\frac{\partial^2\psi}{\partial t^2},\qquad (1)$$

in which a^2 = pressure/ density, and $\psi\ (x, y, z, t)$ is a function whose derivatives as to x, y, z give the velocity-components of the fluid particle that occupies the position (x, y, z) at time t.

2. *Particular solution in polar coördinates.* When (1) is transformed to polar coördinates r, θ, φ, it can, as shown in

* RAYLEIGH, *Theory of Sound*, vol. II., p. 15.

standard works,* be satisfied by a product-function of the form

$$\psi_n = r^{-\frac{1}{2}} J_{\pm(n+\frac{1}{2})}(kr) \,.\, S_n(\theta, \varphi) \,.\, {}^{\sin}_{\cos}\, kat, \tag{2}$$

in which S_n is any surface harmonic of order n, and $J_{\pm(n+\frac{1}{2})}$ is a Bessel function of order $\pm(n+\frac{1}{2})$. Here ψ_n stands for the part of ψ that involves nth order harmonics, and may be written in a more explicit form as the sum of the four terms implied in (2); thus

$$\begin{aligned} r^{\frac{1}{2}}\psi_n = {} & J_m(kr)\,[S_n \sin kat + S'_n \cos kat] + \\ & J_{-m}(kr)\,[S''_n \sin kat + S'''_n \cos kat], \end{aligned} \tag{3}$$

where $m = n+\frac{1}{2}$, and the symbols S_n represent arbitrary nth order harmonics. The conditions of a particular problem may impose restrictions on these coefficients; and the chief object of this paper is to show how to choose the relations between the symbols S_n in order to adapt equation (3) to some of the typical problems in fluid motion to which the functions of Laplace and Bessel are applicable.† In all cases the complete value of ψ is to be built up by putting $n = 0, 1, 2, \cdots$, and determining the arbitraries by the initial distribution of condensation and velocity.

3. *Phase angles.* It will be sometimes convenient to put equation (3) in the form

$$r^{\frac{1}{2}}\psi_n = T_n J_m(kr) \sin(kat + \varepsilon_n) + T'_n J_{-m}(kr) \sin(kat + \varepsilon'_n), \tag{4}$$

wherein

$$T_n \cos \varepsilon_n,\; T_n \sin \varepsilon'_n,\; T_n \cos \varepsilon'_n,\; T'_n \sin \varepsilon'_n$$

take the place of S_n, S'_n, S''_n, S'''_n; the phase angles ε_n, ε'_n, being thus functions of θ, φ.

4. *Lemma.* To find the relations between the phase angles ε_n, ε'_n, and between the coefficients T_n, T'_n, in order that the motion of the medium at a great distance from the origin may be approximately represented by a single wave, either diverging from the pole or converging towards it:

When r is large put

$$J_{\pm m}(kr) = r^{-\frac{1}{2}} \cos(kr - \tfrac{1}{4}\pi \pm \tfrac{1}{2} m\pi), \ddagger$$

dropping a numerical factor; substitute in (4), and change trigonometric products into sums, then

*Gray and Matthews, *Bessel Functions*, pp. 213, 215. *Theory of Sound*, Vol. II., pp. 205, 229, 230, in which the J functions are replaced by two other particular solutions of the equation that $r^{-\frac{1}{2}}J$ satisfies.

† The advantage of the form of solution in equation (3) is that it gives a "realized" result in terms of known functions and without restriction of phase.

‡ Bessel Functions, p. 40.

$$r\psi_n = T_n \sin\left[k(at+r)+\varepsilon_n-\tfrac{1}{4}\pi-\tfrac{1}{2}m\pi\right]+ \\ T'_n \sin\left[k(at+r)+\varepsilon'_n-\tfrac{1}{4}\pi+\tfrac{1}{2}m\pi\right] \tag{5}$$

$$+ \text{ terms involving } at-r.$$

This expresses the superposition of a divergent and a convergent wave, represented by the terms in $at-r$, $at+r$, respectively. In order that the latter wave may be absent, the corresponding terms indicated in (5) must cancel each other for all values of the variables. This requires that $T_n = T'_n$, and also that

$$(\varepsilon'_n+\tfrac{1}{2}m\pi)-(\varepsilon_n-\tfrac{1}{2}m\pi)=\text{an odd multiple of } \pi,$$

say congruent with $(2n+1)\pi$, *i. e.*, with $2m\pi$,

$$\therefore \varepsilon'_n-\varepsilon_n=m\pi, \text{ or a congruent.} \tag{6}$$

Similarly the divergent wave is absent when

$$T_n = T'_n, \quad \varepsilon_n-\varepsilon'_n=m\pi.$$

Thus the velocity potential of a wave which at a great distance from the origin is $\left\{\begin{array}{l}\text{divergent}\\ \text{convergent,}\end{array}\right.$ consists of terms of the form

$$r^{\frac{1}{2}}\varphi_n = T_n\left[J_m(kr)\sin(kat+\varepsilon_n)+J_{-m}(kr)\sin(kat+\varepsilon_n\pm m\pi)\right]. \tag{7}$$

5. *Relations between the symbols S_n, for divergence or convergence.* Put, according to Art. 3,

$$S_n = T_n \cos\varepsilon_n, \qquad S'_n = T_n \sin\varepsilon_n,$$
$$S''_n = T_n \cos(\varepsilon_n\pm m\pi), \quad S'''_n = T_n \sin(\varepsilon_n\pm m\pi),$$

whence the required relations are

$$\therefore \quad S''_n = \mp S'_n \sin m\pi, \quad S'''_n = \pm S_n \sin m\pi. \tag{8}$$

6. *Resolution of ψ_n.* The expression in (3) can be exhibited as the sum of two parts representing divergent and convergent waves. For write

$$r^{\frac{1}{2}}\psi_n = T_n\left[J_m(kr)\sin(kat+\varepsilon_n)+J_{-m}(kr)\sin(kat+\varepsilon_n+m\pi)\right] +$$
$$T'_n\left[J_m(kr)\sin(kat+\varepsilon'_n)+J_{-m}(kr)\sin(kat+\varepsilon'_n-m\pi)\right], \tag{9}$$

the first part representing a divergent wave, and the latter a convergent one, by (7); compare with (3), equate coefficients, and solve; then T_n, T'_n, ε_n, ε'_n are determined by

$$2\,T_n\cos\varepsilon_n = S_n+S'''_n\sin m\pi,\ 2\,T_n\sin\varepsilon_n = S'_n-S''_n\sin m\pi,$$
$$2\,T'_n\cos\varepsilon'_n = S_n-S'''_n\sin m\pi,\ 2\,T'_n\sin\varepsilon'_n = S'_n+S''_n\sin m\pi. \tag{10}$$

7. *Application to spherical vibrator.* The surface of a sphere of radius c is maintained in vibration with given frequency, and the normal surface-velocity is a given harmonic function of latitude and longitude of the form

$$v_n(\theta, \varphi, t) = \bar{S}_n \sin kat + \bar{S}'_n \cos kat, \qquad (11)$$

to find the velocity potential at any point (r, θ, ψ) of the medium at any time t:

Use (3), find the radial velocity-function $\partial\psi_n/\partial r$, and put $r = c$, then

$$v_n = \left(\frac{\partial \psi_n}{\partial r}\right)_{r=c} = F_m\,(kc)\ [S_n \sin kat + S'_n \cos kat] + F_{-m}\,(kc)\ [S''_n \sin kat + S'''_n \cos kat], \qquad (12)$$

in which $\qquad F_m\,(kr) = \frac{\partial}{\partial r}\ .\ \ r^{-\frac{1}{2}}\, J_m\,(kr).$

Identifying (11),(12) gives two equations, and the conditions for divergence in (8), furnish two others to determine S_n, S'_n, S''_n, S'''_n in terms of $\bar{S}_n$, $\bar{S}'_n$; thus

$$\begin{aligned} \gamma\, S_n &= \alpha\, \bar{S}_n + \beta\, \bar{S}'_n, & S''_n &= -\, S'_n \cos n\pi, \\ \gamma\, S'_n &= -\beta\, \bar{S}_n + \alpha\, \bar{S}'_n, & S'''_n &= S_n \cos n\pi, \end{aligned} \qquad (13)$$

where $\alpha = F_m\,(kc)$, $\beta = F_{-m}\,(kc)\,\cos n\pi$, $\gamma = \alpha^2 + \beta^2$; and these values of the symbols S_n substituted in (3) give the required velocity-potential at an external point.*

The internal potential is of different form; for the disturbed region now includes the point $r = 0$, at which $J_{-m}\,(kr)$ becomes infinite, hence the arbitrary coefficients of this function must be zero. Putting $S''_n = S'''_n = 0$, and identifying (11), (12), determines S_n, S'_n, and gives

$$r^{\frac{1}{2}}\psi_n = \frac{1}{F_m\,(kc)}\, J_m\,(kr) \left[\bar{S}_n \sin kat + \bar{S}'_n \cos kat\right]. \qquad (14)$$

8. *Free vibrations inside a fixed spherical envelope.* Here, as in the latter part of Art. 7, the form is

$$r^{\frac{1}{2}}\psi_n = J_m(kr)\left[S_n \sin kat + S'_n \cos kat\right], \qquad (15)$$

but the possible values of the free periods, which depend on k, are to be found from the equation $F_m(kc) = 0$, *i. e.*

$$\frac{\partial}{\partial r}\left[r^{-\frac{1}{2}} J_m(kr)\right]_{r=c} = 0;\dagger \qquad (16)$$

*This solution differs from that given in Theory of Sound, vol. II., p. 207, by being expressed in "realized" form by means of Bessel functions, and by not restricting v_n to be in the same phase over the surface $r = c$.

† *Annals of Mathematics*, vol. 9, No. 1, p. 27.

while the symbols S_n are determined by the harmonic elements of the initial distribution of velocity and condensation.*

9. *Free vibrations between two concentric spherical surfaces.* Since the radial velocity at the surface $r = r_1$ is zero, then

$$F_m(kr_1)\Big[S_n \sin kat + S'_n \cos kat\Big]$$
$$+ F_{-m}(kr_1)\Big[S''_n \sin kat + S'''_n \cos kat\Big] = 0;$$

and there is a similar equation involving r_2.

These must be satisfied for all values of θ, φ, t,

$$\therefore S_n F_m(kr_1) = -S''_n F_{-m}(kr_1);\ S'_n F_m(kr_1) = -S'''_n F_{-m}(kr_1),$$

with two similar equations in r_2,

$$\therefore \frac{S''_n}{S_n} = \frac{S'''_n}{S'_n} = -\frac{F_{-m}(kr_1)}{F_m(kr_1)} = -\frac{F_{-m}(kr_2)}{F_m(kr_2)} = \rho, \text{ say.} \quad (17)$$

The possible values of k, and of the wave length $2\pi/k$, are to be found from the third of these equalities;† and then S''_n, S'''_n are known multiples of S_n, S'_n. Thus (3) takes the form

$$r^{1/2}\psi_n = [J_m(kr) + \rho J_{-m}(kr)]\ (S_n \sin kat + S'_n \cos kat), \quad (18)$$

an equation which, extended to the whole of space, gives a series of nodal spherical surfaces, of which $r = r_1$, and $r = r_2$ are a pair. At such surfaces the superposed divergent and convergent waves interfere.

ADDITIONAL NOTE ON DIVERGENT SERIES.

BY PROFESSOR A. S. CHESSIN.

In a previous note (pp. 72–75) it has been shown that every divergent series oscillating between finite limits can by a proper arrangement of its terms be made convergent. We will now extend those results to the case when one or both limits between which the series oscillates are infinite. To this end it suffices to consider, together with *regular* se-

* The work is exemplified for the case $n = 1$, *Theory of Sound*, pp. 236, 237.

† *Annals of Mathematics*, vol. 9, No. 1, pp. 29, 30.