BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY

THE THIRD SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The Third Summer Meeting of the American Mathematical Society was held in the Lecture Hall of the Society of Natural Sciences at Buffalo, N. Y., on Monday, August 31, and Tuesday, September 1. The Society continued for the present year the policy of affiliation, as regards the Summer Meeting, with the American Association for the Advancement of Science. In deference to the desire of the latter organization, the meeting of the Society was held the week following that of the Association. The register of attendance and the close and undivided attention given to the work of the meeting show that this arrangement was, on the whole, profitable for the Society.

Among those in attendance were: Dr. E. M. Blake, Professor Maxime Bôcher, Mr. J. M. Brooks, Professor F. N. Cole, Professor J. E. Davies, Professor E. W. Davis, Professor A. T. De Lury, Dr. L. E. Dickson, Professor W. P. Durfee, Professor H. T. Eddy, Professor T. S. Fiske, Miss Ida Griffiths, Dr. G. W. Hill, Dr. J. E. Hill, Professor T. F. Holgate, Dr. J. I. Hutchinson, Professor E. W. Hyde, Mr. P. A. Lambert, Dr. G. H. Ling, Professor James McMahon, Professor Mansfield Merriman, Professor E. H. Moore, Professor W. F. Osgood, Professor J. P. Pierpont, Professor Oscar Schmiedel, Dr. Virgil Snyder, Mr. W. M. Strong, Professor L. G. Weld, Professor H. S. White, Professor C. B. Williams, Miss E. C. Williams, Miss M. F. Winston, Professor F. S. Woods and Professor Alexander Ziwet.

Two sessions were held each day, at 10 A. M. and 2.30 P. M. respectively. The President, Dr. G. W. Hill, occupied the chair. On the recommendation of the Council Professor O. W. Anthony, New Windsor College, New Windsor, Md.; Mr. F. H. Clutz, Johns Hopkins University, Baltimore, Md., and Miss E. K. Wentz, Indianapolis, Ind., were elected to membership. Ten nominations for membership were received.

The President read to the Society a communicatiom from the committee in charge of the susquicentennial celebration of the founding of Princeton University, inviting the members of the Society to attend the courses of lectures to be given at Princeton in the week preceding the celebration by Professor Felix Klein and Professor J. J. Thomson.

The following papers were read:

(1) Methods of defining monogenic functions. Dr. E. M. BLAKE.

(2) An existence theorem for a class of linear euthymorphic functions of a single variable. Dr. E. M. BLAKE.

(3) A graphic method for the study of uniform convergence. Professor W. F. OSGOOD.

(4) Uniform convergence and the integration of series term by term. Professor W. F. OSGOOD.

(5) Two triply-infinite systems of simple groups. Dr. L. E. DICKSON.

(6) Ternary algebras. Professor J. B. SHAW.

(7) The geometry upon three surfaces of the seventh order. Dr. J. E. HILL.

(8) A special form of a quartic surface. Dr. J. I. HUTCHINSON.

(9) Note on integral and integro-geometric series. Professor E. D. ROE.

(10) The cross-ratio group of $n!$ $(n-3)$-*ic* Cremona transformations of flat space of $n-3$ dimensions. Professor E. H. MOORE.

(11) Criteria for the reality of nodes in Dupin's cyclides, with a corresponding classification. Dr. VIRGIL SNYDER.

(12) Numerically regular reticulations upon surfaces of deficiency higher than 1. Professor H. S. WHITE.

(13) Loci of the equations $p=\varphi^x e$ and $p=\varphi^x\psi^y e$. Professor E. W. HYDE.

(14) On the hypothesis of the successive transmission of gravity, and the possible perturbative effect on the earth's orbit. Professor JAMES MCMAHON.

(15) The continuity of chance. Professor E. W. DAVIS.

(16) A method of finding, without a table, the number corresponding to a given logarithm. Dr. ARTEMAS MARTIN.

(17) Table giving the first forty roots of the Bessel equation $J_0(x) = 0$, and the corresponding values of $J_1(x)$. Professors B. O. PEIRCE and R. W. WILLSON.

(18) On the projective group. Professor HENRY TABER.

The first of Dr. Blake's papers gives a classification of the methods which have been used for defining monogenic functions, such as algebraic, transcendental, differential and functional equations, assigned singularities, representability upon a given Riemann's surface, etc. Functions φ, defined by an equation $G\left[z, \varphi(z), \varphi\left(\frac{\alpha_1 z+\beta_1}{\gamma_1 z+\delta_1}\right), \cdots \varphi\left(\frac{\alpha_n z+\beta_n}{\gamma_n z+\delta_n}\right)\right]=0$,

where G is algebraic, are suggested as a field which should yield important results. Dr. Blake's second paper contains a proof of the theorem that the multipunctual equation

$$\varphi(z)+\sum_1^n p_\nu(z) \cdot \varphi\left(\frac{\alpha_\nu z}{\gamma_\nu z+1}\right)=0,$$

where
$$|\alpha_\nu|<1 ; 1+\sum_1^n p_\nu(0)=0,$$

and
$$1+\sum_1^n p_\nu(0)\, \alpha_\nu^m \neq 0, (m=1,2, \cdots)$$

possesses (for any arbitrary selection of branches of $p_\nu(z)$, are holomorphic within the domain of $z=0$) one and only one solution $\tau(z)$ which is holomorphic in the domain of $z=0$, and has at that point the value 1. Under the same conditions $\tau(z)$ multiplied by a constant is a solution, and these are the only solutions which are holomorphic at the point $z=0$.

Professor Osgood's first paper will be published in full in the BULLETIN. His second paper, which is to appear in the *American Journal of Mathematics*, is divided into two parts.

In Part I. the most general manner of the convergence or a function $s_n(x)$ satisfying the following conditions is studied by the aid of the geometric method set forth in the foregoing paper: (1) $s_n(x)$ is a continuous function of the real variable x; (2) when x is fixed and n increases without limit, $s_n(x)$ approaches a limit, $f(x)$; (3) $f(x)$ is continuous. Let x_0 be any value of x within the interval of convergence. Three cases can arise: a set of values $x', x'', \cdots$ with $\lim\limits_{i=\infty} x^{(i)}=x_0$ and a corresponding set $n'\, n'' \cdots\cdots$ with $\lim\limits_{i=\infty} n^{(i)}=\infty$ can be so determined that (a) $\lim\limits_{i=\infty}|s_{n(i)}(x^{(i)})|=\infty$; in this case x_0 shall be distinguished as a X-point; (b) $\lim\limits_{i=\infty}|s_{n(i)}(x^{(i)})| \neq f(x_0)$, but perhaps cannot be made infinite;

in this case x_0 shall be designated as a ξ-point; the set of ξ-points thus includes the set of X-points; (c)

$$\lim_{i=\infty} s_{n^{(i)}}(x^{(i)}) = f(x_0),$$

no matter how the above set $(x^{(i)}, n^{(i)})$ be chosen; in this case x_0 is designated as a $\mathfrak{Z}$-point. It is shown that the X-points are dense throughout no interval and that the $\mathfrak{Z}$-points are dense and non-enumerable in every interval. The ξ-points may be dense throughout an interval, but cannot form a continuum. The problem of determining what sets of points can serve as ξ- resp. $\mathfrak{Z}$-points for some $s_n(x)$ is reduced to a problem in the Theory of Sets of Points.

Part II. is devoted to the question: When is

$$\int_0^{x_1} \left[\lim_{n=\infty} s_n(x)\right] dx = \lim_{n=\infty} \left[\int_{x_0}^{x_1} s_n(x)\, dx\right] ?$$

a question in Double Limits. If $s_n(x) = u_1(x) + \cdots + u_n(x)$, then this is equivalent to the question: When can the series $f(x) = u_1(x) + u_2(x) + \cdots$ be integrated term by term? It is found that such integration is always allowable if the interval (x_0, x_1) is free from X-points. If this is not the case,

$$\lim_{n=\infty} \left[\int_{x_0}^{x} s_n(x)\, dx\right],$$

$x_0 \leqq x \leqq x_1$, may be discontinuous; but it is shown by an example that it can also happen that this function is continuous and still never equal to

$$\int_{x_0}^{x} \left[\lim_{n=\infty} s_n(x)\right] dx, \ i.e., \int_{x_0}^{x} f(x)\, dx, \text{ if } x \neq x_0;$$

in other words, it is possible that the term-by-term integral of a series of continuous functions, the value of the given series being a continuous function, may be a continuous function not equal to the integral of the given series. A sufficient condition is deduced that this should not be the case. These theorems give the most general conditions known to the author for the integration of a series term-by-term.

Dr. Dickson's paper contains a generalization for the Galois Field of order p^n, of the system of simple groups obtained by Jordan in the decomposition of the general linear group on m indices and of the group of the Abelian substitutions on $2m$ indices. Two systems of simple groups are obtained whose orders depend on the three independent parameters m, n and p. Hermite's theorem on the analytical representation of substitutions is also generalized for the Galois Field, and the result employed to make a complete determination of quantics of degree $\leqq 6$ suitable to represent substitutions on a power of p letters. Two remarkable classes of quantics of general degree are found, which possess an infinite sweep of suitability for representing substitutions.

Professor J. B. Shaw's paper deals with the general theory of all algebras of three units. After the introduction of the necessary definitions, those algebras are first considered in which one 'factor' only is necessary to determine the product. This part of the paper is essentially a resumé of the theory of the linear vector operator or nomon. Those algebras are then discussed in which the 'product' of *two* multiple quantities is necessary to determine the third. Special classes are considered and also associative algebras. At the end the portion of Lie's continuous groups in this theory is pointed out. In the absence of Professor Shaw his paper was read by Professor Hyde.

In his paper on the geometry upon three surfaces of the seventh order, Dr. J. E. Hill passes by the aid of the general cubo-cubic transformation, in three cases of which the principal systems of the two spaces degenerate, from a general cubic surface to a septic surface. The first surface is distinguished by three triple and three double lines, the second by a triple line and a double quintic, and the third by a triple conic and a double quartic of the second kind. Surface $S_{1^3, 1^3, 1^3, 1^2, 1^2, 1^2}$ possesses 9 simple lines, 36 conics, 9 sheaves and 126 isolated examples of plane cubics, 12 sheaves and 126 isolated examples of plane quartics, 3 sheaves and 21 isolated examples of plane quintics and 9 sheaves of plane sextics. Upon this surface there are also 9 sheaves and 126 isolated examples of twisted cubics; 1 net, 126 sheaves and 162 isolated examples of twisted quartics of the second kind and 9 sheaves of twisted quartics of the first kind; 84 nets, 504 sheaves and 513 isolated examples of twisted quintics, $d = 0$, and 36 nets and 36 sheaves of twisted quintics, $d = 1$; 36 webs, 630 nets, 1,332 sheaves and 756 isolated examples of twisted sextics, $d = 0$; 84 webs, 288 nets and 126 sheaves of twisted sextics, $d = 1$, and 9 webs of twisted sextics, $d = 2$. There are 6 sheaves and 21 isolated examples of tritangent planes and 195 examples of quartitangent planes possessed by this surface, besides some special planes of much higher singularity of contact.

In the same way the surface $S_{1^3, 5^2, 1}$ is shown to possess 13 simple lines,. 5 pairs of conic, 13 plane cubics, 1 sheaf of plane quartics, no plane quintics, and 13 sheaves of plane sextics. This surface still possesses 16 twisted cubics, 209 twisted quartics of the second kind, 16 sheaves and 208 isolated examples of twisted quintics, $d=0$, and 286 twisted quintics, $d=1$, beside twisted curves of higher orders. It has 312 tritangent planes, beside others of higher singularity of tangency.

The surface $S_{2^3, 4^2}$ possesses 9 simple lines, 16 simple conics, no plane cubics, no plane quartics, 16 plane quintics, and 9 sheaves of plane sextics, 90 twisted cubics, 10 sheaves and 546 isolated examples of twisted quartics of the second kind, 126 twisted quartics of the first kind, 144 sheaves and 3,452 isolated examples of twisted quintics, $d=0$, 84 sheaves and 1,260 isolated examples of twisted quintics, $d=1$, and 180 twisted quintics, $d=2$, beside twisted curves of higher orders. This surface possesses 313 tritangent planes and one plane having 3-pointic contact along three sheets of the surface.

The special form of quartic surface considered by Dr. Hutchinson is the locus of the vertex of a cone passing through six given points. If these six points lie in involution on the twisted cubic through them, the coördinates of a point of the surface can be expressed in terms of elliptic functions. The special surface has two additional lines, each of which passes through a double point of the involution.

Professor Roe's paper is devoted to the determination of general formulæ for the summation of integral and integro-geometric series. The summation is affected by the employment of certain numbers $A_{r+\lambda}$, defined as the coefficients of $x^{r+\lambda}$ in the expansion of $(e^x - 1)^r$. The numbers are expressible in determinant form, and their other properties are discussed in extenso. A complete general formula is given for the sum of n terms of an integro-geometric series. In the absence of Professor Roe, his paper was read by the Secretary.

Professor Moore's paper, which is intended for publication in the *Mathematische Annalen*, deals with the cross-ratios of n independent quantities z_i. All of these cross-ratios can be expressed rationally in terms of any $n - 3$ independent ones among them. Starting with the fundamental system of $n - 3$ ratios $[z_n \, z_{n-1} \, z_{n-2} \, z_i]$ $(i = 1, 2, \cdots n - 3)$, by permuting the z's we obtain in all $n!$ fundamental systems of the same type. The expression of these $n!$ systems in terms of (any) one leads to the group $G_{n!}$ in question. For $n = 4$ it is the well-known group of six linear fractional substitutions (collineations) generated by $\lambda' = \frac{1}{\lambda}$, $\lambda' = 1 - \lambda$. For $n \geqq 5$ the $G_{n!}$ is holoedrically isomorphic with the symmetric group in n letters. It contains a subgroup of $(n-1)!$ collineations permuting among themselves certain $n-1$ fundamental points; this $C^n G_{(n-1)!}$ was first given by Klein. The $G_{n!}$ arises by extending the $C^n G_{(n-1)!}$ by an inversion having $n-2$ of the fundamental points as critical points, and the remaining one as a fixed point.

Dr. Snyder defined a Dupin's cyclide, without the use of Line Geometry, by means of three simultaneous linear equations in Lie's hexaspherical coördinates. By associating these three equations with the point-complex and plane-complex, the reality of spheres common to the four complexes, given by the signs of determinants in the coefficients, indicate the presence of nodes.

Professor White discussed the problem of determining the numerically regular reticulations on surfaces of higher deficiency. For a convex surface (deficiency 0) Euler's equation, together with the requirement of regularity, gives three sets of finite integers for vertices, faces and edges of a polyhedron. These by duality become five, corresponding to the five "regular" polyhedrons. On a surface of deficiency > 1 the modified equation of Euler, together with the same limitation of regularity, gives again a finite number of sets of finite integers for vertices, faces, and edges or lines. These sets are of two sorts: *derivative*, obtained from sets belonging to lower deficiencies; and *special*, not so obtainable, peculiar to the deficiency in question. These sets of characteristic numbers *can be realized on concrete models.* The author exhibited a set of models prepared by Mr. O. H. Basquin, comprising thirteen card and six plaster models illustrating deficiency 2, and seven card models showing the special sets for deficiency 3, and explained Mr. Basquin's method for obtaining derivative sets for any deficiency from those for lower deficiencies.

Professor Hyde discussed certain loci defined by the equation $p = \varphi^x e$ and $p = \varphi^x \psi^y e$, where p is a variable point, e a fixed point, x and y variable scalars, and φ and ψ linear point functions of the form $\varphi(q) = \sum (A_k e_k \cdot e_k \,|\, q)$, ($k = 0, 1, 2$ or $k = 0, 1, 2, 3$, according as plane or solid space is under consideration). As x varies from $-\infty$ to $+\infty$ the first equation defines a curve described by p. This curve was discussed for different positions of e, the equation of the tangent plane, the velocity of p for any value of x, etc., were determined both in two and in three dimensional space. The second equation defines a surface, which under certain conditions is a *skew* surface wholly inside the tetrahedron of reference, under other conditions a *cone* similarly limited. The reciprocal curves and surfaces were briefly discussed, and also the loci of the equations $p = \varphi^u e$ and $p = \varphi^u \psi^v e$, where $u = f_1 x$ and $v = f_2 y$, the first equation giving with suitable values of f_1, φ, and e curves terminated by arbitrary points, and the other surface having corners at arbitrary points.

Professor McMahon presented an investigation of the following hypothesis. Suppose that the sun is moving in a straight line with velocity u and that the whole solar system shares this translatory motion. Suppose also that the gravitational influence issues continually from the sun in waves that move outward with velocity v (perhaps equal to the velocity of light), and that when any wave reaches the earth the latter is attracted toward the wave center or point of space from which the wave issued. This effective center of acceleration is at a distance from the sun which varies between the limits $ka(1 - e)$ and $ka(1 + e)$ where k is the ratio of u to v, a is the semiaxis major and e the eccentricity of the earth's orbit. Then the orbit of the earth relatively to the sun is that which would be due to a center of force that performs small oscillations about its mean position. The law of this oscillatory motion was determined, and the equations of acceleration of the earth in its orbit, along and perpendicular to the radius vector were corrected for this small disturbance. Appropriate solutions of the resulting differential equations were given as far as terms in ke^2. The most important perturbative terms were examined, and their effect on the orbit determined.

Professor Davis' paper embodied an examination of the idea of continuity leading to Dedekind's definition of a continuous row of points, the one-to-one correspondence between point row and number series, integers, fractions, incommensurables, transcendents, etc. No *law* can define the positions of *all* points on a line. A perfect distribution must contain as a part of itself a chance distribution. But there must be continuity; the mind requires it to explain discontinuity. There must also be chance to explain the discontinuity of law. One-to-one correspondence exists between the points on a finite and an infinite line, an infinite surface, infinite three-way and n-way space, all space in all time, all movements, all qualities, the universe throughout all time. The examination of the idea of chance shows the the latter presupposes an infinitely refined agnosticism, the limit of all possible or conceivable increase of knowledge. The theory of the Unknowable, the Absolute, the Will and wills were cognized under this philosophical aspect.

The object of Dr. Martin's paper was to deduce from the ordinary logarithmic series by proper modifications rapidly converging series from which the number corresponding to any given logarithm may be computed without tables. Four forms of series were given available under different conditions.

Dr. R. W. Willson and Professor B. O. Peirce presented a table giving the first forty roots of the Bessel equation $J_0(x) = 0$ and the corresponding values of $J_1(x)$. The first ten values of x for which $J_0(x)$ vanishes have been given to ten places of decimals by Meissel. The next thirty and the values of $J_1(x)$ corresponding to the first forty roots have been computed by the authors by means of Vega's ten-place tables of logarithms, except in the few cases where a greater number of places was necessary, and then recourse was had to Thoman's tables. The computation has been gone over twice.

Professor Taber's paper contains a theory of the special linear homogeneous group in n variables constituting a generalization of Study's theory of such a group for $n=2$. The results show that there are as many species of transformations of the special linear homogeneous group in n variables as there are factors of n. The paper also contains a theory of certain other sub-groups of the general projective group, analogous to Study's theory of the special linear homogeneous group of the plane.

F. N. COLE.

COLUMBIA UNIVERSITY.

CELESTIAL MECHANICS.

Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac. Vols. V., VI., VII. Washington, 1894–5.

The progress made in modern times toward the practical solution of the problems of celestial mechanics has been in no way behind that of the more theoretical investigations. New and more efficient methods of attacking the problems