THE BUFFALO COLLOQUIUM.*

On April 16, 1896, with the approval of the Council, the following circular was issued to the members of the American Mathematical Society:

To the Members of the American Mathematical Society:

At the instance of certain members of the Society, the Council has given its approval to a proposal for a meeting auxiliary to the Summer meeting, to continue for one week subsequent to the regular session, and to be designated as a Colloquium or Conference. The proposal and the reasons for it are briefly the following:

GROUNDS FOR THE PROPOSAL. The objects now attained by the Summer meeting are twofold: an opportunity is offered for presenting before discriminating and interested auditors the results of research in special fields, and personal acquaintance and mutual helpfulness are promoted among the members in attendance. These two are the prime objects of such a gathering. It is believed, however, that a third no less desirable result lies within reach. From the concise, unrelated papers presented at any meeting, only few derive substantial benefit. The mind of the hearer is too unprepared, the impression is of too short duration to produce accurate knowledge of either the content or the method. The half-hour paper, the twenty-minute paper, or the paper read by title, are forgotten almost before they are finished. The one-hour lectures are more effective, but they too are weakened by the fact of complete novelty. Positive and exact knowledge, scientific knowledge, is rarely increased in these short and stimulating sessions.

On the other hand, the courses of lectures in our best universities, even with topics changing at intervals of a few weeks, do give exact knowledge and furnish a substantial basis for reading and investigation. Lectures followed by reading yield much larger returns for the time expended than unaided reading.

THE PLAN FOR A COLLOQUIUM. To extend the time of a lecture to two hours, and to multiply this time by three or by six, would be practicable within the limits of one week. An expert lecturer could present, in six two-hour lectures, a moderately extensive chapter in some one branch of Mathematics. With some new matter, much that is old could be mingled, and digests of recent or too much neglected publi-

*Reported by THOMAS S. FISKE.

cations. There would be time for some elementary details as well as for more profound discussions. In short, lectures could be made profitable to all who have a general knowledge of the higher Mathematics. The Council has appointed a committee to make the necessary arrangements, and if twenty or more members of this Society desire it, it is proposed to invite each year, beginning with the present summer, some one, two, or three of our eminent mathematical scholars and lecturers to prepare such a short course upon selected topics, usually those upon which they have been working during the year previous. One course of six lectures and possibly another of three would then be arranged on a program for the week following the regular summer meeting, and usually at the same place. This would leave space for discussions, collateral reading and study, for such subsidiary lectures as any members might offer, and for short excursions or other recreation.

Some honorarium should of course be offered the lecturers. It will be understood that each member who signifies his intention to participate in a colloquium agrees to be liable for an equal share, not to exceed five dollars, of the expenses.

This plan, in its general features, has already the endorsement of the members of the Society whose names appear below. You are requested to notify the Secretary before May 10th whether you will attend the proposed colloquium following our meeting on August 31st and September 1st, at Buffalo, N. Y. If so, will you specify some two or three lecturers and topics whose selection would gratify you personally. Such suggestions will be useful for subsequent summers, if not, for any reason, adopted this year.

If enough favorable replies are received to warrant it, the Committee of the Council will attempt to provide a program, announcing it in the May or June number of the BULLETIN. Further plans and means will be a proper subject for discussion at the August meeting of this Society.

HENRY S. WHITE,
E. H. MOORE,
W. F. OSGOOD,
THOMAS S. FISKE,
F. N. COLE,
ALEXANDER ZIWET,
F. MORLEY.

In a circular published July 14th, by the Committee in charge of the arrangements for the Summer Meeting, the following announcement appeared:

"The Committee in charge of the preliminary arrangements for the Summer Meeting is able to report that definite action has now been taken for holding a Colloquium auxiliary to the Summer Meeting and immediately following the regular session. The Committee takes pleasure in announcing that it has secured, as lecturers for the Colloquium, Professor Maxime Bôcher, of Harvard, and Professor James Pierpont, of Yale, each of whom has consented to offer a course of six one to two-hour lectures. Professor Bôcher's subject will be *Linear Differential Equations and Their Applications;* Professor Pierpont's the *Galois Theory of Equations.* Through the courtesy of the lecturers, the Committee is permitted to give below a preliminary outline, subject to modification, of the ground to be covered in each course.

"Members of the Society intending to take part in the Colloquium are requested to meet in the Lecture Hall of the Society of Natural Sciences, Buffalo, on Wednesday, September 2d, at 9 : 30 A. M. The management of the Colloquium rests entirely with the participants, subject to the arrangements already concluded by the Committee with the lecturers. The Committee suggests tentatively, as a basis for action, that the Colloquium be held on the four days, Wednesday to Saturday, September 2d to September 5th, three lectures to be given each day, the lectures to be given alternately by the two lecturers, Professor Bôcher beginning on Wednesday at 10 A. M., Professor Pierpont following at 3 P. M., Professor Bôcher lecturing again at 8 P. M., and so on."

On the morning of September 3d the following persons met to participate in the Colloquium: Professor M. Bôcher, Professor J. E. Davies, Professor H. T. Eddy, Professor T. S. Fiske, Miss Ida Griffiths, Dr. J. E. Hill, Professor J. McMahon, Professor W. F. Osgood, Professor J. Pierpont, Professor Oscar Schmiedel, Professor H. S. White, Miss Ella C. Williams and Professor A. Ziwet.

Professor H. S. White was elected presiding officer. It was decided that three lectures should be given on each day, the first at 9 A. M., the second at 11 A. M., and the third at 3 P. M. On one day, however, the afternoon lecture was postponed until evening, in order that those attending the Colloquium might have an opportunity of visiting Niagara Falls. The assessment levied on each of the participants in the Colloquium was fixed at three dollars.

At the close of the last lecture those present expressed their thanks to the presiding officer for his efforts in behalf of the Colloquium, and a motion was adopted recommending to the Council that arrangements be made for a Colloquium in connection with the next Summer Meeting of the Society. The lecturers have kindly prepared the following brief abstracts of their courses :

PROFESSOR BÔCHER'S COURSE.

The lecturer began by insisting on the necessity, in the study of differential equations, of proving everything and not allowing oneself to be misled into thinking that anything is proved in the ordinary text-books on the subject, such, for instance, as Forsyth ; all that is there done being to explain a series of devices by which more or less general solutions of certain special differential equations can be obtained, while the general theorems (for instance, the theorem concerning the number of arbitrary constants in the general solution) are not proved and are frequently stated in a misleading manner.

The work of the course was restricted to equations of the form :

$$\frac{d^n y}{dx^n} + p(x) \cdot \frac{d^{n-1} y}{dx^{n-1}} + q(x) \cdot \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots\cdots + t(x) \cdot y = 0,$$

and for the sake of simplicity it was assumed that the coefficients $p(x), \cdots t(x)$ are single-valued functions. It was pointed out at the start that two different points of view may be taken: 1st, the variable x may be regarded as complex and the coefficients $p(x), \cdots t(x)$ as analytic (monogenic) functions, and we may seek analytic functions y which satisfy the equation; or 2d, $p(x), \cdots t(x)$ may be regarded as real functions, not necessarily analytic, of the real variable x and we seek any real function y which satisfies the equation. The distinction which is here pointed out forms the chief, though not the only, difference between what may be called the classical theory of linear differential equations as expounded in the treatises of Craig, Heffter and Schlesinger, and various sides of the subject which have not yet become classical. The classical theories being already easily accessible, the treatment of them was restricted in these lectures to as narrow compass as seemed compatible with the very great importance of some of the points involved. On the classical side of the theory the following matters were taken up.

The existence of solutions in the neighborhood of non-singular points of the differential equations (*i. e.*, of points at which no one of the coefficients $p(x), \cdots t(x)$ has a singular point) was established by substituting into the equation a power series with undetermined coefficients and showing that these coefficients can always be determined (the first n remaining arbitrary), so that the series will satisfy the equation and will converge within a circle reaching out to the nearest singular point of the equation. It was pointed out that even in this simple case Frobenius' proof* is preferable to the proof which is usually given† and which is nothing but the application to the special case of linear equations of Cauchy's general proof of the existence of solutions of differential equations. After pointing out, by the method of analytic extension, that the solutions thus found are the only analytic solutions which the equation can have, the question of fundamental systems of solutions and of linear dependence were taken up the treatment differing from that given by Schlesinger (p. 28–34) in various respects. The subject of regular singular points (*Punkte der Bestimmtheit* according to Fuchs) was briefly considered‡ as were also the subjects of the group of the equation and of irregular points.

On the non-classical side the existence of solutions was treated by the method of successive approximations used as a proof of this existence theorem for the first time apparently by Peano in 1887,§ but already used as a method of actually finding solutions by Liouville (for equations of the second order) in 1836‖ and by Caqué in 1864.¶ It was pointed out that the method also applies to the case where the variable is complex, and there gives a means of representing all branches of the solution instead of merely a portion of one branch, as is the case for solutions in the form of power series.** Returning to the case of real equations

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* Cf. SCHLESINGER: *Handbuch der Theorie der linearen Differentialgleichungen*, Vol. I., p. 164, where, however, the proof is given for *regular* points which include the non-singular points as special cases.

† Cf. SCHLESINGER'S *Handbuch*, Vol. I., p. 21; and KLEIN: *Hypergeometrische Funktion* (Lithographed), p. 49.

‡ For the method of presentation cf. the pamphlet by the lecturer entitled *The Regular Points of Linear Differential Equations of the Second Order*, and recently published by Harvard University.

§ Translated into French Math. Ann., Vol. XXXII. The method was subsequently generalized by Picard to cover the case of any (nonlinear) differential equation, cf. Picard's *Traité d'Analyse*, Vol. II., p. 301, or a translation by Prof. Fiske in the first volume of the BULLETIN.

‖ Liouville's Journal, Vol. I., p. 255.

¶ Liouville's Journal, 2 Série Vol. IX. Caqué's method was simplified and generalized by Fuchs in 1870; cf. Schlesinger's *Handbuch*, p. 370.

** This seems to have been first noticed by Fuchs in the article just referred to.

the theorem which was established by this method of successive approximations was that if in the interval from $x=a$ to $x=b$ the coefficients of the linear equation of the nth order are real and continuous functions of x, a real solution y, will exist, which is continuous throughout the interval $a\ b$, as are also its first $n-1$ derivatives, and which has, together with its first $n-1$ derivatives, arbitrarily assigned values at an arbitrarily chosen point c of the interval ab. Quite as important as this theorem is the complementary one that there cannot exist two different solutions which together with their first $n-1$ derivatives have the same values at c, and which together with these derivatives are continuous throughout ab. *

The next question taken up was the dependence of the solutions of the differential equation upon parameters which enter into the coefficients of the equation. Under certain simple conditions the solutions will be continuous functions of the parameters or, if the variables are complex and the conditions are slightly modified, analytic functions. The theorems here referred to are of importance in many applications to physical questions as the coefficients of the equations which occur in such questions (Bessel's or Legendre's equation, for instance) actually involve parameters. Closely connected with the subject is a paper by Sturm in the first volume of Liouville's Journal some of the chief results of which were treated and applied to a mechanical problem.

The last lecture was devoted to various questions which, although referring, in part at least, to the case of complex variables, have found no place in the classical text-books. The first was Stieltjes' treatment of the generalized problem of Lamé's polynomials. This treatment depends upon a mechanical problem in the equilibrium of particles and was introduced into the lectures to illustrate not so much the importance of differential equations to applications as the usefulness of applications in the study of differential equations. The subject was presented in an even somewhat more general form than the lecturer had used in his book: *Ueber die Reihenentwicklungen der Potentialtheorie*, p. 215.† Finally the subject of the *resolvents* of differential equations

* For two different proofs of this theorem cf. Picard's *Traité d'Analyse*, Vol. II., p. 299, and Jordan's *Cours d'Analyse* (revised edition), Vol. III., p. 93.

† Reproduced in Klein's lithographed notes: *Ueber lineare Differentialgleichungen der zweiten Ordnung*, p. 202.

was briefly touched upon * while the very important questions connected with the conformal representation by means of Schwarz's s-functions were dismissed with a mere mention, this subject being now tolerably accessible chiefly through the writings of Klein and his pupils.†

In conclusion, it should, perhaps, be mentioned that in the lectures differential equations of the *second* order were used almost exclusively. This was necessary in some parts of the course where the results do not apply to equations of higher order, while in other cases (for instance, in the discussion of the logarithmic cases of regular or irregular points) the restriction to equations of the second order made it possible to avoid somewhat lengthy discussions, without, however, obscuring the main facts. In most cases, however, this restriction had as its only object to make the formulæ used shorter, and, therefore, more easily grasped but not essentially simpler. Moreover it should not be forgotten that equations of the second order are the ones which actually occur in physical and astronomical problems.

PROFESSOR PIERPONT'S COURSE.

Professor Pierpont commenced by developing the few simple algebraical and substitution-theoretical notions and theorems underlying the Galoisian theory. This accomplished he discussed the Galoisian resolvent and group of an equation for a given domain of rationality, and determined the groups of certain equations which were to be employed later. It was next shown how the Galoisian theory enables us to make the solution of a given equation depend upon a system of resolvents whose roots were rational in the roots of the given equation while their coefficients were rationally known; and among the unlimited number of such systems those corresponding to a series of composition were particularly examined. The groups of such resolvents were shown to be simple, those equations corresponding to subgroups of prime indices were found to be cyclic and their algebraic solution was effected. From this followed that if the factors of composition were all primes the equation could be solved algebraically. As an illustration, the solu-

* For two examples of such resolvents cf. Klein's : *Hypergeometrische Funktion*, p. 250 and p. 269. Another equally important resolvent is the one (linear of the third order) satisfied by the product of any two particular solutions of a linear equation of the second order.

†Cf., for instance, Klein's *Hypergeometrische Funktion* from p. 250 to the end.

tion of the biquadratic by means of a series of composition was effected.

To penetrate more deeply into the Galoisian theory we must consider the effect produced on the Galoisian group of an equation on adjoining the roots of resolvents whose roots are not necessarily rational in the roots of the given equation. Here a theorem of Kronecker's is fundamental:

If $f=0$ and $g=0$ be two rational irreducible equations in a given domain of degrees resp. m and n, and if the adjunction of a root y_0 of $g=0$ makes f reducible, the adjunction of a root x_0 of $f=0$ makes g reducible. If α be the degree of the irreducible factor $f(x, y_0)$ having x_0 for root, and if β be the degree of the irreducible factor $g(y, x_0)$ having y_0 for root then

$$m\beta = n\alpha.$$

From this theorem a number of corollaries were easily deduced:

Cor. I. If the adjunction of a root of an irreducible equation $g=0$ of degree n reduces the group of an equation $f=0$ to a subgroup of index r, then n is a multiple of r and hence never less than r.

Cor. II. If $n=r$ and this is the case when n is prime the roots of $g=0$ are rational in those of $f=0$.

Cor. III. If the adjunction of the roots of $g=0$ reduce the group F of $f=0$ to F_1, the adjunction of the roots of $f=0$ reduces the group G of $g=0$ to G_1. The index of F_1 under F and G_1 under G is the same, the groups F_1 and G_1 are invariant subgroups and the quotient groups $F|F_1$, $G|G_1$, are holoedric isomorph, *i. e.*, abstractly identical.

Cor. IV. When G is simple the roots of $g=0$ are rational in those of $f=0$, and $G=F|F_1$.

From Cor. I. follows that whenever a reduction of the group of an equation takes place on adjoining a root of a binomial equation of prime degree p, the pth roots of unity lying already in the domain, it must be reduced to an invariant subgroup of index p. From this follows that it is not only sufficient but also necessary that the factors of composition be all primes in order that an equation can be solved algebraically.

From this corollary follows also Abel's theorem that whenever an equation can be solved algebraically it is possible to give the expression of the root such a form that all radicals entering it are rational in the roots of the given equation and the roots of unity.

From this finally follows the theorem that the general equation of degree greater than four does not possess an algebraic solution. In regard to this, attention was called to Dirichlet's criticism that while no algebraic solution exists while the coefficients were independent variables, still every concrete equation, with integral coefficients might possess an algebraic solution. In reply to this, reference was made to a paper of Hilbert's in the 110th volume of *Crelle*, where it is shown that this is not possible.

The problem of the algebraic solution of general equations beyond the fourth degree is only one of a number of celebrated problems of the past which the Galoisian theory easily disposed of. Among these Professor Pierpont treated the following:

1. The Delian problem or the duplication of the cube.

2. The trisection of a given angle and the general case dividing by n.

3. Gauss' theorem on the geometrical construction of regular polygons.

4. The casus irreducibilis.

To this end it was necessary to demonstrate the theorem that it is possible only then to lay off by rule and compass a line when its length can be expressed rationally in the roots of a sequence of equations of second degree, the original domain being $R(1)$; and also to deduce two new corollaries from Kronecker's theorem, viz.:

Cor. V. Irreducible equations of degree 2^m only can be solved by square roots.

Cor. VI. Irreducible equations with real roots of degree 2^m only can be solved by real radicals alone.

In developing the Galoisian theory, Professor Pierpont had taken pains to show that all the steps required by the theory could in every concrete case be actually executed. Two problems, however, remained: the determination of the Galoisian resolvent, and the group of an equation for a given domain. No method had been indicated to obtain the resolvent, and the method given to find the group depended upon a knowledge of properties of the roots of the given equation which often might be altogether unknown, *e. g.*, in case of an equation with integers for coefficients, the domain being generated by some algebraic number. Here the methods used to dispose of these fundamental problems were those indicated in Kronecker's Festschrift and developed by Molk, Stolz, Bolza and Weber. It was first shown how an integral rational function of several variables could be decomposed into its irreducible factors in a natural domain R, and it was demonstrated that this decomposition was unique. Such a method permits us to find the Galoisian resolvent of an equation for a natural domain; to find it for a derived domain R', it was first shown that every such domain could be replaced by one generated by a single irrationality y_0 root of an irreducible equation $g(y) = 0$, whose coefficients lay in R. The irreducible factors of a polynom $F(t)$ for the domain $R' = (R, y_0)$ may now be found as Wirtinger observed, as follows: Replace t by $t + uy_0$, where u is an indeterminate; if now

$$F(t + uy_0) = G(t, y_0)\, H(t, y_0)$$

the norm

$$N = \Pi F(t + uy_k) = \Pi G(t, y_k) \Pi H(t, y_k),$$

the products being extended over all the roots of $g = 0$, is the product of two rational factors in R. Should now N be irreducible in R, $F(t)$ is certainly irreducible in R'. To break $F(t)$ into factors rational in R', we have simply to find the greatest common divisor of a divisor of N and $F(t + uy_0)$.

Having shown how the Galoisian resolvent for a given domain may be found, we obtain the Galoisian group as follows: Take

$$V = u_1 x_1 + \cdots + u_n x_n$$

to build the Galoisian resolvent, the u's being indeterminate. Let

$$F(t; u_1, u_2 \cdots u_n)$$

be the Galoisian resolvent, considered as a function of the u's; let

$$\gamma = \begin{pmatrix} u_1 & u_2 & \cdots\cdots & u_n \\ u_{i_1} & u_{i_2} & & u_{i_n} \end{pmatrix}$$

be a substitution, leaving F unchanged, then

$$g = \begin{pmatrix} x_1 & x_2 & \cdots\cdots & x_n \\ x_{i_1} & x_{i_2} & & x_{i_n} \end{pmatrix}$$

is a substitution of the Galoisian group and this embraces only such substitutions.

At this point only one lecture was left and it was decided that this be devoted to the equations upon which the division of the argument of the elliptic function depends; a method essentially the same as that given by Weber was employed. Attention, however, was called to a fundamental paper of Hölder's in the 34th volume of the *Annalen*. it is important from the Galoisian standpoint: 1, as showing the character of the essential elements of any system of resolvents in which the roots of the given equation can be rationally expressed; 2, as making it imperative to enlarge the notion of a group from a substitution-group whose elements are concrete substitutions on the roots of an equation to a group whose elements are not explicitly given, but merely the laws of their combination.

Professor Pierpont regretted that time did not permit him to develop the theory of finite groups from this abstract standpoint and to touch upon some of the beautiful results obtained by Frobenius, Hölder, Cole and others. The importance of these methods and theories not only for the Galoisian theory, but for many other branches of mathematics, makes it desirable that they be made the subject of a future colloquium.

A GEOMETRICAL METHOD FOR THE TREATMENT OF UNIFORM CONVERGENCE AND CERTAIN DOUBLE LIMITS.

Presented at the Third Summer Meeting of the American Mathematical Society.

BY PROFESSOR W. F. OSGOOD.

The geometrical representation of functions by curves and surfaces is of two-fold importance; for not only does it represent to the eye by means of a concrete picture relations which would otherwise appear only in abstract arithmetic form, but this picture in its turn makes evident new facts and points out at the same time the course that the arithmetic proof of the theorems thus suggested would naturally take. The value of this method for the purposes of instruction alike in elementary and advanced infinitesimal calculus, as well as in analysis generally, can hardly be overestimated. How can the conception of the function be better explained than by such an example as a temperature curve? What better means is there for making clear the idea of the implicit function — y defined implicitly as a function of x by the equation $f(x, y) = 0$ — than by cutting the surface $z = f(x, y)$ by the plane $z = 0$? And how valuable is the surface $u = \varphi(x, y)$ when the differential of a function of two independent variables is introduced!