(*Mathematische Annalen*, volume 44), are the principal contributions to the theory of the twisted sextic; with the exception of the first, which deals with the classification of sextic curves in space, these all treat of the twisted sextic of deficiency one.

Of papers upon twisted curves of orders higher than six, Weyr's article in the sixty-ninth volume of the *Sitzungsberichte der math. natur. Klasse der Academie zu Wien* ("Ueber Raumcurven siebenter Ordnung"), is the only one in which a general treatment is found. Section 26 contains, altogether, twenty-two references.

Section 27 (special forms of twisted curves), containing 140 references, embraces the list of all articles, references to which could not find a fit place in the preceding sections.

To the observant student, such a bibliography as the one here briefly sketched, is interesting in many ways other than those at first apparent. It shows the trend of the different schools and their exoduses from one system or line of thought to another; the swarming of disciples after masters as the latter have changed the directions or methods oı their researches; the fluctuating enthusiasm in progressive fields of investigation; the numerous accidental duplications of discoveries and demonstrations, especially between different schools and languages; and, most important of all, it lays bare the "rag ends" that the masters, grown tired of one field of investigation, have left convenient for the disciple to lay hold of and unravel. Finally, it is believed, that to know well that which has already been done best fits one to do that which is, as yet, undone.

COLUMBIA UNIVERSITY.

LINEAR DIFFERENTIAL EQUATIONS.

Handbuch der Theorie der linearen Differentialgleichungen. Von Professor Dr. LUDWIG SCHLESINGER, Privatdocenten an der Universität zu Berlin. In zwei Bänden. Erster Band. Leipzig, Teubner, 1895. 8vo, xx+ 486 pp.

The literature of the theory of linear differential equations has increased so rapidly during the last quarter of a century, and much of it has become of such importance that a detailed and methodical presentation of the subject such as is contained in the volume before us will be welcomed by large numbers of mathematicians and advanced students. While the author has attempted to make the book as accessible as possible to different classes of readers by developing as far as he needs them various subjects not coming directly under the heading of differential equations, the book is in no sense an introduction to the theory. It is a work to be referred to by persons who already know the subject in outline at least and wish detailed information in some special direction, not to be worked through chapter by chapter.

The first thing which strikes us on opening the volume is the arrangement of the table of contents, which covers thirteen pages and for which the contents of Lacroix's *Calculus* has served as a model. The interesting feature here is the extensive list of references (to original memoirs, etc.), which is given in connection with each section. This will explain in part the remarkable freedom of the pages of the book from foot notes as these references are not repeated. The advantages of this method are obvious, but it also has the disadvantage that it is impossible to make the precise bearing of the references as clear when they are given in this form as if they were immediately connected with the subject in hand.

The volume opens with a historical introduction due in part to Paul Günther, who was to have coöperated with Professor Schlesinger in the publication of the book, but who died shortly after the work was undertaken. The greater part of this introduction is devoted to an account of the historical development of the theory of analytic functions and of the general theory of differential equations, only the last three pages treating of the case of *linear* equations. Here and in a few historical statements scattered through the book we find an error in historical perspective which is so often met with elsewhere that it seems necessary for us to refer to it.

The founder of the modern theory of linear differential equations was Riemann. Although his paper of 1857 on the hypergeometric series refers to a very special class of differential equations only, the methods there developed were the ones which, according to Riemann's suggestion, have been used in the further development of the theory. These methods, like most others due to Riemann, were not generally understood at the time. In 1865, however, Fuchs, inspired by the lectures of Weierstrass,* as well as by Rie-

*It is here stated (cf. *Crelle* Vol. 66, p. 122; see also a statement by Thomé on p. 322 of the same volume) that the theorem concerning the development of solutions about non-singular points is reproduced from Weierstrass's lectures. This proof is explicitly ascribed by Schlesinger to Fuchs.

mann's paper, treated in an easily intelligible manner two important special problems. It was not, however, in the memoir of 1857 alone that Riemann laid the foundation of the modern theory of linear differential equations. In a posthumous paper on minimum surfaces published in 1867, very important geometrical methods involving linear differential equations of the second order were introduced which, as developed by Schwarz,* form the basis of the theory of automorphic functions. However much Fuchs may have done in his numerous contributions towards the development of the theory of linear differential equations, he is in no way its founder as he is termed by Schlesinger and by many others.

After the historical introduction comes a nine page *Introduction*, in which some matters concerning differential equations in general and the singular points of analytic functions are explained.

Besides these introductions the volume before us consists of eight parts (Abschnitte), each of which is subdivided into chapters and these again into sections with headings.

In Part I., which consists of only fourteen pages, Weierstrass's proof of the fundamental existence theorem for linear differential equations is first given† and the analytic extension of the elements of the function thus obtained is considered, the general case in which the coefficients of the equation are multiple valued functions being at once taken up. Fundamental systems and the linear dependence of solutions are then treated.

Part II. is devoted to the more formal sides of the theory which bring out the analogy of linear differential equations to algebraic equations. Among the questions considered are the determination of the common solutions of two differential equations ; the breaking up of homogeneous linear expressions into symbolic factors ; the reduction of the order of a homogeneous linear differential equation when one of its solutions is known ; the theory of multipliers and ad-

---

* "Ueber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt." *Crelle* Vol. 75, 1872. The methods referred to had been in part independently discovered by Schwarz in 1867; cf. Schwarz's collected works, Vol. I., p. 109.

† An error which occurs on p. 123 of Fuchs's memoir is here copied. There will be no constants $M_k > |p_k(x)|$ throughout a circle surrounding a non-singular point and reaching out to the nearest singular point. It is necessary to draw a second circle concentric with the first and a little smaller, as is done in a similar case on p. 165, where Frobenius serves as the model.

joint differential equations; the fundamental theorems concerning non-homogeneous differential equations. Finally a brief discussion of the conception of reducibility as applied to linear differential equations is given.

Part III. refers to the behavior of the solutions of the differential equation in a doubly connected region within which the coefficients are single valued. The necessary theorems concerning the composition of linear substitutions are first developed, no specific knowledge except of the theory of determinants being assumed. The fundamental equation is defined and discussed from various points of view, and is finally applied to the development of solutions of the differential equation about points in the neighborhood of which the coefficients of the equation have the character of algebraic functions.

In Part IV. the theory of regular singular points (Punkte der Bestimmtheit) is considered.* Assuming that the coefficients of the equation behave like rational functions in the neighborhood of the point in question, a necessary form for these coefficients at a regular point is obtained, and it is then shown that this form is also sufficient to ensure the regularity of the point. There follows a detailed study of the solutions of the equation in the neighborhood of a regular point. Finally the results obtained are extended in two directions, first to the case of non-homogeneous equations and secondly to the case in which the coefficients of the homogeneous equation have the character of algebraic functions in the neighborhood of the point in question.

Part V. is entitled: "Differential equations of the Fuchsian class," *i. e.*, equations which are everywhere regular. It is first shown that the more general case in which the coefficients are algebraic functions can be reduced to the case in which they are rational functions, and this case alone is then considered. The results of part IV. are applied to the equations in question and are extended by the consideration of the question of the convergence of the series used on their circles of convergence. The general form of the differential equation which is everywhere regular and for which the singular points and their exponents are given is then obtained, and after certain simple special cases, including the equation with constant coefficients, have been treated, the hypergeometric differential equation is taken up and, after being reduced to the unsymmetrical

* Concerning the terminology here used, cf. the first foot note on p. 89 of this volume of the BULLETIN.

form in which Gauss used it, treated at some length. Although rather a matter of form than of substance, it should be mentioned that neither here nor elsewhere is it clearly brought out that the point at infinity does not play a part essentially different in the theory of linear differential equations from that played by any other point. This makes a symmetrical development of the theory of the hypergeometric function quite impossible.

Part VI. treats of the representation of solutions in a region which is bounded by two concentric circles and within which the coefficients are single valued analytic functions, including the important special case of the development about an irregular point. The method of infinite determinants is first taken up,* and then a method of Hamburger depending upon conformal transformation. Each of these methods while giving a complete solution of the problem involves a determination of constants by transcendental processes. The other two methods which are here treated apply in special cases only and even then usually give merely particular solutions. They involve, however, only algebraic determinations of constants. The first of these methods refers to the case in which some of the solutions are regular at an irregular point while the second concerns Thomé's "normal solutions."

While the series which have so far been used to represent the solutions of differential equations converge within regions bounded by one circle or by two concentric circles only, the question is taken up in Part VII. of obtaining solutions by means of infinite processes which converge throughout the whole plane, the case of equations with rational coefficients being alone explicitly considered. Caqué's method as generalized by Fuchs is first explained and a few applications of it are given. The rest of this part is devoted to the discussion of the neighborhood of an irregular point for equations with rational coefficients, Poincaré's discussion of the semi-convergent developments in the neighborhood of such points being, however, barely touched upon.

The eighth and last part is again concerned with equations with rational coefficients, the problem here being to give methods for the computation of the coefficients of the

* Concerning the mathematical connection between Hill's original use of these determinants and the use to which they are here put cf. a paper by Burkhardt: Ueber einige mathematische Resultate neuerer astronomischer Untersuchungen, insbesondere über irreguläre Integrale linearer Differentialgleichungen. Papers of the Chicago Mathematical Congress, pp. 20–25.

substitutions which express a fundamental system of solutions in the neighborhood of one point in terms of a fundamental system in the neighborhood of another point when we pass from one point to the other by a given path. Several methods are explained and the part closes with an application to the differential equation satisfied by the hypergeometric series.

The foregoing brief summary of the contents of the volume conveys no idea of the mass of detail given. This has not, especially in the later parts, been carried to the extent of including all known results concerning the subjects treated. The amount of detail given is, however, still very great, so great in fact as to obscure at times the underlying ideas, and for this reason, we repeat, the book will be found most useful to those who already have a clear grasp of the leading principles of the subject.

At various points in the book several more or less essentially distinct methods of treatment were open to the writer. In such cases it is to be regretted that Professor Schlessinger should often have been satisfied with the exposition of one of these methods even though, as is usually the case, it is the best. Advanced students for whom the volume before us is written, refer to a handbook not merely to find out what the facts in a certain branch of the subject are and how they can be established, but what their relations with one another and with other subjects are, and each new method of treatment suggests new relations. A few examples will make clear what we mean.

The fundamental existence theorem is established in Part I., by means of Weierstrass's proof of convergence of the development into a power series. This proof is simple and historically interesting. Frobenius's proof of convergence when applied to the special case of non-singular points is, however, still simpler. Even more striking than this is the proof of this existence theorem which can be based upon Caqué's solution given in Part VII. The possibility of this method is not even hinted at nor is any reference given.* Another matter is the necessary and sufficient condition that a set of functions should be linearly independent. This condition is deduced at the beginning of Part II. from the results of Part I., but it is not pointed out that this theorem can be proved directly and can then serve as a foundation for much of the work of Part I., making it in particular unnecessary to prove directly that the circle of convergence of

* See for references p. 53 of this volume of the BULLETIN.

the developments into power series about a non singular point reaches out to the nearest singular point of the differential equation. A last example of the same kind concerns the development of solutions both about regular and about irregular points in the cases in which logarithms enter. The method originally used by Fuchs in these cases is not even referred to in the text.* In all these cases and in many more of the same sort a brief explanation of the general outlines of these alternative methods, together with references for the details, would have been sufficient.

Somewhat in line with the question just considered, but involving far more important matters is the treatment accorded to Riemann's theory of the hypergeometric function. The differential equation which this function satisfies is treated as an important example of the general theory and the chief results of Riemann and his predecessors are obtained, but no attempt is made to convey to the reader anything of the spirit of Riemann's remarkable memoir. This is the more to be regretted as the methods used by Riemann have not, as has sometimes been suggested, been superseded by more modern methods; on the contrary the general theory of linear differential equations has not as yet grown up to the point where all of Riemann's methods can be generally applied.

We have confined ourselves so far to the criticism of details and have refrained from any general discussion of the scope, arrangement, and completeness of the work as a whole. This course has been made necessary by the fact that only the first of the two volumes of which the Handbook is to consist is as yet before us. We cannot, however, refrain from referring to one matter of a general character and that a very important one, the entire omission of the theory of the real solutions of real differential equations. We refer here not merely to the proofs of the existence of solutions when the coefficients of the equations are not assumed to be analytic,† but also to the important class of

* A modification of Frobenius's method for the treatment of regular points has recently been published by Kneser (*Math. Ann.*, Vol. 47, p. 408). The object of this modification is to avoid the use of Weierstrass's theorem that a uniformly convergent series of analytic functions may be differentiated term by term. It is not as well known as it should be that while the proof of this theorem as given by Weierstrass is rather complicated, its proof by means of Cauchy's integral is immediate (cf. for instance Demartres: Cours d'Analyse, Vol. 2, p. 74). There can, therefore, be no object in avoiding the use of this theorem except for the purist who wishes to base everything with Weierstrass on the use of power series.

† For references see pages 53 and 54 of the present volume of the BULLETIN.

theorems represented by Sturm's famous paper in the first volume of *Liouville's Journal.* When we consider the great importance of these questions both theoretically and in physical applications it is hard to justify their omission in what claims to be a Handbook of the whole theory of linear differential equations. This omission may, perhaps, be in part made good in the second volume.

Although Professor Schlesinger's treatise fails to meet some of the demands which it seems to us may fairly be made of a Handbook. it is certain, owing to the great amount of information which it contains in accessible form, to fill an important place in every mathematical library.

MAXIME BÔCHER.

HARVARD UNIVERSITY,
December, 1896.

TABLE OF THE FIRST FORTY ROOTS OF THE BESSEL EQUATION $J_0(x) = 0$ WITH THE CORRESPONDING VALUES OF $J_1(x)$.

Presented to the American Mathematical Society at its Third Summer Meeting, September 1, 1896.

BY R. W. WILLSON AND B. O. PEIRCE.

THE first ten values of x for which Bessel's function of the zeroth order, $J_0(x)$, vanishes have been given to ten places of decimals by Meissel.* The next thirty roots of the equation, $J_0(x) = 0$ and the values of $J_1(x)$ corresponding to these forty roots have been computed by us by means of Vega's ten place table of logarithms† except in the few cases where a greater number of places was necessary, and for these we have had recourse to Thoman's tables.‡ All the values have been checked by duplicate computation and the first four values of $J_1(x)$ by comparison with Meissel's tables.

* *Math. Abhandlungen der K. Akad. der Wissenschaften zu Berlin*, 1888.

† Thesaurus Logarithmorum Completus, Lipsiae, 1794.

‡ Tables de Logarithmes à 27 Décimales pour les Calculs de Précision, Paris, 1867.