complementary with respect to n of the given combination. Suppose now we form the n_m combinations of any n numbers m at a time and consider the set of combinations formed by combining each of these n_m combinations with their complementaries in such a way that the numbers in any combination arranged in their natural order are immediately followed by the numbers in the complementary combination arranged in the same way. The paper then gives an expression for the number of inversions in any combination of this set and also the excess of the number of combinations in the set which have an even number of inversions over those which have an odd number.

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THE THEOREMS OF OSCILLATION OF STURM AND KLEIN. (FIRST PAPER.)

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(Read before the American Mathematical Society at the Meeting of December 29, 1897.)

IN the first volume of *Liouville's Journal* (1836) Sturm has deduced certain properties of the real solutions of linear differential equations of the second order which are of fundamental importance both in pure and in applied mathematics. The opinion has been expressed* that Sturm's work cannot be regarded as rigorous and that other methods must be substituted for his, for instance the method of successive approximations recently employed by Picard for establishing some of these theorems. In one sense it is true that Sturm's work is not rigorous, as hardly any work in analysis done during the first half of the present century shows an appreciation of the difficulties connected with the conception of continuity. The work of Sturm may, however, be made perfectly rigorous without serious trouble and with no real modification of method. In the first two sections of the present paper I have proved such of Sturm's results as are necessary to establish his theorem of oscillation.† In doing this I have departed somewhat from his

* Cf. the first paragraph of Picard's note in the *Comptes Rendus* for February, 1894, and also Klein, Lineare Differentialgleichungen der zweiten Ordnung (lithographed 1894) p. 266: "In der That genügen die Existenzbeweise, wie sie Sturm und Liouville führen, keineswegs den heutigen Anforderungen d r Strenge. Man wird verlangen alle die von ihnen gegebenen Entwickelungen in neuer Weise abzuleiten."

† This name is due to Klein.

order of presentation (although even this is a matter of convenience rather than necessity) but his methods are essentially preserved.

I have not thought it desirable to complicate the presentation by preserving at all points the generality of Sturm's memoir.* I have not, however, thought it well to restrict the functions with which we deal to being analytic, for although the proofs would then have appeared simpler this simplicity would have been gained at the expense of their elementary character. On the other hand, I have restricted all the functions with which we deal to being continuous, this being, I suppose, what Sturm intended to do. The questions, some of them very important, which refer to cases in which these functions are discontinuous either within or at an extremity of the intervals in which we consider them, I hope to return to on a future occasion.

Sturm's theorem of oscillation relates to a differential equation which involves a variable parameter. This theorem has been extended by Klein to certain equations involving more than one parameter. We shall confine ourselves in § 3 to Lamé's equation and even then shall consider only the simplest, but, at the same time, the most important case of Klein's theorem of oscillation as Klein himself originally did (*Mathematische Annalen*, vol. 18, 1881). Klein's proof rested entirely on geometric intuition and the form in which I have since presented the proof † is, although the different steps are somewhat more sharply defined, of the same sort. ‡ I have given the proof here in what I hope will be found to be a perfectly rigorous analytical form which however rests, as I have briefly indicated in foot notes, upon the same kind of geometric considerations as Klein's original proof.

For the convenience of such readers as are not thoroughly familiar with the methods of exact analysis or with the elements of the theory of linear differential equations, I recapitulate here a few facts of which we shall have to make use.

(A) If throughout and on the boundary of a certain fi-

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* Thus a and a' are here regarded as constant in §2, instead of being functions of λ. Cf. also footnote, p. 300.

† Cf. my dissertation: Ueber die Reihenentwickelungen der Potentialtheorie, Göttinger Preisschrift, 1891, and a book published under the same title (to which I shall refer in § 3 as Reihenentwickelungen) in 1894 by Teubner. See also Klein: Lineare Differentialgleichungen der zweiten Ordnung, 1894.

‡ An analytic form of proof is also given by Pockels in which, however, questions of continuity are disregarded. See p. 118 of his book: Ueber die Differentialgleichung $\Delta u + k^2 u = 0$. Leipzig, Teubner, 1891.

nite region $f(x, y, z, \dots)$ is a continuous function of $(x, y, z, \dots)$ it will be *uniformly continuous* there.* From this it follows immediately that if when $a \leqq x \leqq b$ and $c \leqq y \leqq d$ $f(x, y)$ is a continuous function of (x, y) and if when $y = y_0$ $(c \leqq y_0 \leqq d)$ $f(x, y)$ does not vanish when $a \leqq x \leqq b$ then it is possible to find a positive quantity ε so small that when $|y - y_0| < \varepsilon$ (and also $c \leqq y \leqq d$) $f(x, y)$ does not vanish when $a \leqq x \leqq b$.

(B) If when $a \leqq x \leqq b$, $\lambda_1 \leqq \lambda \leqq \lambda_2$, $\mu_1 \leqq \mu \leqq \mu_2$ $f(x, \lambda, \mu)$ is a single valued continuous function of (x, λ, μ) then in the same intervals $\int_a^x f(x, \lambda, \mu)\, dx$ will be a continuous function of (x, λ, μ).†

(C) If when $a \leqq x \leqq b$ $p(x)$ and $q(x)$ are single valued continuous functions of x one and only one function $y(x)$ exists, which at every point of the interval ab satisfies the differential equation:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

(and which, therefore, has continuous first and second derivatives throughout ab) and which at the point $x_0 (a \leqq x_0 \leqq b)$ satisfies the relations $y(x_0) = \alpha$, $y'(x_0) = \alpha'$ (α and α' arbitrary real constants).‡

In particular if $\alpha = \alpha' = 0$, y is identically zero.

(D) If the functions p and q in (C) involve a parameter λ in such a way that when $a \leqq x \leqq b$ and $\lambda_2 \leqq \lambda \leqq \lambda_1$ they are continuous functions of (x, λ) then the function y determined in (C) will be in the same region a continuous function of (x, λ). §

(E) The function y determined in (C) cannot have an infinite number of roots in the interval ab unless $\alpha = \alpha' = 0$. For if it had, these roots would have at least one limiting point $x_1 (a \leqq x_1 \leqq b)$ and since y is continuous at x_1 $y(x_1) = 0$. Moreover by Rolle's theorem there will be in the neighbor-

* Cf. Jordan: Cours d'Analyse, 2d edition, vol. I, p. 48; Harkness and Morley: Theory of Functions, §64; or Picard: Traité d'Analyse, vol. I, p. 3 and p. 90 (where, however, the term *uniform continuity* is not mentioned). In the last two works the important condition that the boundary must belong to the region of continuity is not explicitly stated.

† I am not able to give a reference to a proof of this theorem. Such a proof may be given in a manner precisely similar to that in which it is ordinarily proved that the above integral is a continuous function of (λ, μ) and also of x. See Harnack's Calculus §146, VIII, and §151.

‡ Cf. Picard: Traité d'Analyse vol. II., Chap. 11; or Jordan: Cours d'Analyse, vol. III, p. 88.

§ This follows immediately from the method of successive approximations. See for a special case Picard: Traité d'Analyse, vol. III, p. 93.

hood of x_1 an infinite number of points at which $y' = 0$, and therefore since $y'(x)$ is continuous $y'(x_1) = 0$.* From the remark at the end of (C) we see that y and y' cannot both vanish at x_1.

(F) If x_0 is a root of the function y determined in (C) $(a \leqq x_0 \leqq b)$, then for sufficiently small values of ε $y(x_0 + \varepsilon)$ and $y(x_0 - \varepsilon)$ have opposite signs since $y'(x_0) \neq 0$; and if x_0', x_0'' are two successive roots, $y'(x_0')$ and $y'(x_0'')$ have opposite signs.

(G) By the change of dependent variable:

$$\bar{y} = e^{\frac{1}{2}\int p dx} \cdot y$$

or by the change of independent variable: †

$$t = \int e^{-\int p dx} dx$$

the differential equation in (C) may be reduced to the binomial form:

$$\frac{d^2 y}{dx^2} = \varphi(x) \cdot y.$$

We shall confine our attention to equations of this form in the following §§ 1, 2.

§1. *The Comparison of Corresponding Solutions of Two Differential Equations.*

We will begin by establishing the following fundamental *Theorem of Comparison:*

I. *If in the differential equations:*

$$(1) \qquad \frac{d^2 y}{dx^2} = \varphi_1(x) \cdot y$$

$$(2) \qquad \frac{d^2 y}{dx^2} = \varphi_2(x) \cdot y$$

$\varphi_1(x)$ *and* $\varphi_2(x)$ *are single valued continuous functions when* $a \leqq x \leqq b$ *and if throughout this interval* $\varphi_1(x) \geqq \varphi_2(x)$ *(the equal-*

* This follows more simply still from the fact that $[y(x_1 + \triangle x) - y(x_1)] / \triangle x$ keeps vanishing as $\triangle x$ approaches zero. The proof given in the text is however more convenient than the one just indicated in the case of equations of higher order tha ı the second.

† It should be noticed that since $dt / dx > 0$, an interval ab of the x-axis throughout which p is continuous corresponds in a one to one manner to an interval of the t-axis, and that any set of points in the first interval follow each other in the same order as the corresponding points in the second interval.

ity sign not holding for all values of x in the neighborhood of a); if moreover y_1 and y_2 denote solutions of (1) *and* (2) *respectively which satisfy the conditions $y_1(a) = y_2(a) = \alpha$, $y_1'(a) = y_2'(a) = \alpha'$ and if y_1 has n roots $x_1, x_2, \cdots, x_n$ such that $a < x_1 < x_2 \cdots < x_n \leqq b$; then y_2 will have at least n roots between a and b and the ith $(i = 1, 2, \cdots, n)$ of these roots measured from a is less than x_i.*

Let us consider the differential equation:

$$\text{(3)} \qquad \frac{d^2y}{dx^2} = \left[\varphi_2(x) + \lambda\left(\varphi_1(x) - \varphi_2(x)\right)\right] y.$$

This equation reduces when $\lambda = 1$ to (1) and when $\lambda = 0$ to (2). Let $y(x, \lambda)$ be the solution of (3) which satisfies the conditions:

$$y(a, \lambda) = \alpha, \quad y'(a, \lambda) = \alpha',$$

so that $y(x, 1) = y_1(x)$ and $y(x, 0) = y_2(x)$.

For what follows it is important to see that we can always find a constant a' which exceeds a by so little that when $a < x \leqq a'$ and $0 \leqq \lambda \leqq 1$ $y(x, \lambda) \neq 0$. This is at once obvious from (A) when α is not zero. When $\alpha = 0$ we get our result in a similar manner by considering $y'(x, \lambda)$.

Now consider the differential equations satisfied by $y(x, \lambda_1)$ and $y(x, \lambda_2)$ respectively. Multiply the first of these equations by $y(x, \lambda_2)$ the second by $y(x, \lambda_1)$ and subtract. This gives:

$$y(x, \lambda_2)\, y''(x, \lambda_1) - y(x, \lambda_1)\, y''(x, \lambda_2) =$$

$$(\lambda_1 - \lambda_2)\, [\varphi_1(x) - \varphi_2(x)]\, y(x, \lambda_1)\, y(x, \lambda_2).$$

Let us integrate this equation from $x = a$ to $x = x$:

$$y(x, \lambda_2)\, y'(x, \lambda_1) - y(x, \lambda_1)\, y'(x, \lambda_2) =$$

$$\text{(4)} \qquad (\lambda_1 - \lambda_2) \int_a^x [\varphi_1(x) - \varphi_2(x)]\, y(x, \lambda_1)\, y(x, \lambda_2)\, dx.$$

The integral which on the right hand side of this equation is multiplied into $\lambda_1 - \lambda_2$ is by (B) a continuous function of $(x, \lambda_1, \lambda_2)$ when $a' \leqq x \leqq b$, $0 \leqq \lambda_1 \leqq 1$, $0 \leqq \lambda_2 \leqq 1$. Moreover when $\lambda_1 = \lambda_2$ it is positive within the limits above indicated (it cannot be zero since it is at least as great as the integral from a to a'). Accordingly, it follows from (A) that it is possible to find a positive quantity ε independent of x, λ_1, and λ_2 and so small that when $|\lambda_1 - \lambda_2| < \varepsilon$ the above integral is positive throughout the region mentioned. Let us then insert between the values $\lambda = 1$ and $\lambda = 0$ a number of other values $k_1, k_2, \cdots, k_m$ in such a way that all the differences $1-k_1$, k_1-k_2, k_2-k_3, $\cdots$, k_m-0 are less than ϵ. Then if we let λ_1 and λ_2 be two successive values taken from the set 1, k_1, k_2, $\cdots$, k_m, 0 the integral just discussed will be positive when $a' \leqq x \leqq b$.

Let us begin by letting $\lambda_1 = 1$, $\lambda_2 = k_1$, $x = x_i$. Then since $y(x_i, 1) = 0$ the second term on the left hand side of (4) drops out, while the right hand side is positive. We see then that $y(x_i, k_1)$ and $y'(x_i, 1)$ have the same sign. In the same way $y(x_{i-1}, k_1)$ and $y'(x_{i-1}, 1)$ have the same sign. But $y'(x_i, 1)$ and $y'(x_{i-1}, 1)$ have opposite signs (see (F)), and therefore $y(x_i, k_1)$ and $y(x_{i-1}, k_1)$ have opposite signs. Accordingly $y(x, k_1)$ has at least one root between x_{i-1} and x_i. By similar reasoning $y(x, k_1)$ has at least one root between a' and x_1 (since $y(a', 1)$ and $y(a', k_1)$ have the same sign while $y'(x_1, 1)$ and therefore $y(x_1, k_1)$ has the opposite sign from them).

There cannot be more than one root in any one of the intervals just considered. For if there were let μ and ν be two successive roots of $y(x, k_1)$ between which no root of $y(x, 1)$ lies. We see then by using (4) as before that $y(\mu, 1)$ and $y(\nu, 1)$ have respectively opposite signs from $y'(\mu, k_1)$ and $y'(\nu, k_1)$ while these last two quantities have opposite signs from each other. $y(\mu, 1)$ and $y(\nu, 1)$ must then have opposite signs. But this is impossible since $y(x, 1)$ has no root between μ and ν.

Our theorem now follows at once. For we have just proved that $y(x, k_1)$ has at least n roots between a and b and that if x_i' $(i = 1, 2, \cdots, n)$ denotes the ith of these roots from a $x_i' < x_i$. In precisely the same way it follows that $y(x, k_2)$ has at least n roots between a and b and that if

$$x_i''(i = 1, 2, \cdots, n)$$

denotes the ith of these roots from a $x_i'' < x_i' < x_i$. Proceeding in this way we finally get the theorem concerning the roots of $y(x, 0)$ above stated.

To the theorem just established may be added a second *Theorem of Comparison:*

II. *The notation and conditions* being the same as in Theorem I, if neither $y_1(b)$ nor $y_2(b)$ is zero and if $y_1(x)$ and $y_2(x)$ each has just n roots between a and b (n may be zero), then:*

$$y_1'(b) / y_1(b) > y_2'(b) / y_2(b).$$

* It is easy to show that the restriction we made in Theorem I, that the equality sign in the relation $\phi_1(x) \geqq \phi_2(x)$ shall not hold for all points in the neighborhood of a may here be replaced by the condition that it must not hold for all points of the interval ab.

In the following proof we will use the same notation as in the proof of Theorem I.

In the first place it is clear that each of the functions $y(x, k_1)$, $y(x, k_2)$, $\cdots$, $y(x, k_m)$ has just n roots between a and b and does not vanish when $x = b$. For by Theorem I it must have at least n roots greater than a and less than b since this is true of $y(x, 1)$; and it cannot have more than n such roots nor can it vanish when $x = b$ since then by Theorem I $y(x, 0)$ would have at least $n + 1$ roots between a and b. It follows then from (F) that the functions $y(x, 1)$, $y(x, k_1)$, $\cdots$, $y(x, k_m)$, $y(x, 0)$ have changed sign the same number of times in passing from $x = a'$ to $x = b$, and since they all have the same sign when $x = a'$ they must all have the same sign when $x = b$.

The method we will use to prove our theorem is to establish the continued inequality:

$$\frac{y'(b, 1)}{y(b, 1)} > \frac{y'(b, k_1)}{y(b, k_1)} > \frac{y'(b, k_2)}{y(b, k_2)} > \cdots > \frac{y'(b, 0)}{y(b, 0)}.$$

The first step of this inequality will be established if we can prove that the difference between the two first terms:

$$\frac{y(b, k_1)\, y'(b, 1) - y(b, 1)\, y'(b, k_1)}{y(b, 1)\, y(b, k_1)}$$

is positive. We have just seen that $y(b, 1)$ and $y(b, k_1)$ have the same sign, so that the denominator is positive while the numerator is at once seen to be positive when we let $\lambda_1 = 1$, $\lambda_2 = k_1$ in (4). In precisely the same way we prove the other inequalities above written.

The two preceding theorems enable us to compare an equation we wish to investigate with one about which something is known. The equation most frequently taken as standard of comparison is:

$$\frac{d^2y}{dx^2} = cy \tag{5}$$

where c is a constant. We give now some theorems which can be obtained by such a comparison and which we shall need in the following sections.

When c is positive the solution of (5) which satisfies the conditions $y(a) = \alpha$, $y'(a) = \alpha'$ is:

$$y = \frac{\alpha\sqrt{c} + \alpha'}{2\sqrt{c}}\, e^{\sqrt{c}\,(x-a)} + \frac{\alpha\sqrt{c} - \alpha'}{2\sqrt{c}}\, e^{-\sqrt{c}\,(x-a)}.$$

It is easily seen that when $\alpha = 0$ y vanishes only when $x = a$, and that when $\alpha \neq 0$ and $c \geqq \frac{\alpha'^2}{\alpha^2}$ y does not vanish at all. We may then deduce the following proposition from Theorem I:

III. *If $\varphi(x)$ is single valued and continuous when $a \leqq x \leqq b$ and if y is the solution of:*

$$\frac{d^2y}{dx^2} = \varphi(x)\, y$$

which satisfies the conditions $y(a) = \alpha$, $y'(a) = \alpha'$ then, provided that throughout the interval ab $\varphi(x) \geqq \alpha'^2/\alpha^2$ (or if $\alpha = 0$ provided that $\varphi(x) \geqq 0$), $y(x)$ will not vanish when $a < x \leqq b$.

By applying Theorem II we get:

IV. *If y is determined as in Theorem III and if $\varphi(x) \geqq c$ (the equality sign not holding for all values of x in the neighborhood of a) then provided that $c \geqq \alpha'^2/\alpha^2$ (or if $\alpha = 0$ provided that $c \geqq 0$) we have when $a < x \leqq b$:*

$$\frac{y'(x)}{y(x)} > \sqrt{c}\,\frac{(\alpha\sqrt{c} + \alpha')e^{\sqrt{c}(x-a)} - (\alpha\sqrt{c} - \alpha')e^{-\sqrt{c}(x-a)}}{(\alpha\sqrt{c} + \alpha')e^{\sqrt{c}(x-a)} + (\alpha\sqrt{c} - \alpha')e^{-\sqrt{c}(x-a)}}.$$

We may add to Theorems III and IV as a corollary whose truth may also be seen directly:

If $\varphi(x) \geqq 0$ and either $\alpha = 0$ or *$\alpha' = 0$, then when $a < x \leqq b$ $y(x)$ does not vanish and $y'/y > 0$.*

If on the other hand c is negative, any solution of (5) may be written in the form:

$$y = C_1 \cos\sqrt{-c}\,x + C_2 \sin\sqrt{-c}\,x,$$

and since this function has an infinite number of roots situated at intervals of $\pi/\sqrt{-c}$ we get the theorem:

V. *If y is determined as before and if throughout the interval ab:*

$$\varphi(x) < -\frac{n^2\pi^2}{(b-a)^2}$$

then y will have at least n roots in the interval ab.

§2. *On the Solutions of Differential Equations involving one Parameter.*

We have been dealing so far with differential equations which do not involve parameters.* We come now to some theorems concerning the solutions of the equation:

$$(6) \qquad \frac{d^2y}{dx^2} = \varphi(x, \lambda)y$$

in which we will suppose that φ is a continuous function of (x, λ) when $a \leqq x \leqq b$ and $\lambda_1 \geqq \lambda \geqq \lambda_2$. Let us denote by $y(x, \lambda)$ the solution of (6) which satisfies the conditions $y(a, \lambda) = \alpha$, $y'(a, \lambda) = \alpha'$ (α and α' independent of λ). We get then the theorem:

VI. *If when $a \leqq x \leqq b$ and $\lambda_1 \geqq \lambda' > \lambda'' \geqq \lambda_2$ $\varphi(x, \lambda') \geqq \varphi(x, \lambda'')$ (the equality sign not holding for all values of x in the neighborhood of a) and if $y(x, \lambda_1)$ has n roots in the interval $a < x \leqq b$ then $y(x, \lambda)$ $(\lambda_1 \geqq \lambda \geqq \lambda_2)$ will have at least n roots in this interval. Moreover, if we call these roots (or if there are more than n the first n of them) $x_1(\lambda), x_2(\lambda), \cdots, x_n(\lambda)$* where $a < x_1 < x_2 < \cdots < x_n \leqq b$ *then $x_i(\lambda)$ is a single valued and continuous function of λ when $\lambda_1 \geqq \lambda \geqq \lambda_2$.*

Everything here stated follows at once from Theorem I except the continuity of $x_i(\lambda)$. In order to establish this continuity it is sufficient (and necessary) to show that, λ' being any value of λ in the interval $\lambda_1\lambda_2$, no matter how small a positive quantity δ we may take it is possible to find a positive quantity ε so small that when

$$|\lambda - \lambda'| < \varepsilon, \ |x_i(\lambda) - x_i(\lambda')| < \delta \ (i = 1, 2, \cdots, n).$$

To show this let δ be taken so small that in the intervals

$$|x_i(\lambda') - x| \leqq \delta \ (i = 1, 2, \cdots, n) \quad y'(x, \lambda') \neq 0.$$

These intervals we will denote by $I_1, I_2, \cdots, I_n$. The possibility of this choice of δ is obvious from the fact that

$$y'(x_i(\lambda'), \lambda') \neq 0$$

and from the continuity of $y'(x, \lambda')$. Let us now notice that just as on p. 299 it is possible to find a quantity a' such that when $a < x \leqq a'$ and $\lambda_1 \geqq \lambda \geqq \lambda_2$ $y(x,\lambda) \neq 0$; and consider the intervals $a' \leqq x \leqq x_1(\lambda') - \delta$, $x_1(\lambda') + \delta \leqq x \leqq x_2(\lambda') - \delta$, $\cdots$, $x_{n-1}(\lambda') + \delta \leqq x \leqq x_n(\lambda') - \delta$ which we will call $J_1, J_2, \cdots, J_n$. Since $y(x, \lambda')$ does not vanish in any of these intervals while $y'(x, \lambda')$ does not vanish in any of the intervals $I_1, \cdots, I_n$ it is possible, (see (A)) to choose ε so small that when

* The introduction which we found it convenient to make of a parameter λ in the proof of Theorems I and II does not affect this statement.

$|\lambda - \lambda'| < \varepsilon$ $y(x, \lambda)$ does not vanish in any of the intervals J nor at the point $x_n(\lambda') + \delta$ and $y'(x, \lambda)$ does not vanish in any of the intervals I. This being done it is clear that when $|\lambda - \lambda'| < \varepsilon$ $y(x,\lambda)$ has one and only one root in each of the intervals I. For if it had more than one root in any of these intervals $y'(x,\lambda)$ would have, by Rolle's theorem, to vanish in this interval. And it must have at least one root in each interval I_i since $y(x, \lambda')$ and therefore $y(x, \lambda)$ has opposite signs in the intervals J_i and J_{i+1} (or when $i = n$ in the interval J_n and at the point $x_n(\lambda') + \delta$). Moreover since $y(x, \lambda)$ has no roots in the intervals J or in the interval $a < x \leqq a'$ it follows that the root which we have just seen lies in the interval I_i is the root $x_i(\lambda)$. Our theorem is thus proved.*

VII. *If in (6) when $a \leqq x \leqq b$ and $\lambda_1 \geqq \lambda' > \lambda'' \geqq \lambda_2$*

$$\varphi(x, \lambda') \geqq \varphi(x, \lambda'')$$

(the equality sign not holding for all points in the neighborhood of a)† and if $y(x, \lambda_1)$ has just n roots (n may be zero) when $a < x \leqq b$ and $y(x, \lambda_2)$ has just $n + m$ roots in this interval then there will be one and only one value of λ ($\lambda_1 > \lambda \geqq \lambda_2$) for which $y(x, \lambda)$ vanishes when $x = b$ and has just $n + k$ roots ($0 < k < m$) between a and b.

In order to prove this let us define the function $\varphi(x, \lambda)$ when $x > b$ by the formula $\varphi(x, \lambda) = \varphi(b,\lambda) - (x - b)$. The function as thus defined is a continuous function of (x,λ) when $\lambda_1 \geqq \lambda \geqq \lambda_2$ and $a \leqq x$; and throughout this whole region it satisfies the relation $\varphi(x,\lambda') \geqq \varphi(x,\lambda'')$ when $\lambda' > \lambda''$. Moreover when x increases beyond b $\varphi(x,\lambda)$ decreases indefinitely so that by Theorem V $y(x,\lambda)$ has for all values of $\lambda(\lambda_1 \geqq \lambda \geqq \lambda_2)$ an infinite number of roots greater than b. Let us call the first $m + n$ roots of $y(x,\lambda)$ which are greater than a

$$x_1(\lambda), x_2(\lambda), \cdots, x_{m+n}(\lambda).$$

Then by Theorem VI $x_{m+k}(\lambda)$ is a single valued and continuous function of λ which constantly decreases as λ varies from λ_1 to λ_2. Therefore since $x_{m+k}(\lambda_1) > b$ and $x_{m+k}(\lambda_2) \leqq b$ there must be one and only one value of λ ($\lambda_1 > \lambda \geqq \lambda_2$) for which $x_{m+k}(\lambda) = b$.

We come now to Sturm's theorem of oscillation:

* Compare the proof just given with the treatment of implicit functions in Jordan's Cours d'Analyse 2d edition, vol. I., p. 80.

† Here, as in the proof of Theorem II, all that need be required is that the equality sign shall not hold for all values of x in the interval ab. This remark applies to the next theorem also.

VIII. *If when* $a \leqq x \leqq b$ *and* $L > \lambda > l$ $\varphi(x, \lambda)$ *is a single valued continuous function of* (x, λ) *and if when* $a \leqq x \leqq b$ *and* $L > \lambda' > \lambda'' > l$ $\varphi(x, \lambda') \geqq \varphi(x, \lambda'')$ *(the equality sign not holding for all values of* x *in the neighborhood of* a*)*;* *and finally if no matter how large the quantity* M *and how small the quantity* m *may be chosen it is possible to take* λ *on the one hand so near to* L *that for all values of* x *in the interval* ab $\varphi(x, \lambda) > M$, *and on the other hand so near to* l *that for all valves of* x *in the interval* ab $\varphi(x, \lambda) < m$; *then there is one and only one value of* λ $(L > \lambda > l)$ *for which the differential equation :*

$$\frac{d^2y}{dx^2} = \varphi(x, \lambda)\, y$$

has a solution which has n *roots greater than* a *and less than* b *and which satisfies the conditions :*

$$\alpha' y(a) - \alpha y'(a) = 0, \tag{1}$$

$$\beta' y(b) - \beta y'(b) = 0. \tag{2}$$

Here n is an arbitrarily chosen integer positive or zero, and $\alpha, \alpha', \beta, \beta'$ are arbitrarily chosen real quantities restricted merely by the fact that α and α' must not both be zero and β and β' must not both be zero. Moreover, it is not necessary to require that L and l should be finite; we may have $L = +\infty$ or $l = -\infty$ or both of these cases may occur at once.

In the special case in which $\beta = 0$ the truth of this theorem follows at once from Theorems III, V, and VII. From this special case the general theorem may be deduced as follows :

By means of the special case of the theorem just referred to it is possible to find two values of λ (λ' and λ'') for each of which a solution of the differential equation exists which satisfies condition (1) and which vanishes when $x = b$ while the solution corresponding to λ' has just $n - 1$ roots and the solution corresponding to λ'' just n roots greater than a and less than b. From Theorem I it is clear that $\lambda' > \lambda''$ and that when $\lambda' > \lambda \geqq \lambda''$ and only then the solution $y(x, \lambda)$ which satisfies condition (1) has just n roots greater than a and less than b. If then we can prove that in this interval there is one and only one value of λ for which condition (2) is satisfied our theorem will be proved.

Now when $\lambda' > \lambda > \lambda''$ $y(b, \lambda) \neq 0$ as otherwise by Theorem

* See the preceding footnote.

I $y(x, \lambda'')$ would have at least $n + 1$ roots greater than a and less than b. Accordingly $y'(b, \lambda) / y(b, \lambda)$ is a continuous function of λ between these limits. Moreover by Theorem II this ratio continually increases as λ varies from λ'' to λ', and it must increase from $-\infty$ to $+\infty$ since its numerical value clearly becomes infinite as λ approaches either λ' or λ''. The above mentioned ratio must therefore take on every value, and in particular the value β' / β, once and only once in the interval in question.

The case $n = 0$ is not covered by the above proof. Here however we may as before find a value λ'' such that $y(b, \lambda'') = 0$ while $y(x, \lambda'')$ does not vanish when $a < x < b$, and it is clear as before that when $L > \lambda \geqq \lambda''$ and only then $y(x, \lambda) \neq 0$ when $a < x < b$, and that $y(b, \lambda) \neq 0$ when $L > \lambda > \lambda''$. Accordingly $y'(b, \lambda) / y(b, \lambda)$ is continuous and constantly increases as λ varies from λ'' to L. Moreover its numerical value becomes infinite when λ approaches λ'' and by the formula in Theorem IV the same is true when λ approaches L, so that as before it takes on the value β' / β for one and only one value of λ.

A special case of the last theorem deserves mention as it is the one which is of by far the greatest use, viz., the case in which $\alpha = 0$ or $\alpha' = 0$ and $\beta = 0$ or $\beta' = 0$. Here, as is at once seen by reference to the corollary stated after Theorem IV, we may replace the condition stated in VIII concerning M by the condition that it must be possible to take λ so near to L that for all values of x between a and b $\varphi(x, \lambda) \geqq 0$.

We added finally the theorem (whose truth is immediately obvious from Theorems I and II):

IX. *If two values (λ_1 and λ_2) of λ are determined by Theorem VIII by using in each case the same values of α and α' and if the values of n, β, β' corresponding to λ_1 are n_1, β_1, β'_1 and the values corresponding to λ_2 are n_2, β_2, β_2' then:*
(1) *if $n_1 > n_2$ we shall have $\lambda_1 < \lambda_2$;*
(2) *if $n_1 = n_2$ and $\beta_1 = 0$ while $\beta_2 \neq 0$ then $\lambda_1 < \lambda_2$;*
(3) *if $n_1 = n_2$, $\beta_1 \neq 0$, $\beta_2 \neq 0$ and $\beta_1' / \beta_1 < \beta_2' / \beta_2$ then $\lambda_1 < \lambda_2$.**

We have assumed throughout this and the preceding section that $a < b$. The contrary assumption might have been treated in the same way although many of the theorems

*It is upon this theorem that Klein's determination of the relative magnitude of the $2n + 1$ values of B which in the theory of Lamé's polynomials correspond to one and the same value of $A = n(n + 1)$ rests. *Cf.* Lineare Differentialgleichungen der zweiten Ordnung, p. 341–346. I take this occasion of mentioning the fact, which I pointed out to Professor Klein in June, 1893, that the theorem just referred to follows at once from the discussion of Poincaré, *Acta Mathematica*, vol. 7, p. 306.

would then require slight modification. The easiest way of getting the theorems in this case is perhaps to change the independent variable x by the formula $x' = -x$.

§ 3. *Lamé's Equation.*

We will write Lamé's equation in the form :

$$\frac{d^2y}{dx^2} + \frac{1}{2}\left(\frac{1}{x-e_1} + \frac{1}{x-e_2} + \frac{1}{x-e_3}\right)\frac{dy}{dx}$$
$$- \frac{Ax+B}{4(x-e_1)(x-e_2)(x-e_3)}y = 0.*$$

We shall regard the singular points e_1, e_2, e_3 as real and unequal ($e_1 < e_2 < e_3$) and constant, while A and B will be regarded as real parameters. The equation can be reduced to the binomial form :

$$\frac{d^2y}{dt^2} = (Ax+B)\,y$$

by the change of independent variable (see (G)) :

$$t = \int_c^x \frac{dx}{2\sqrt{(x-e_1)(x-e_2)(x-e_3)}}.$$

The lower limit of integration c is arbitrary. Let us first consider a finite segment ab of the x-axis where either $e_1 \leqq a < b \leqq e_2$ or $e_3 \leqq a < b$.† If we let in the first case $e_1 \leqq c \leqq e_2$ and in the second case $e_3 \leqq c$ the integral t will be finite and real for all points of the segment ab. The segment ab corresponds then in a one to one manner to the segment of the t-axis from $t(a)$ to $t(b)$. If then we notice that when we assign to A an arbitrary value the function $Ax + B$ increases with B for all points of the segment $t(a)$ to $t(b)$ and (to put it roughly) increases from $-\infty$ to $+\infty$ as B varies from $-\infty$ to $+\infty$ we get at once from Theorem VIII :

X. *If we assign to A an arbitrary value there will be one and only one value of B ($B = F(A)$) for which Lamé's equation*

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* For the relation between the notation here used and Lamé's notation see Reihenentwickelungen, p. 113, footnote.

† The following discussion applies equally well to the case in which the segment ab covers a part or the whole of the interval $e_1 e_2$ or $e_3\,\infty$ in which it lies more than once (provided merely that it remains of finite length), *cf.* Reihenentwickelungen, p. 123 and following. Only slight verbal alterations would be necessary in the text to include this case.

has a solution y which has n roots in the segment $a < x < b$ *and at the ends of the segment satisfies the relations:*

$$(1) \qquad \alpha' y(a) - \alpha \frac{dy(a)}{dt} = 0$$

$$(2) \qquad \beta' y(a) - \beta \frac{dy(a)}{dt} = 0.*$$

Here n may be any integer positive or zero while α, α', β, β' are any real quantities provided that we have neither $\alpha = \alpha' = 0$ nor $\beta = \beta' = 0$. The function $F(A)$ depends, of course, upon all of these quantities, and if we indicate the argument A only it is because in the questions which we shall take up the other quantities are regarded as fixed.† We proceed now to the deduction of some properties of the function $F(A)$ concerning which we, as yet, know only that it is defined and single valued for all values of A.

XI. A_1 *and* A_2 *being any two different values of* A:

$$a < \frac{F(A_2) - F(A_1)}{A_1 - A_2} < b.‡$$

For if we call the middle term of this inequality x_0 we have:

$$[A_1 x + F(A_1)] - [A_2 x + F(A_2)] = (A_1 - A_2)(x - x_0)$$

and if the above inequality did not hold we should have:

either $\qquad A_1 x + F(A_1) \geqq A_2 x + F(A_2) \quad (a \leqq x \leqq b)$

or $\qquad A_1 x + F(A_1) \leqq A_2 x + F(A_2) \quad (a \leqq x \leqq b)$

the equality sign holding at most when $x = a$ or when $x = b$. Either of these inequalities is seen to be impossible when we compare the equations:

$$\frac{d^2y}{dt^2} = [A_1 x + F(A_1)]y \qquad \frac{d^2y}{dt^2} = [A_2 x + F(A_2)]y$$

by means of Theorems I and II.

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* Except when a or b coincides with e_1, e_2, e_3 we may clearly replace dy/dt in these conditions by dy/dx.

† A question in which this is not the case is the one referred to in the footnote on p. 306.

‡ This is nothing but an analytical statement of the fact that the two lines $y = A_1 x + F(A_1)$ and $y = A_2 x + F(A_2)$ intersect within the infinite strip bounded by the ordinates erected at a and b. *Cf.* Reihenentwickelungen, p. 128, 1st line. The proof we here give is also nothing but an analytical statement of the proof there suggested.

XII. *$F(A)$ is continuous for every value of A.*

Let A_1 be the value of A for which we wish to prove $F(A)$ continuous. Then we must prove that no matter how small the positive quantity δ may be chosen we can find a positive quantity ε such that when

$$|A_2 - A_1| < \varepsilon \quad |F(A_2) - F(A_1)| < \delta.$$

Now consider the quantities $|a|$ and $|b|$ and call the largest of the two h. Then the last theorem shows that:

$$\frac{|F(A_2) - F(A_1)|}{|A_1 - A_2|} < h$$

and therefore if we let $\varepsilon = \delta / h$ the inequality we wish to establish is satisfied.*

We proceed now to four lemmas.

LEMMA 1. *It is possible to find a positive quantity M so large that when $A > M$ $Ab + F(A) > 0$.*

For instance we may let

$$M = \frac{2(n+1)^2\pi^2}{(b-a)\left[t\left(\frac{a+b}{2}\right) - t(a)\right]^2}.$$

For if there were a value of A greater than this value for which $Ab + F(A) \leqq 0$ we should have for this value of A:

$$\frac{-(n+1)^2\pi^2}{\left[t\left(\frac{a+b}{2}\right) - t(a)\right]^2} > A\left(\frac{a-b}{2}\right) \geqq A\left(\frac{a+b}{2}\right) + F(A);$$

and therefore when $\quad a \leqq x \leqq \dfrac{a+b}{2}$

i. e., when $\quad t(a) \leqq t \leqq t\left(\dfrac{a+b}{2}\right)$,

$$Ax + F(A) < -\frac{(n+1)^2\pi^2}{\left[t\left(\frac{a+b}{2}\right) - t(a)\right]^2}.$$

* We have here proved that $F(A)$ is *uniformly* continuous (since the value found for ε is independent of A) for all values of A. This would not follow from the mere continuity, since the interval in question is infinite.

Every solution of the equation :

$$\frac{d^2 y}{dt^2} = [Ax + F(A)]\, y$$

must therefore by Theorem V have at least $n + 1$ roots in the segment $t(a) < t < t(b)$ while by hypothesis one solution of this equation has only n roots in this segment.

LEMMA 2. *It is possible to find a positive quantity M so large that when $A < -M$ $\;Aa + F(A) > 0$.*

Letting

$$M = \frac{2(n+1)^2\pi^2}{(b-a)\left[t(b) - t\left(\frac{a+b}{2}\right)\right]^2}$$

the proof is precisely similar to the proof just given.

Up to this point we have placed no restriction on the constants α, α', β, β' except that neither α and α' nor β and β' shall both be zero. Now, however, in order to avoid all complication in the proofs and also because the cases to which we thus restrict ourselves are by far the most important *we will assume that either $\alpha = 0$ or $\alpha' = 0$ and either $\beta = 0$ or $\beta' = 0$*, *i. e.*, we assume $\alpha\alpha' = 0$ and $\beta\beta' = 0$. It is not hard to show that this restriction is not necessary for the truth of Theorem XIII (and therefore for the truth of Klein's theorem of oscillation) but the following two Lemmas would have to be slightly changed in form* and would be decidedly less easy to prove without this restriction.

LEMMA 3. *If $\alpha\alpha' = 0$ and $\beta\beta' = 0$ then when $A > 0$*

$$Aa + F(A) < 0.$$

For otherwise we should have when $a < x \leqq b$

$$Ax + F(A) > 0$$

and this is seen to be impossible from the Corollary to Theorem IV on account of the conditions at a and b.

LEMMA 4. *If $\alpha\alpha' = 0$ and $\beta\beta' = 0$ then when $A < 0$*

$$Ab + F(A) < 0.$$

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* The inequalities there given would then be true only when $A > M$ and $A < -M$ respectively when M is a certain positive quantity. They are, however, true as stated, even without the restrictions on α, α', β, β' when $n \geqq 2$, and here the proof given in the text applies.

For otherwise we should have when $a \leqq x < b$

$$Ax + F(A) > 0$$

and this is seen to be impossible as before.

XIII. *If* $\alpha\alpha' = 0$ *and* $\beta\beta' = 0$* *it is possible to find a positive quantity* M *such that when* $A > M$ *and also when* $A < -M$:

$$a < -\frac{F(A)}{A} < b.$$

This follows at once from the preceding Lemmas.

We have so far supposed the segment ab to lie either in the interval from e_1 to e_2 or in the interval from e_3 to $+\infty$. Consider now a segment ab where a and b satisfy one of the relations: $a < b \leqq e_1$ or $e_2 \leqq a < b \leqq e_3$. If then we take c so that in the first case $c \leqq e_1$ in the second case $e_2 \leqq c \leqq e_3$ the integral t will be pure imaginary throughout the segment ab. If we bring in the real variable τ by the relation $t = i\tau$ we get as the binomial form of the equation:

$$\frac{d^2y}{d\tau^2} = -(Ax + B)y.$$

The fact that $Ax + B$ has the negative instead of the positive sign will make some changes in the proofs of the foregoing propositions and in particular the statements of the four Lemmas will have to be somewhat changed, *but it is easy to see that Theorems X, XI, XII, XIII remain true if* τ *is substituted for* t *in them.*

We are now in a position to prove Klein's *Theorem of Oscillation* (in its simplest form):

XIV. *Let* $a_1 b_1$ *and* $a_2 b_2$ *be any two segments of the x-axis such that* $a_1 < b_1 \leqq a_2 < b_2$ *and such that none of the inequalities* $a_1 < e_i < b_1$ $(i = 1, 2, 3)$ *or* $a_2 < e_i < b_2$ $(i = 1, 2, 3)$ *are true, and let us assign to each of the points* a_1, b_1, a_2, b_2 *one of the conditions* $y = 0$ *or* $\frac{dy}{dt} = 0$; *and let* n_1 *and* n_2 *be any two integers positive or zero. Then there exists one and only one pair of values* A, B *for which Lame''s equation has a first solution which satisfies the prescribed conditions at* a_1 *and* b_1 *and has* n_1 *roots in the segment* $a_1 < x < b_1$, *and a second solution which satisfies the prescribed conditions at* a_2 *and* b_2 *and has* n_2 *roots in the segment* $a_2 < x < b_2$.

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* These restrictions, as above stated, are not essential to the truth, but merely to the proof here given of the theorem.

Let us call the values of B determined by applying Theorem X to the segments $a_1 b_1$ and $a_2 b_2$ respectively $F_1(A)$ and $F_2(A)$. Our theorem will be established if we can prove that there is one and only one value of A for which $F_1(A) = F_2(A)$.

That there cannot be more than one such value is obvious at once from Theorem XI. For if A_1 and A_2 are two such values we find by applying Theorem XI in succession to the segments $a_1 b_1$ and $a_2 b_2$:

$$a_1 < \frac{F_1(A_2) - F_1(A_1)}{A_1 - A_2} < b_1 \leqq a_2 < \frac{F_2(A_2) - F_2(A_1)}{A_1 - A_2} < b_2.$$

But these inequalities contain a contradiction, since the second and fifth terms are equal.*

In order to show that there does exist one value of A for which $F_1(A) = F_2(A)$ let us apply Theorem XIII in succession to the segment $a_1 b_1$ and $a_2 b_2$. We thus see that when A has a sufficiently large positive value:

$$a_1 < -\frac{F_1(A)}{A} < b_1 \leqq a_2 < -\frac{F_2(A)}{A} < b_2,$$

and therefore: $F_1(A) > F_2(A)$. On the other hand, when A is negative and its numerical value is sufficiently large we see in the same way that: $F_1(A) < F_2(A)$. Accordingly $F_1(A) - F_2(A)$ is positive when A has a large positive value and negative when A has a numerically large negative value and must, therefore, since by Theorem XII it is continuous for all values of A, be zero for some value of A.†

As has already been mentioned the theorem of oscillation still holds if for the simple conditions at a_1, b_1, a_2, b_2 we substitute more complicated ones of the form $a'y - a\, dy/dt = 0$, and in fact the proof of the negative part of the theorem applies immediately to this case as does also the proof of the positive part when $n_1 \geqq 2$ and $n_2 \geqq 2$; and even in the special cases in which n_1 or n_2 has the value 0 or 1 the proof requires no change when once Theorem XIII has been

*For a geometrical statement of this proof see Reihenentwickelungen, p. 130, second footnote

† Geometrically this proof means that since the line $y = Ax + B$ will, when sufficiently steep, cut the axis of x in the segment $a_1 b_1$ or $a_2 b_2$ according as one or the other of these segments is used to determine it, the line determined by $a_1 b_1$ will when A is large and positive liea bove, when A is numerically large and negative lie below, the line determined by $a_2 b_2$ and must therefore for some intermediate value of A coincide with it.

established for these cases. On the other hand the theorem of oscillation for other differential equations which like Lamé's involve two parameters* may be established by reasoning almost identical with that here used, the difference again coming in only in the four Lemmas. I hope soon to return to these and other similar questions.

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SOME EXAMPLES OF DIFFERENTIAL INVARIANTS.

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(Read before the American Mathematical Society at the Meeting of December 29, 1897.)

IN the following paper certain invariants for projective transformations are given. The derivation, according to Lie's methods, is given in full for the plane, and the method for the corresponding problem in space of three dimensions is sketched in, and the results of the solution are given. It is believed that all the invariants given are new.

For an infinitesimal point transformation of the xy plane x and y receive the increments

$$\delta x = \xi(x, y)\, \delta t, \qquad \delta y = \eta(x, y)\, \delta t,$$

respectively, where δt is an infinitesimal independent of x and y. This infinitesimal transformation is represented by the symbol

$$Xf = \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}.$$

The increment of any function $\varphi(x, y)$ is then

$$\delta\varphi = \frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y = \left(\frac{\partial\varphi}{\partial x}\xi + \frac{\partial\varphi}{\partial y}\eta\right)\delta t = X\varphi \cdot \delta t.$$

If, then, φ is to be invariant for the transformation Xf, we have as a necessary and sufficient condition $X\varphi = 0$. Lie

*For instance Lamé's generalized equation. See Reihenentwickelungen, p. 125.