established for these cases. On the other hand the theorem of oscillation for other differential equations which like Lamé's involve two parameters* may be established by reasoning almost identical with that here used, the difference again coming in only in the four Lemmas. I hope soon to return to these and other similar questions.

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SOME EXAMPLES OF DIFFERENTIAL INVARIANTS.

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IN the following paper certain invariants for projective transformations are given. The derivation, according to Lie's methods, is given in full for the plane, and the method for the corresponding problem in space of three dimensions is sketched in, and the results of the solution are given. It is believed that all the invariants given are new.

For an infinitesimal point transformation of the xy plane x and y receive the increments

$$\delta x = \xi (x, y)\, \delta t, \qquad \delta y = \eta (x, y)\, \delta t,$$

respectively, where δt is an infinitesimal independent of x and y. This infinitesimal transformation is represented by the symbol

$$Xf = \xi (x, y) \frac{\partial f}{\partial x} + \eta (x, y) \frac{\partial f}{\partial y}.$$

The increment of any function $\varphi (x, y)$ is then

$$\delta\varphi = \frac{\partial \varphi}{\partial x}\delta x + \frac{\partial \varphi}{\partial y}\delta y = \left(\frac{\partial \varphi}{\partial x}\xi + \frac{\partial \varphi}{\partial y}\eta\right)\delta t = X\varphi \cdot \delta t.$$

If, then, φ is to be invariant for the transformation Xf, we have as a necessary and sufficient condition $X\varphi = 0$. Lie

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*For instance Lamé's generalized equation. See Reihenentwickelungen, p. 125.

has further shown that if $\varphi(x, y)$ be invariant for the infinitesimal transformation Xf, that it is also invariant for the finite transformations defined by Xf. This follows at once from the fact that any one of these finite transformations changes $\varphi(x, y)$ into φ_1, where*

$$\varphi_1 = \varphi + \frac{t}{1!} X\varphi + \frac{t^2}{2!} X(X\varphi) + \frac{t^3}{3!} X[X(X\varphi)] + \cdots$$

If $X\varphi = 0$ this reduces to $\varphi_1 = \varphi$, so that $\varphi(x, y)$ is invariant for the finite transformation. When, therefore, we desire to find the invariants for the infinitesimal transformation Xf we have simply to solve the partial differential equation

$$\xi(x, y) \frac{\partial\varphi}{\partial x} + \eta(x, y) \frac{\partial\varphi}{\partial y} = 0.$$

In this case there is only one invariant, all others being functions of this one.

The transformation Xf transforms not only the points (x, y), but also every figure in the plane. Thus if we consider a curve, its points, slope, radius of curvature, etc., are transformed. The manner in which the points are transformed is given by Xf, and knowing how the individual points are transformed we can compute how

$$\frac{dy}{dx} = y', \quad \frac{d^2y}{dx^2} = y'', \cdots$$

are transformed. It is easily shown that †

$$\delta y' = \left[\frac{d\eta}{dx} - y' \frac{d\xi}{dx}\right]\delta t$$

$$= \left[\frac{\partial\eta}{\partial x} + y'\left(\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x}\right) - y'^2 \frac{\partial\xi}{\partial y}\right]\delta t = \eta_1(x, y, y')\delta t,$$

$$\delta y'' = \left[\frac{d\eta_1}{dx} - y'' \frac{d\xi}{dx}\right]\delta t = \eta_2(x, y, y', y'')\delta t.$$

The symbol

$$X'f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta_1 \frac{\partial f}{\partial y}$$

* Lie: Differentialgleichungen mit bekannten infinitesimalen Transformationen, herausgegeben von Scheffers, p. 58, Theorem 4, and p. 62.

† Lie: Differentialgleichungen, p. 272, p. 358.

shows then how x, y, and y' are transformed, and therefore how any function of them is transformed. This is known as the *once extended* transformation. Just as before the increment of $\varphi(x, y, y')$ is $X'\varphi \cdot \delta t$. Similarly, the *twice* extended transformation is

$$X''f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta_1 \frac{\partial f}{\partial y'} + \eta_2 \frac{\partial f}{\partial y''},$$

and $X''\varphi \cdot \delta t$ is the increment of $\varphi(x, y, y', y'')$. To find the *differential invariants* of the nth order for the transformation Xf we must extend Xf n times, and then the solutions of the partial differential equation $X^{(n)}f = 0$ are the desired invariants. In solving this equation $x, y, y', y'', \cdots, y^{(n)}$ are to be treated as independent variables. If the function is to be a simultaneous invariant for a number of transformations $X_1f, X_2f, \cdots, X_rf$, we have to seek the solutions of the simultaneous system of partial differential equations

$$X_1^{(n)}f = 0,\ X_2^{(n)}f = 0, \cdots, X_r^{(n)}f = 0.$$

Of course there may be no common solutions of this system of differential equations. In particular, if $X_1f, \cdots, X_rf$ are the infinitesimal transformations of a group with r parameters, the system of differential equations just mentioned is complete. The number of variables involved is $n + 2$, and therefore in this case there are at least $n + 2 - r$ common solutions of the r equations. If the equations are linearly dependent there are more solutions, there being $n + 2 - r + k$, when the equations are connected by k linear relations.

If we consider a configuration involving a number of points $(x_1, y_1), (x_2, y_2), \cdots$, instead of only one point, we must write instead of Xf the following:

$$\xi(x_1 y_1) \frac{\partial f}{\partial x_1} + \eta(x_1 y_1) \frac{\partial f}{\partial y_1} + \xi(x_2 y_2) \frac{\partial f}{\partial x_2} + \eta(x_2 y_2) \frac{\partial f}{\partial y_2} + \cdots$$

To find the differential invariants $\varphi(x_1, y_1, x_2, y_2, y_1', y_2', y_1'', y_2'', \cdots)$ this transformation must be extended just as before, and the solutions sought of the partial differential equation obtained by equating this extended transformation to zero.

We have now stated the principles required for the solution of the following problem:

Given two curves in the xy plane, with a point on each. The plane is subjected to the general projective transformation. Required the differential invariants of the second order.

It is well known that the general projective group in the plane has eight parameters, and that the eight independent infinitesimal transformations may be written in the form*

$$X_1f=\frac{\partial f}{\partial x},\ X_2f=\frac{\partial f}{\partial y},\ X_3f=x\frac{\partial f}{\partial x},\ X_4f=x\frac{\partial f}{\partial y},\ X_5f=y\frac{\partial f}{\partial x},$$

$$X_6f=y\frac{\partial f}{\partial y},\quad X_7f=x^2\frac{\partial f}{\partial x}+xy\frac{\partial f}{\partial y},\quad X_8f=xy\frac{\partial f}{\partial x}+y^2\frac{\partial f}{\partial y}.$$

When we consider the differential invariants of the second order for two points, we have eight variables, viz.

$$x_1,\ y_1,\ y_1',\ y_1'',\ x_2,\ y_2,\ y_2',\ y_2''.$$

Therefore by extending each of the transformations

$$X_1f, \cdots, X_8f$$

twice we obtain a complete system of eight equations involving eight variables. The transformations are extended by means of the formulæ

$$\delta y' = \left[\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\right]\delta t = \eta_1\delta t,$$

$$\delta y'' = \left[\frac{d\eta_1}{dx} - y''\frac{d\xi}{dx}\right]\delta t = \eta_2\delta t.$$

For example, for X_8f we have, $\xi = xy$, $\eta = y^2$, and

$$\eta_1 = 2yy' - y'(xy'+y) = yy' - xy'^2, \quad \eta_2 = -3xy'y''.$$

The complete system of partial differential equations is thus found to be :

$$0 = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2},$$

$$0 = \frac{\partial f}{\partial y_1} + \frac{\partial f}{\partial y_2},$$

$$0 = x_1\frac{\partial f}{\partial x_1} - y_1'\frac{\partial f}{\partial y_1'} - 2y_1''\frac{\partial f}{\partial y_1''} + x_2\frac{\partial f}{\partial x_2} - y_2'\frac{\partial f}{\partial y_2'} - 2y_2''\frac{\partial f}{\partial y_2''},$$

$$0 = x_1\frac{\partial f}{\partial y_1} + \frac{\partial f}{\partial y_1'} + x_2\frac{\partial f}{\partial y_2} + \frac{\partial f}{\partial y_2'},$$

* Lie : Continuierliche Gruppen, herausgegeben von Scheffers, p. 26.

$$0 = y_1 \frac{\partial f}{\partial x_1} - y_1'^2 \frac{\partial f}{\partial y_1'} - 3\, y_1' y_1'' \frac{\partial f}{\partial y_1''}$$

$$+ y_2 \frac{\partial f}{\partial x_2} - y_2'^2 \frac{\partial f}{\partial y_2'} - 3\, y_2' y_2'' \frac{\partial f}{\partial y_2''},$$

$$0 = y_1 \frac{\partial f}{\partial y_1} + y_1' \frac{\partial f}{\partial y_1'} + y_1'' \frac{\partial f}{\partial y_1''} + y_2 \frac{\partial f}{\partial y_2} + y_2' \frac{\partial f}{\partial y_2'} + y_2'' \frac{\partial f}{\partial y_2''},$$

$$0 = x_1^2 \frac{\partial f}{\partial x_1} + x_1 y_1 \frac{\partial f}{\partial y_1} + (y_1 - x_1 y_1') \frac{\partial f}{\partial y_1'} - 3\, x_1 y_1'' \frac{\partial f}{\partial y_1''}$$

$$+ x_2^2 \frac{\partial f}{\partial x_2} + x_2 y_2 \frac{\partial f}{\partial y_2} + (y_2 - x_2 y_2') \frac{\partial f}{\partial y_2'} - 3\, x_2 y_2'' \frac{\partial f}{\partial y_2''},$$

$$0 = x_1 y_1 \frac{\partial f}{\partial x_1} + y_1^2 \frac{\partial f}{\partial y_1} + y_1' (y_1 - x_1 y_1') \frac{\partial f}{\partial y_1'} - 3\, x_1 y_1' y_1'' \frac{\partial f}{\partial y_1''}$$

$$+ x_2 y_2 \frac{\partial f}{\partial x_2} + y_2^2 \frac{\partial f}{\partial y_2} + y_2' (y_2 - x_2 y_2') \frac{\partial f}{\partial y_2'} - 3\, x_2 y_2' y_2'' \frac{\partial f}{\partial y_2''}.$$

If these eight equations are independent there are no common solutions, and consequently no invariant of the second order. It is found however, that the determinant of the coefficients vanishes identically, that is:

$$\begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
x_1 & 0 & -y_1' & -2y_1'' & x_2 & 0 & -y_2' & -2y_2'' \\
0 & x_1 & 1 & 0 & 0 & x_2 & 1 & 0 \\
y_1 & 0 & -y_1'^2 & -3y_1'y_1'' & y_2 & 0 & -y_2'^2 & -3y_2'y_2'' \\
0 & y_1 & y_1' & y_1'' & 0 & y_2 & y_2' & y_2'' \\
x_1^2 & x_1y_1 & y_1-x_1y_1' & -3x_1y_1'' & x_2^2 & x_2y_2 & y_2-x_2y_2' & -3x_2y_2'' \\
x_1y_1 & y_1^2 & y_1'(y_1-x_1y_1') & -3x_1y_1'y_1'' & x_2y_2 & y_2^2 & y_2'(y_2-x_2y_2') & -3x_2y_2'y_2''
\end{vmatrix} \equiv 0$$

Therefore the equations are connected by a linear relation. Moreover, all the seven rowed determinants do not vanish identically; for instance, the upper left hand corner one is equal to

$$y_1'' \, [y_2 - y_1 - y_1' (x_2 - x_1)] \; [y_2 - y_1 - y_2' (x_2 - x_1)]$$

$$[y_2 - y_1 - y_1' (x_2 - x_1) + y_2 - y_1 - y_2' (x_2 - x_1)].$$

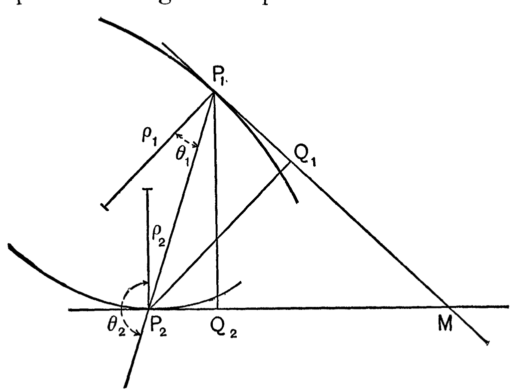
Seven of the equations are therefore linearly independent, and consequently there is one and only one common solution of this system of partial differential equations. By the method of finding all the solutions of one of the equations, and substituting them as new independent variables in the remaining equations, and repeating the process, this one solution is readily shown to be

$$f = \frac{y_1''}{y_2''}\left[\frac{y_2 - y_1 - y_2'(x_2 - x_1)}{y_2 - y_1 - y_1'(x_2 - x_1)}\right]^3$$

This is the only invariant of the second order of two points, for the general projective transformation of the plane. To interpret this geometrically, we have:

$$\frac{[y_2 - y_1 - y_2'(x_2 - x_1)]^3}{y_2''} = \frac{(1 + y_2')^{\frac{3}{2}}}{y_2''}\left[\frac{y_2 - y_1 - y_2'(x_2 - x_2)}{(1 + y_2')^{\frac{1}{2}}}\right]^3$$
$$= \rho_1 \cdot \overline{P_1 Q_1}^3,$$

where ρ_2 is the radius of curvature at (x_2, y_2), or P_2, of the curve through this point, and $\overline{P_1 Q_2}$ is the length of the perpendicular let fall from (x_1, y_1), or P_1, to the tangent at (x_2, y_2). Let the tangents at P_1 and P_2 intersect at M (see figure), θ_1 be the angle at P_1 between the normal to the



curve at that point and the line $P_1 P_2$, and θ_2 the corresponding angle at P_2. Then the invariant is

$$\frac{\rho_2}{\rho_1}\left(\frac{P_1 Q_2}{P_2 Q_1}\right)^3 = \frac{\rho_2}{\rho_1}\left(\frac{P_1 M}{P_2 M}\right)^3 = \frac{\rho_2 \cos^3 \theta_2}{\rho_1 \cos^3 \theta_1}.$$

Discussion of the Invariant for a Special Case.

Let us suppose that the two points lie on the same curve, and that the invariant has a constant value however the points be chosen. Interchanging the two points the invariant takes on the reciprocal value, hence its value must be ± 1. Now let the points be taken infinitely near to each other. Then, dropping the subscripts for the point (x_1, y_1), we have

$$x_2 = x + \delta x,$$

$$y_2 = y + y'\delta x + \frac{y''}{2!}\delta x^2 + \frac{y'''}{3!}\delta x^3 + \frac{y^{\text{iv}}}{4!}\delta x^4 + \frac{y^{\text{v}}}{5!}\delta x^5 + \cdots,$$

$$y_2' = y' + y''\delta x + \frac{y'''}{2!}\delta x^2 + \frac{y^{\text{iv}}}{3!}\delta x^3 + \frac{y^{\text{v}}}{4!}\delta x^4 + \cdots,$$

$$y_2'' = y'' + y'''\delta x + \frac{y^{\text{iv}}}{2!}\delta x^2 + \frac{y^{\text{v}}}{3!}\delta x^3 + \cdots,$$

and
$$\frac{y_2''}{y''}\left[\frac{y_2 - y - y'(x_2 - x)}{y_2 - y - y_2'(x_2 - x)}\right]^3$$

$$= -1 - \frac{\delta x^3}{y''^3}\left[9y^{\text{v}}y'' - 45y''y'''y^{\text{iv}} + 40y'''^3\right] + \cdots$$

This shows that the invariant cannot have the constant value $+1$; if it has the constant value -1 the two points must lie on the curve whose differential equation is

$$9y^{\text{v}}y'' - 45y''y'''y^{\text{iv}} + 40y'''^3 = 0,$$

that is, they must lie on a conic. Conversely, if the two points do lie on a conic the value of the invariant is easily shown to be -1. The case in which the conic degenerates into a pair of right lines has no interest, as the invariant then becomes indeterminate. We have, therefore:

If two points lie on a curve and the invariant

$$\frac{\rho_2 \cos^3\theta_2}{\rho_1 \cos^3\theta_1}$$

has a constant value for all positions of the points on the curve, then this constant value must be -1, *and the curve is an undegenerate conic.*

That is, we have for a conic

$$\rho_1 \cos^3\theta_1 + \rho_2 \cos^3\theta_2 = 0.$$

This latter result may also be deduced as a special case of a general theorem due to Reiss.* Let an algebraic curve of the nth degree be cut by any right line in the points $M_1, M_2, \cdots, M_n$. Denote the radius of curvature at the point M_i by ρ_i, and the angle between the normal at M_i and the transversal by θ_i. Then the formula referred to is

$$\sum_1^n{}_i \frac{1}{\rho_i \cos^3\theta_i} = 0.$$

*Correspondance mathématique et physique de Quetelet, vol IX., p. 152.

When the curve is a conic this evidently reduces to

$$\rho_1 \cos^3 \theta_1 + \rho_2 \cos^3 \theta_2 = 0.$$

That, in general, $\dfrac{\rho_2 \cos^3 \theta_2}{\rho_1 \cos^3 \theta_1}$ is invariant for all projective transformations of the plane, whether the two points be on the same curve or different curves, seems to have been hitherto unnoticed.

Analogous Invariant for Surfaces.

In space of three dimensions consider two surfaces, and a point on each. Let the surfaces be subjected to the general projective transformation, and let us seek the differential invariants of the second order. The general projective group in three dimensions has fifteen infinitesimal transformations. If we write

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2},$$

the invariant will be a function of $x_1, y_1, z_1, p_1, q_1, r_1, s_1, t_1$, and $x_2, \cdots, t_2$; that is, it will involve sixteen variables. Each of the fifteen infinitesimal transformations must then be twice extended. If the point transformation be

$$Xf = \xi\,(x, y, z)\frac{\partial f}{\partial x} + \eta\,(x, y, z)\frac{\partial f}{\partial y} + \zeta\,(x, y, z)\frac{\partial f}{\partial x},$$

the twice extended transformation is

$$X''f = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z} + \varphi\frac{\partial f}{\partial p} + \psi\frac{\partial f}{\partial q}$$
$$+ \rho\frac{\partial f}{\partial r} + \sigma\frac{\partial f}{\partial s} + \tau\frac{\partial f}{\partial t},$$

where we have,

$$\varphi = \frac{\partial \zeta}{\partial x} + p\frac{\partial \zeta}{\partial z} - p\left(\frac{\partial \xi}{\partial x} + p\frac{\partial \xi}{\partial z}\right) - q\left(\frac{\partial \eta}{\partial x} + p\frac{\partial \eta}{\partial z}\right),$$

$$\psi = \frac{\partial \zeta}{\partial y} + q\frac{\partial \zeta}{\partial z} - p\left(\frac{\partial \xi}{\partial y} + q\frac{\partial \xi}{\partial z}\right) - q\left(\frac{\partial \eta}{\partial y} + q\frac{\partial \eta}{\partial z}\right),$$

$$\rho = \frac{\partial \varphi}{\partial x} + p\frac{\partial \varphi}{\partial z} + r\frac{\partial \varphi}{\partial p} + s\frac{\partial \varphi}{\partial q}$$
$$- r\left(\frac{\partial \xi}{\partial x} + p\frac{\partial \xi}{\partial z}\right) - s\left(\frac{\partial \eta}{\partial x} + p\frac{\partial \eta}{\partial z}\right),$$

$$\sigma = \frac{\partial \varphi}{\partial y} + q\frac{\partial \varphi}{\partial z} + s\frac{\partial \varphi}{\partial p} + t\frac{\partial \varphi}{\partial q}$$

$$- r\left(\frac{\partial \xi}{\partial y} + q\frac{\partial \xi}{\partial z}\right) - s\left(\frac{\partial \eta}{\partial y} + q\frac{\partial \eta}{\partial z}\right),$$

or
$$\sigma = \frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial z} + r\frac{\partial \psi}{\partial p} + s\frac{\partial \psi}{\partial q}$$

$$- s\left(\frac{\partial \xi}{\partial x} + p\frac{\partial \xi}{\partial z}\right) - t\left(\frac{\partial \eta}{\partial x} + p\frac{\partial \eta}{\partial z}\right),$$

$$\tau = \frac{\partial \psi}{\partial y} + q\frac{\partial \psi}{\partial z} + s\frac{\partial \psi}{\partial p} + t\frac{\partial \psi}{\partial q}$$

$$- s\left(\frac{\partial \xi}{\partial y} + q\frac{\partial \xi}{\partial z}\right) - t\left(\frac{\partial \eta}{\partial y} + q\frac{\partial \eta}{\partial z}\right)$$

Using these formulas to extend the group, we obtain a complete system of 15 partial differential equations in 16 variables. The equations are, however, *not independent*, there being one relation between them. They have therefore $16 - 15 + 1 = 2$ solutions, and therefore there are two invariants of the second order. One of these is at once deducible geometrically. At each of the points draw the two principal tangents to the surface. Pass a plane through the line joining the two points, and each of these tangents. Then, since principal tangents are carried over by a projective transformation into principal tangents, the anharmonic ratio of these four planes must be invariant. This first invariant, which we shall call I_1, has no great interest. The second invariant may also be derived geometrically, by making use of the invariant found for plane curves, but the process is very long. It is much shorter to actually integrate the system of fifteen differential equations above mentioned. We find that

$$I_2 = \frac{s_1^2 - r_1 t_1}{s_2^2 - r_2 t_2}\left[\frac{z_2 - z_1 - p_2(x_2 - x_1) - q_2(y_2 - y_1)}{z_2 - z_1 - p_1(x_2 - x_1) - q_1(y_2 - y_1)}\right]^4$$

is the second solution. The geometrical meaning of this is that

$$I_2 = \frac{R_1 R_1' \cos^4 \theta_1}{R_2 R_2' \cos^4 \theta_2}$$

is invariant for a projective transformation, where R_1, R_1 are the principal radii of curvature at (x_1, y_1, z_1), and θ_1 is the angle between the normal at (x_1, y_1, z_1) and the line joining (x_1, y_1, z_1) with (x_2, y_2, z_2) ; and R_2, R'_2 and θ_2 are the corresponding quantities at (x_2, y_2, z_2).

If the two points lie on one surface, and the invariant I_2 have a constant value however the points be chosen on this surface, then $I_2 = +1$. If the surface be a quadric surface which is not a cone, it is easily shown that $I_2 = 1$.

The analytical expression for the first invariant, I_1, is the following:

$$I_1 = \frac{\left\{\begin{array}{l} (z_2 - z_1)^2 (r_1 t_2 + r_2 t_1 - 2 s_1 s_2) \\ + 2(z_2 - z_1)(x_2 - x_1)[(p_1 + p_2) s_1 s_2 - p_1 r_2 t_1 - p_2 r_1 t_2 \\ \qquad + (q_2 - q_1)(r_1 s_2 - r_2 s_1)] \\ + 2(z_2 - z_1)(y_2 - y_1)[(q_1 + q_2) s_1 s_2 - q_1 r_1 t_2 - q_2 r_2 t_1 \\ + (p_2 - p_1)(s_2 t_1 - s_1 t_2)] + (x_2 - x_1)^2 [p_1^2 r_2 t_1 + p_2^2 r_1 t_2 \\ - 2 p_1 p_2 s_1 s_2 + 2(q_2 - q_1)(p_1 r_2 s_1 - p_2 r_1 s_2) + (q_2 - q_1)^2 r_2 r_1] \\ + 2(x_2 - x_1)(y_2 - y_1)\left[\begin{array}{l} (p_1 q_2 + p_2 q_1 - 2 p_1 q_1 - 2 p_2 q_2) s_1 s_2 \\ + p_1 q_2 r_2 t_1 + p_2 q_1 r_1 t_2 \end{array}\right. \\ \qquad \left. \begin{array}{l} + (p_2 - p_1)(p_2 s_1 t_2 - p_1 s_2 t_1) \\ + (q_2 - q_1)(q_2 r_2 s_1 - q_1 r_1 s_2) \end{array}\right] \\ \qquad + (y_2 - y_1)^2 [q_1^2 r_1 t_2 + q_2^2 r_2 t_1 - 2 q_1 q_2 s_1 s_2 \\ \qquad + 2(p_2 - p_1)(q_1 s_1 t_2 - q_2 s_2 t_1) + (p_2 - p_1)^2 t_1 t_2] \end{array}\right\}}{\begin{array}{c} [z_2 - z_1 - p_1 (x_2 - x_1) - q_1 (y_2 - y_1)] \sqrt{(s_1^2 - r_1 t_1)} \\ [z_2 - z_1 - p_2 (x_2 - x_1) - q_2 (y_2 - y_1)] \sqrt{(s_2^2 - r_2 t_2)} \end{array}}$$

This is *not* the anharmonic ratio of the four planes mentioned above, but that ratio is

$$-\tfrac{1}{4} I_1 + \tfrac{1}{2}.$$

This anharmonic ratio is readily computed when we notice that the anharmonic ratio of four planes, all passing through two common points (a_1, b_1, c_1) and (a_2, b_2, c_2), and one plane through each of the four points $(x_1, y_1, z_1), \cdots, (x_4, y_4, z_4)$ is

$$\frac{D_{12} D_{34}}{D_{14} D_{32}}$$

where

$$D_{ik} = \begin{vmatrix} a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ x_i & y_i & z_i & 1 \\ x_k & y_k & z_k & 1 \end{vmatrix}$$

If the two points lie on a quadric (not a cone) we find $I_1 = 2$, so that the anharmonic ratio of the four planes is zero. The two pairs of planes reduce to a single pair for quadric surfaces, as is also evident from geometrical considerations.

LEIPZIG,
December 7, 1897.