where

$$\alpha = \sqrt{b^2 - c^2}, \quad \beta = \sqrt{c^2 - a^2}, \quad \gamma = \sqrt{a^2 - b^2}.$$

Transform to a new system of coördinates x_1, x_2, x_3, x_4, by means of the equations

$$x_1 = (c - b)\left[a\alpha\beta x - (b\beta^2 - c\gamma^2)y + iac\alpha\gamma w\right],$$
$$x_2 = (b + c)\left[a\alpha\beta x - (b\beta^2 + c\gamma^2)y + iac\alpha\gamma w\right],$$
$$x_3 = (a + c)\left[a\alpha x - b\beta y + c\gamma z\right],$$
$$x_4 = (c - a)\left[a\alpha x - b\beta y - c\gamma z\right].$$

With respect to this new system, the coördinates of the six chosen nodes are (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (e_1, e_2, e_3, e_4), (e_2, e_1, e_4, e_3), where

$$e_1 = \beta\,(b^2 - c^2)\,(c - a), \quad e_3 = (a^2 - c^2)\,(c + b),$$
$$e_2 = \beta\,(b^2 - c^2)\,(c + a), \quad e_4 = (a^2 - c^2)\,(c - b).$$

These six points form an involution of the kind described. (See *Annals of Mathematics*, vol. 11, p. 159.)

NOTE ON INTEGRATING FACTORS.

BY MR. PAUL SAUREL.

(Read before the American Mathematical Society at the Meeting of February 26, 1898.)

If the differential equation

$$X_1dx_1 + X_2dx_2 + \cdots + X_ndx_n = 0, \quad n \gtreqless 3, \qquad \text{(A)}$$

be integrable, and if u = constant be the integral of this equation, then, as is well known, there exists a function M such that

$$du \equiv MX_1dx_1 + MX_2dx_2 + \cdots + MX_ndx_n.$$

And as

$$\frac{\partial u}{\partial x_1} \equiv MX_1, \quad \frac{\partial u}{\partial x_2} \equiv MX_2, \cdots \frac{\partial u}{\partial x_n} \equiv MX_n,$$

and
$$\frac{\partial^2 u}{\partial x_r \partial x_s} \equiv \frac{\partial^2 u}{\partial x_s \partial x_r},$$

M must satisfy the conditions

$$\frac{\partial}{\partial x_r}\left(MX_s\right) \equiv \frac{\partial}{\partial x_s}\left(MX_r\right). \qquad (1)$$

The object of this note is to establish the conditions for the existence of an integrating factor which shall contain but one of the variables, and to show that such an integrating factor, if it exists, is unique.

If M be a function of only one of the variables, x_1 for example, the conditions (1) yield the $n-1$ equations

$$M\frac{\partial X_1}{\partial x_r} = M\frac{\partial X_r}{\partial x_1} + X_r\frac{dM}{dx_1}, \qquad r = 2, 3, \cdots, n;$$

or, in better form,

$$\frac{\dfrac{\partial X_1}{\partial x_r} - \dfrac{\partial X_r}{\partial x_1}}{X_r} = \frac{\dfrac{dM}{dx_1}}{M}, \quad r = 2, 3, \cdots, n; \qquad (\text{I})$$

and the $\frac{1}{2}(n-1)(n-2)$ equations

$$M\frac{\partial X_s}{\partial x_r} = M\frac{\partial X_r}{\partial x_s},$$

or
$$\frac{\partial X_s}{\partial x_r} = \frac{\partial X_r}{\partial x_s}, \quad r \neq s; \quad r, s = 2, 3, \cdots, n. \qquad (\text{II})$$

In equations (I) the right-hand member is a function of x_1 alone, and the left-hand members are therefore all equal to the same function of x_1.

Thus the conditions necessary for the existence of an integrating factor which shall be a function of x_1 alone, are that the left-hand members of equations (I) shall be identically equal to one and the same function of x_1, and that equations (II) shall hold identically,— $\frac{1}{2}n(n-1)$ conditions in all.

These are also the sufficient conditions. For, if they be satisfied, put
$$\frac{\dfrac{dM}{dx_1}}{M}$$

equal to one of the equal expressions

$$\frac{\dfrac{\partial X_1}{\partial x_r} - \dfrac{\partial X_r}{\partial x_1}}{X_r}, \qquad r = 2, 3, \cdots, n,$$

and integrate; then is M, except as to an arbitrary constant factor, uniquely determined. And the function thus determined is an integrating factor, for the conditions (1) are all satisfied in virtue of equations (I) and (II).

If the proof given above be applied to the case of an equation containing two variables, it will show that the necessary and sufficient condition for an integrating factor containing x_1 alone is that

$$\frac{\dfrac{\partial X_1}{\partial x_2} - \dfrac{\partial X_2}{\partial x_1}}{X_2} = f_1(x_1).$$

If at the same time the condition for an integrating factor containing x_2 alone be satisfied, viz:

$$\frac{\dfrac{\partial X_2}{\partial x_1} - \dfrac{\partial X_1}{\partial x_2}}{X_1} = f_2(x_2),$$

then

$$\frac{X_1}{f_1(x_1)} = -\frac{X_2}{f_2(x_2)},$$

and the differential equation may be put into the form

$$f_1(x_1)dx_1 - f_2(x_2)dx_2 = 0$$

in which the variables are separate.

The uniqueness of the integrating factor containing but one of the variables may also be obtained very simply from the known theorem that the ratio of any two integrating factors is an integral of the differential equation. For, since this ratio contains all the variables, if only one of the variables appears in the first integrating factor, the second must contain all the remaining variables, and may contain all the variables.

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