The last two chapters of the book, on cyclifying surfaces and evolutoids, have not undergone much change.

The work seems to be unusually free from serious misprints or inaccuracies. The following corrections might perhaps be worth mentioning : p. 16, footnote : Dr. Kneser's first paper appeared in Vol. XXXI (not XXI). the second will be found in Vol. XXXIV, at p. 204 (not XXIV, p. 506), of the *Mathematische Annalen* ; p. 61, end of §2 : the last expression for tan μ is obviously wrong, the last but one should have in the denominator ds instead of $d\sigma$; p. 91: the proof of the relation $A' = B + (\frac{1}{2}\pi - \alpha)$ seems unduly long, as the triangle APB (Fig. 34), all of whose sides are infinitesimal of the same order while P is a right angle, gives at once $\frac{1}{2}\pi + B$ for the exterior angle $\alpha + A'$ at A ; p. 110, l. 11 from foot of page : read $= 1$ for $- 1$, and in the numerator of α read $- 1/r$ for $1/r$; p. 137, l. 15 from top : for Hauptnormalen read Binormalen.

The concluding remarks of the work give an interesting outlook on problems awaiting solution in the geometrical theory of tortuous curves. Coming as they do from one who has made a special study of the subject they will be read with great interest by all workers in this field.

ALEXANDER ZIWET.

UNIVERSITY OF MICHIGAN,
February, 26, 1898.

PAGE'S DIFFERENTIAL EQUATIONS.

Ordinary Differential Equations. An elementary text-book, with an introduction to Lie's theory of the group of one parameter. By JAMES MORRIS PAGE, PH.D., Adjunct Professor of Mathematics, University of Virginia. The Macmillan Company, New York, 1897. 12mo, xviii + 226 pp.

This little volume is what it purports to be,—an elementary text-book with an introduction to Lie's elementary methods of integration as applied to ordinary differential equations. The contents fall into twelve chapters devoted to the following subjects in order : I. Genesis of the ordinary differential equation in two variables, pp. 1–9 ; II. The simultaneous system and the equivalent linear partial differential equation, pp. 10–24 ; III. The fundamental theorems of Lie's theory of the group of one parameter, pp. 25–59 ; IV. Connection between Euler's integrating factor and Lie's infinitesimal transformation, pp. 62–97 ; V. Geometrical applications of the integrating factor, orthogonal trajectories and isothermal systems, pp. 100–107 ; VI. Differential equations of the first order but not of the first degree, singular solutions, pp. 109–118 ; VII. Riccati's equation and Clairaut's equation, pp. 119–131 ; VIII. Total differential equations of the first order and degree in three variables which are derivable from a single primitive, pp. 132–139 ; IX. Ordinary differential equations of the second order in two variables, pp. 140–162 ; X. Ordinary differential equations of the mth order in two variables, pp. 164–174 ; XI. The general linear differential equation in two variables, pp. 175–189 ; XII. Methods for the integration of the simultaneous system, pp. 201–211.

The author finds himself under the usual obligations which the maker of an elementary text-book has to face ; these he acknowledges in the preface with a care that ought to satisfy the most scrupulous, but throughout the text one misses a number of references to Lie's lectures on differential equations and to the treatises of Boole and Forsyth which would have been helpful to both student and instructor in their supplementary reading. The few direct references to Lie's early theorems are the citations to the original memoirs in the Verhandlungen of the Christiania Scientific Society as given in Lie's published lectures ; the latter are accessible to American readers while the former are not ; double references might not have been amiss in these cases.

The arrangement of the material is a natural one and the exposition is clear. A detailed account of the subject matter is hardly called for here. That which is of most interest is the portion of the work devoted to Lie's theory of the group of one parameter. This part of the work, with the exception of the treatment of singular solutions mentioned below, is drawn from Lie's lectures on differential equations.* In this section of the book the author follows Lie not only in matter but also in method ; this faithfulness to the original enhances the value of the book to the student who desires to go more deeply into Lie's theory of continuous groups. The presentation is largely geometrical. Numerous illustrative examples are interspersed throughout the text. Here it might be mentioned that each chapter is followed by a list of problems ; answers to these are collected together at the end of the book just

* Lie : Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen, bearbeitet und herausgegeben von Scheffers, Leipzig, 1891.

before a well-ordered index. The material due exclusively to Lie is that of chapters III–V and IX–XII inclusive. The theorems and processes made use of in these chapters were discussed somewhat at length in a recent review of Lie's differential equations in the BULLETIN.*

It might have been well to give the complete proof of Lie's fundamental theorem relative to the equivalence of the notions infinitesimal transformation and one parameter group. The proof requires an elementary theorem of the theory of functions, but the student can well afford to make the acquaintance of this theorem at the stage when he is reading this book. The existence theorem is assumed and this other more elementary theorem of the theory of functions might have been assumed in the same connection. This fundamental equivalence theorem is left half demonstrated when it might have been put in a form thoroughly intelligible to beginners.

The notion lineal element in the sense used by Lie is of course introduced, but one questions why the notion element association of lineal elements is not brought into play. The latter is one of the most useful of the geometrical notions of Lie and its rôle could have been indicated on a single page.

The value of the book would have been largely increased had the author given footnotes indicative of the historical relations of the various subjects studied. Footnotes of another kind would also have been especially helpful, namely those indicating limitations, extensions and connections. For example, the student learns that a differential equation of the first order admits of an indefinite number of infinitesimal transformations, but in the study of differential equations of the second order no reference is made to the corresponding theorems : 1° that an ordinary differential equation of the second order admits at most of eight independent infinitesimal point transformations ; 2° that every differential equation of the second order can be transformed into every other differential equation of that order by contact transformations. In fact, the student is given no hint as to the existence of a contact transformation, not even in the most opportune place when finding what is technically known as the extension of a point transformation. These extended point transformations are the most special form of contact transformations ; five lines would have sufficed to introduce the general notion at this point. On the other

*Second series, vol. IV., No. 4, January, 1898. Lie's Differential Equations, pp. 155–167.

hand an admirable and noteworthy feature of Page's book is that no opportunity is lost to indicate where it is possible to extend given theorems in two and three variables to n variables.

No one detail of the subject of ordinary differential equations has been the victim of such unsatisfactory and cumbersome treatment as that of singular solutions. The method of the book before us offers relief. In three pages the author has developed a very neat method of finding the singular solution of an ordinary differential equation when such solutions exist. This method for determining singular solutions is a beautiful one from a theoretical standpoint, but its practical value is impaired because of the fact that the problem of finding the group of which a given differential equation admits is itself a problem of integration not capable of general solution. The author fails to call attention to certain elementary properties of singular solutions which the method may be made to yield in a simple manner. Of these, for example, are 1° every singular solution satisfies an infinite number of differential equations; 2° the existence of a singular solution depends upon the form of the equation. *

The translation of Lie's terminology has found in most cases fortunate English equivalents. The spirit of the original is certainly preserved in adopting the term pathcurve for Bahncurve, to name a single instance. Differential equations of the different orders which are invariant under known one parameter groups are designated as Lie's differential equations. This is a graceful way of recognizing permanently the creator of the theory of continuous groups; but with greater propriety the groups themselves could have been called Lie groups and this too in greater harmony with existing designations, putting the Lie groups of the transformation theory of differential equations alongside the Galois groups of the substitution theory of algebraic equations.

Page's book is intended for beginners. It is a pioneer in English. Its author was one of Lie's earliest pupils. Its object is a worthy one and merits the appreciation of American students of mathematics. The technicalities touched

* It may be of interest to note here that Taylor seems to have been the first to observe that a differential equation may have solutions not comprised in the general integral. He remarked this in 1715 by finding that $y^2 = 1 + x^2$ is a singular solution of $y^2 - 2xyy' + (1 + x^2)y'^2 - 1 = 0$. See pp. 270 and 360 of Frenet's Recueil d'exercices sur le calcul infinitesimal, quatrième édition, 1882.

upon here with a critical pen are to be regarded as trivial when compared with the work as a whole. In the hands of a teacher who will supplement the exercises of the text with more of those illustrative of the older methods on the one hand, and who will drill his students vigorously in reckoning with Lie's fundamental operator, the infinitesimal transformation, until they acquire facility in forming commutators,* on the other, the book is fully capable of realizing its double object of introducing the beginner† to two of the widest and most fruitful domains of mathematics—the theory of continuous groups and the theory of differential equations.

The mechanical make up of the book is up to the standard of the Macmillan press. A few trivial typographical errors will not escape the careful reader; the most flagrant probably is the occurrence of three in two consecutive lines of the preface.

The list of mathematical text-books in English is being continually and wisely augmented by the publications of The Macmillan Company, but with all that is good in the new, American students would welcome re-edited reprints of the classic volumes of Boole in the field of differential equations.

EDGAR ODELL LOVETT.

PRINCETON, N. J.,
24 *February*, 1898.

SHORTER NOTICES.

Annuaire pour l' An 1898, publié par le Bureau des Longitudes. Paris, Gauthier-Villars et Fils.

One or two alterations which have been made in the *Annuaire* for the current year call for mention. A few of the more technical astronomical details concerning variable and double stars and small planets have been omitted, while the

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* This term has been suggested as the equivalent of Lie's *Klammerausdruck* to represent the operation $(U_1U_2)f \equiv U_1(U_2f) - U_2(U_1f)$, where U_1f and U_2f are two infinitesimal transformations; its use would have spared numerous circumlocutions in the text.

† The reader will observe that in the above opinion the writer takes exception to one advanced in *Nature* (current volume, 10 February, 1898), where the usefulness of Page's book as an introductory volume for beginners is questioned. No text-book can supply both text and teacher. In the hands of an instructor who is alive to both sides of the subject the book is susceptible of successful application to the needs of those studying the subject for the first time.