BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY

THE FIFTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Fifth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY was held at the Massachusetts Institute of Technology, Boston, Mass., on Friday and Saturday, August 19 and 20, 1898. The Society renewed this year its usual policy of affiliation, so far as concerns the Summer Meeting, with the American Association for the Advancement of Science. This arrangement presents several advantages and involves no sacrifice of independence on either side. A more satisfactory and efficient condition would result if the Association should at some time decide to resolve itself into a federation of scientific societies, a policy which has a strong support. Meanwhile very cordial relations exist between the Society and Section A of the Association. Members of either body were free to attend and participate in the scientific proceedings of the other. The date of the Society's meeting was so chosen as to avoid conflict with that of the Section; and to accommodate those members of the Society who were in attendance on the Colloquium the Section set apart a special day for the reading of its chief mathematical papers.

In attendance and in number of papers presented the Fifth Summer Meeting surpassed any of its predecessors. The total attendance at the three sessions exceeded sixty-five, and included the following fifty-three members of the Society:

Professor A. L. Baker, Dr. C. C. Barnum, Professor M. Bôcher, Professor W. E. Byerly, Professor C. H. Chandler, Professor A. S. Chessin, Dr. J. B. Chittenden, Dr. A. Cohen, Professor F. N. Cole, Professor L. L. Conant, Professor C. L. Doolittle, Professor L. W. Dowling, Professor H. T. Eddy, Professor T. S. Fiske, Dr. A. B. Frizell, Dr. J. W. Glover, Miss Ida Griffiths, Mr. G. H. Hallett, Mr. H.

E. Hawkes, Professor Ellen Hayes, Dr. J. I. Hutchinson, Professor H. Jacoby, Mr. C. J. Keyser, Professor P. Ladue, Professor P. A. Lambert, Professor G. Lanza, Dr. G. H. Ling, Professor F. H. Loud, Mr. F. M. McGaw, Professor J. McMahon, Dr. A. Macfarlane, Professor M. Merriman, Professor W. H. Metzler, Professor E. H. Moore, Dr. D. A. Murray, Professor J. C. Nagle, Professor S. Newcomb, Professor W. F. Osgood, Professor B. O. Peirce, Mr. D. L. Pettegrew, Professor J. Pierpont, Professor C. Puryear, Professor P. F. Smith, Dr. V. Snyder, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor A. G. Webster, Professor H. S. White, Professor C. B. Williams, Miss E. C. Williams, Professor F. S. Woods, Professor R. S. Woodward, and Professor T. W. D. Worthen.

The opening session began at 10 a. m. on Friday. The President, Professor Simon Newcomb, occupied the chair and delivered a brief address on the remarkable growth of mathematical science in this country. Having only strictly scientific business before it, the meeting was able to complete the lengthy programme in three sessions, finishing at noon on Saturday. At the last two sessions Vice-Presidents R. S. Woodward and E. H. Moore presided. The Council announced the election of the following persons to membership in the Society: Mr. H. E. Hawkes, Yale University; Dr. F. H. Safford, Harvard University; Dr. F. Schlesinger, Yerkes Observatory; Dr. J. Westlund, Yale University. Six applications for membership were received. At the meeting of the Council on Friday evening, a committee consisting of Professors T. S. Fiske, Simon Newcomb, E. H. Moore, M. Bôcher, and J. Pierpont was appointed to consider the question of securing improved facilities for the publication of original mathematical papers in this country.

The following papers were read at the meeting:

(1) Dr. E. M. Blake: "On the ruled surfaces generated by the plane movements whose centrodes are congruent conics tangent at homologous points." (Illustrated by models.)

(2) Professor T. F. Holgate: "A second locus connected with a system of coaxial circles."

(3) Dr. J. I. Hutchinson: "On the Hessian of the cubic surface."

(4) Dr. Virgil Snyder: "Asymptotic lines on cubic scrolls."

(5) Professor Alexander Chessin: "Relative motion considered as disturbed absolute motion."

(6) Professor A. L. Baker: "Fundamental algebraic operations."

(7) Professor ALEXANDER CHESSIN: "On the development of the perturbative function in terms of the mean anomalies."

(8) Professor E. O. LOVETT: "Note on the differential invariants of a system of $m + 1$ points by projective transformation."

(9) Professor W. F. OSGOOD: "Note on the extension of the Poincaré-Goursat proof of a theorem of Weierstrass's."

(10) Professor W. F. OSGOOD: "Supplementary note on a single-valued function with a natural boundary, whose inverse is also single-valued."

(11) Professor MAXIME BÔCHER: "The theorems of oscillation of Sturm and Klein."

(12) Professor A. L. BAKER: "Space concepts in mathematics."

(13) Dr. T. P. HALL: "An algebra of space."

(14) Professor E. H. MOORE: "The subgroups of the generalized modular group."

(15) Professor L. L. CONANT: "An application of the theory of substitutions."

(16) Dr. J. H. BOYD: "A method for finding an approximate integral for any differential equation of the second order."

(17) Dr. H. F. STECKER: "Non-euclidean cubics."

(18) Dr. G. A. MILLER: "On the simple isomorphisms of a Hamiltonian group to itself."

(19) Dr. L. E. DICKSON: "A new triply-infinite system of simple groups obtained by a twofold generalization of Jordan's first hypoabelian group."

(20) Dr. L. E. DICKSON: "Construction of a linear homogeneous group in C_q^m variables, isomorphic to any given linear homogeneous group in m variables."

(21) Dr. JACOB WESTLUND: "On a class of equations of transformation."

(22) Professor F. MORLEY: "A generalization of Desargues's theorem."

(23) Dr. E. L. STABLER: "A rule for finding the day of the week corresponding to a given date."

(24) Dr. ARTEMAS MARTIN: "Evolution by logarithms."

(25) Dr. ARTEMAS MARTIN: "A method of finding, without tables, the number corresponding to a given logarithm —II."

Dr. Boyd was introduced by Professor C. B. Williams. Dr. Blake's and Professor Morley's papers were read by Professor Fiske, Professor Holgate's by Professor White, Dr. Stecker's by Professor Woods, and Dr. Westlund's by Professor Pierpont. In the absence of the authors the papers of Professor Lovett, Dr. Hall, Dr. Miller, Dr. Dickson, Dr. Stabler, and Dr. Martin were read by title.

Abstracts of those papers which are not intended for publication in the BULLETIN are given below.

Dr. Blake's paper is intended for publication in the *Annals of Mathematics.* The movements considered are defined as follows: upon a plane α' containing a conic C' moves a coincident plane α containing a congruent conic C, in such a manner that the two conics are always tangent at homologous points. The locus of a point carried by α is a unicursal curve of the fourth order when C and C' are central conics and a unicursal curve of the third order when they are parabolas. The properties of these curves taken from the writings of Roberts, Cayley, and others are given in brief. The locus of a straight line carried by α is a quartic scroll with a nodal circle at infinity and a nodal straight line when C and C' are central conics; and a cubic scroll with a nodal straight line when they are parabolas. These surfaces admit of considerable variety with regard to the reality and aggregation of their pinch points. They are described under thirteen types. The study of them from the point of view of their mechanical generation is believed to be new.

Professor Chessin's papers will be published in the *American Journal of Mathematics.* The following is an outline of the first paper: Let a space R move within a space A and call these spaces respectively the *relative* and the *absolute* spaces. The motion of any system S may be considered as taking place either in the space A or in the space R. In the first case the motion of S is called *absolute*, in the second *relative.* The motion of R within A is called space motion. If there were no space motion the motion of S in R would be identical with its motion in A. We may therefore consider the space motion as a *perturbation of the absolute motion*, the latter being the *undisturbed* motion, while the relative motion will become the *disturbed* motion. The object of the paper presented was to give an expression for the perturbative function which enables us to pass from the absolute to the relative motion in a way similar to that followed in the general theory of perturbations. The expressions obtained for the perturbative function Ω show that it is composed of two parts, one independent of the equations of constraint to which S

may be subject and the other directly depending on these equations. It was particularly to give expressions for the latter part of Ω that the present investigation was made, expressions for the other part having been obtained before.

In the *Astronomical Journal* (1894, Nos. 326 and 332) Professor Chessin has given a method which greatly reduces the work of computation and simplifies the development of the perturbative function in terms of the eccentric anomalies as given by Professor Simon Newcomb in the Astronomical Papers, vol. 3, Part I (see also Tisserand, Mécanique Céleste, vol. 4, p. 313). In his second paper before the Society a similar method is given by Professor Chessin for the development of the perturbative function in terms of the mean anomalies. The method applies as well to the development of Le Verrier, as to that given by Professor Newcomb (Astronomical Papers, vol. 5, Part I), although it is here applied only to the latter development. It is estimated that the actual work of computation is thus reduced to about one-fifth of what it would be if the current methods were used.

Professor Lovett's paper will be published in part in the *American Journal of Mathematics* and in part in the *Bulletin des Sciences Mathematiques.* Let a system of $m + 1$ points in $n + 1$ dimensional space be given by the coördinates

$$x_{\nu,0},\ x_{\nu,1},\cdots,\ x_{\nu,n}, \qquad (\nu = 0, 1, \cdots, m);$$

and let

$$x_{\nu,0} = x_{\nu,0}\,(x_{\nu,1}, \cdots, x_{\nu,n}), \quad x_{\nu,0,j} = \frac{\partial x_{\nu,0}}{\partial x_{\nu,j}}, \quad x_{\nu,0,j,k} = \frac{\partial^2 x_{\nu\,0}}{\partial x_j \cdot \partial x_k} = x_{\nu\,0,k,j}.$$

The general projective group of this $n + 1$ dimensional space is generated by the $(n + 1)\,(n + 3)$ independent infinitesimal transformations

$$\frac{\partial f}{\partial x_i}, \quad x_i \frac{\partial f}{\partial x_k}, \quad x_i \sum x_j \frac{\partial f}{\partial x_j}, \qquad (i, j, k = 0, 1, \cdots, n).$$

By forming the second extensions of these point transformations by the method of Lie and equating them to zero, we have a complete system of partial differential equations whose integration will determine the projective differential invariants of the second order. The integration of this system for the case of $m + 1$ points yields the result that the following $m + 1$ forms are absolute invariants by the general projective group:

$$\sum_0^m {}_i \frac{|x_{k,0,1,1}, x_{k,0,2,2}, \cdots, x_{k,0,n,n}|}{|x_{i,0,1,1}, x_{i,0,2,2}, \cdots, x_{i,0,n,n}|} \left\{ \frac{x_{i,0} - x_{k,0} - \sum_1^n {}_j (x_{i,j} - x_{k,j}) x_{i,0,j}}{x_{i,0} - x_{k,0} - \sum_1^n {}_i (x_{i,j} - x_{k,j}) x_{k,0,j}} \right\}^{n+2},$$

$$(k = 0, 1, \cdots, m).$$

These invariants are interpreted geometrically in the following manner: Take $m + 1$ hypersurfaces (n dimensional manifoldnesses) in the $n + 1$ dimensional space, perfectly arbitrarily chosen except that a hypersurface is to pass through each of the $n + 1$ points of the given system; let $\rho_{i,1}, \cdots, \rho_{i,n}$ be the principal radii of curvature of the hypersurface through the point $(x_{i,0}, x_{i,1}, \cdots, x_{i,n})$ at this point; take any point $(x_{k,0}, x_{k,1}, \cdots, x_{k,n})$ of the system and join it by straight lines to all the other points of the system; let θ_i be the angle at $(x_{k,0}, \cdots, x_{k,n})$ between the normal to the hypersurface through this point and the line joining it to $(x_{i,0}, \cdots, x_{i,n})$, and let φ_i be the angle between the latter line and the normal to the surface through $(x_{i,0}, \cdots, x_{i,n})$ at the point; then the above invariants show that the forms

$$\sum {}_i \frac{\prod_{j=1}^{j=n} \rho_{k,j} \cos^{n+2} \theta_i}{\prod_{j=1}^{j=n} \rho_{i,j} \cos^{n+2} \varphi_i}$$

are absolute constants. When the $m + 1$ points lie on a surface of the $(m + 1)$th degree and on a straight line simultaneously

$$\sum_0^m {}_i \frac{1}{\prod_{j=1}^{j=n} \rho_{i,j} \cos^{n+2} \varphi_i} = 0.$$

In Professor Baker's first paper it is shown that, assuming the weight of a point as its defining property, there are only six possible operations performable on it. The processes of complex numbers and quaternions are not extensions but applications of these operations. In his second paper Professor Baker traces the fertility of many mathematical ideas to four fundamental characteristics of space.

The algebra of space considered by Dr. Hall is a vector algebra whose laws of operation are derived from geometric definitions of vector addition and vector multiplication.

Every real algebraic transformation corresponds to, and represents, a possible motion in space; and at any stage of the process algebraic expressions may be expressed geometrically, and vice versa. The algebra is developed far enough to enable it to be used in considering the properties of loci of one, two, and three dimensions.

Professor Moore's paper will be published in the *Mathematische Annalen.* The chief features of the paper are outlined in the following abstract. The modular group Γ of all unimodular substitutions $(\alpha, \beta, \gamma, \delta)$

$$\omega' = \frac{\alpha\omega + \beta}{\gamma\omega + \delta} \qquad (\alpha\delta - \beta\gamma = 1)$$

of the complex variable ω, where the $\alpha, \beta, \gamma, \delta$ are rational integers, has for every rational prime q a self-conjugate subgroup $\Gamma_{\mu(q)}$ of finite index $\mu(q)$ containing all substitutions $(\alpha, \beta, \gamma, \delta)$ for which $\alpha \equiv 1$, $\beta \equiv 0$, $\gamma \equiv 0$, $\delta \equiv 1$ (mod. q). The corresponding quotient-group $\Gamma / \Gamma_{\mu(q)}$ is conveniently given as the say finite modular group $G^{q+1}_{\mu(q)}$ of substitutions $(\alpha, \beta, \gamma, \delta)$ on the $q + 1$ marks $\omega (\omega = \infty, 0, 1, \cdots, q-1)$, where the $\alpha, \beta, \gamma, \delta$ are integers taken modulo q. By generalizing from the Galoïs field of rank 1 to that of rank n we have the generalized finite modular group $G^{q^n+1}_{M(q^n)}$ of order $M(q^n) = q^n(q^{2n} - 1)$ or $q^n(q^{2n} - 1) / 2$ according as $q = 2$ or $q > 2$. Mathieu first exhibited this group and studied its cyclic subgroups (*Comptes Rendus*, 1858, 1859; *Liouville's Journal*, 1860, p. 39). That (except for the cases $q^n = 2^1, 3^1$) it is simple was proved by Moore (Mathematical Papers of the Chicago Congress, 1893; in abstract, *Bulletin of the New York Mathematical Society*, December, 1893), and independently, but imperfectly, by Burnside (*Proceedings of the London Mathematical Society*, February, 1894).

In the present paper all subgroups of the $G^{q^n+1}_{M(q^n)}$ are determined; for the case $n = 1$ this was done by Gierster (*Mathematische Annalen*, vol. 18 (1881)). The subgroups are of three kinds: (1) metacyclic or solvable groups; (2) (3) groups of the abstract character of certain groups $G^{q^n+1}_{M(q^n)}$ or of certain groups $G^{q^n+1}_{2M(q^n)}$ $(q > 2)$, a group $G^{q^n+1}_{2M(q^n)}$ being obtained by extending the $G^{q^n+1}_{M(q^n)}$ by the substitution $\omega' = \rho\omega$, where ρ is a primitive root of the $GF[q^n]$. Thus the doubly infinite system of simple groups

$$G_{M(q^n)}^{q^n+1} \qquad (q^n \neq 2^1, 3^1)$$

determines by the decomposition of the subgroups of its constituent groups, apart from the simple groups of prime order, only simple groups of the original system. An equation of degree $q^n + 1 (q^n \neq 2^1, 3^1)$ whose Galois group is the $G_{M(q^n)}^{q^n+1}$ has resolvents of degree D, $D < q^n + 1$, only in the cases $q^n = 5^1, 7^1, 11^1, 3^2$, when D is respectively 5, 7, 11, 6. For $n = 1$ this is a noted theorem of Galois and Gierster.

The group $\Gamma[\Omega]$ of all unimodular substitutions $(\alpha, \beta, \gamma, \delta)$ where the coefficients are algebraic integers of an algebraic field Ω [Körper Ω] has for every prime functional π [Weber] of absolute norm q^n, a self-conjugate subgroup $\Gamma_{m(q^n)}[\Omega]$. The quotient-group is the group $G_{M(q^n)}^{q^n+1}$. The substitutions of the modular group Γ leave invariant a modular function $J(\omega)$. The transcendental modular equation $J = J(\omega)$ has algebraic resolvents, modular equations, with Galois groups $G_{M(q)=\mu(q)}^{q+1}$. The group $\Gamma[\Omega]$, for field Ω of degree > 1, is improperly discontinuous and has no corresponding automorphic function. If however an automorphic function $K(\omega)$ belongs to a group whose substitutions have coefficients of an algebraic field Ω, then the generalized modular equation $K = K(\omega)$ has algebraic resolvents whose Galois groups are certain generalized finite modular groups $G_{M(q^n)}^{q^n+1}$ or certain of their subgroups. This remark connects the present paper with recent work of Fricke and Bianchi on certain special classes of automorphic functions (Cf. Fricke-Klein, Automorphe Functionen I, p. 585 fg., 1897).

Professor Conant gave some illustrations of substitution properties by means of the shuffling of the cards of an ordinary whist pack.

Dr. Boyd's paper furnishes a method for finding an approximate general integral of the differential equations

$$(a) \qquad \frac{dy}{dx} + P_0 + P_1 y + P_2 y^2 = 0,$$

$$(b) \qquad P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0,$$

where the P's are any functions of x. The object of the paper is first to reduce the differential equations indicated under (a), (b), to the *normal Riccatian form*

$$\frac{dz}{dx} + z^2 = F(x),$$

then to develop a process, by the repetition of which an approximate integral is found. The paper assumes that an approximate integral is found when it is shown that the remainder, left in the equation after the so-called approximate integral has been substituted, becomes as small as we choose when the process has been repeated often enough.

Dr. Stecker's paper presented a discussion of the properties of non-euclidean cubics. Among the results obtained were the following: The ratio of the sines of the non-euclidean distances from any tangent of a cubic to either focal line and to the pole of that line with respect to the absolute equals $\sqrt{\mu}$. For any focal line and pole there are twelve tangents to the cubic such that the ratio of the sines of the non-euclidean distances from focal line and pole is constant. The ratio of the sines of the non-euclidean distances from any tangent of a cubic to any pair of focal lines equals $\sqrt{\frac{\mu}{\mu_1}}$ times the ratio of the sines of the non-euclidean distances to the poles of those lines with respect to the absolute. The non-euclidean distances from the point of contact of a cubic with any absolute tangent to the intersections with that tangent of any pair of focal lines are equal; the absolute tangent to the cubic makes equal angles with lines joining its point of contact to either pair of foci. The ratio of the product of the cosines of the non-euclidean distances from any point of a cubic to the poles, with respect to the absolute, of the tangents at three collinear points of inflexion to the cube of the cosine of the non-euclidean distance to the pole of the line through the three points of inflexion is constant. If two of the above tangents are absolute tangents, then the ratio of the square of the cosine of the non-euclidean distance from any point of a cubic to the intersection of such a pair of absolute tangents to the product of the cosines of the non-euclidean distances to their poles with respect to the absolute is constant. The relation between the cosines of the non-euclidean distances of any point of a cubic from the poles with respect to the absolute of any pair of focal lines (no two passing through the same absolute point) is expressed by the vanishing of the determinant

$$\begin{vmatrix} \lambda\Omega_{xo} & \cos xo \\ \lambda_1\Omega_{yo} & \cos yo \end{vmatrix}$$

for every point of the cubic. The product of $\lambda\Omega_{xo}$ into the cosine of the non-euclidean distance from any point of a cubic to the pole with respect to the absolute of any focal line is the same in value for all focal lines. If a is the intersection of two absolute tangents and x, y, z the poles, with respect to the absolute, respectively of the chord of contact, third tangent, and satellite line, then the ratio of any two products of the form $\cos^2 ao \cos^2 xo \cos yo \cos zo$ is constant for all points of the cubic for any pair of absolute tangents. The ratio of the product of the sines of the non-euclidean distances from any point of a cubic to the focal lines of any pair equals $\frac{\lambda}{\lambda_1}$ times the corresponding ratios for any other pair. The determinant

$$\begin{vmatrix} \sin\frac{1}{xo} & \sin\frac{1}{yo} & \lambda \\ \sin\frac{1}{x_1o} & \sin\frac{1}{y_1o} & \lambda_1 \end{vmatrix}$$

vanishes for any pair of focal lines of a cubic. Among other properties discussed was a class of relations concerning first and second polars with respect to the cubic of fixed lines related to the cubic and absolute.

By far the simplest definition of the first hypoabelian group consists in regarding it as the group of linear substitutions on m pairs of indices ξ_i, η_i taken modulo 2, which leave invariant the function

$$\xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_m\eta_m.$$

By Jordan's proof this group has a subgroup of index two which is *simple* if $m > 2$. The first generalization of this group, made by Dr. Dickson while at Paris and presented a year and a half ago to the *Quarterly Journal*, has since been greatly simplified and published in the July number of the BULLETIN. The present abstract of Dr. Dickson's first paper, to be presented for publication to the London Mathematical Society, announces a further generalization. Use is made of the notion of a Galois field, viz., the totality of polynomials built upon the root of a congruence of degree n and irreducible modulo p. These polynomials, of which p^n are distinct, have the property that the addition, subtraction, multiplication, or division (except by zero) of two polynomials gives always an unique third polynomial, belonging to the set. The generalization may now be described as the replacement of the field of integers modulo 2 by the Galois field of order p^n, n any integer and p any prime. Consider the group $L_{m,n,p}$ of all linear substitutions

$$S\begin{cases} \xi_i' = \sum_{j=1}^{m}(\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \\ \eta_i' = \sum_{j=1}^{m}(\beta_{ij}\xi_j + \delta_{ij}\eta_j), \end{cases} \qquad (i = 1, \cdots, m).$$

(in which coefficients and indices belong to the Galois field of order p^n) which leave invariant

$$\xi_1\eta_1 + \cdots + \xi_m\eta_m$$

The quadratic conditions thus imposed on the coefficients are readily written down and the fact verified that the determinant of S must be ± 1. It is shown that the group $L_{m,n,p}$ may be generated from the substitutions

$$Q_{i,j,\lambda}: \quad \xi_i' = \xi_i + \lambda\xi_j, \quad \eta_j' = \eta_j - \lambda\eta_i,$$
$$T_{i,\lambda}: \quad \xi_i' = \lambda\xi_i, \quad \eta_i' = \lambda^{-1}\eta_i,$$
$$E_i = (\xi_i\eta_i);$$

that it has a subgroup $L'_{m,n,p}$ of index two and order

$$O'_{m,n,p} =$$
$$[p^{nm} - 1][(p^{n(2m-2)} - 1)p^{n(2m-2)}][(p^{n(2m-4)} - 1)p^{n(2m-4)}]$$
$$\cdots [(p^{2n} - 1)p^{2n}].$$

which is extended to the main group by one of the transpositions $E_j \equiv (\xi_j\eta_j)$. Thus, if $p > 2$, $L'_{m,n,p}$ contains all the substitutions of L having determinant $+1$; while for $p = 2$, it is readily proven to contain all satisfying the relation

$$\sum_{i,j}^{1,\ldots,m} \alpha_{ij}\delta_{ij} = m.$$

The group $L'_{m,n,p}$ contains the invariant subgroup (of index one or two)

$$L''_{m,n,p} \equiv \{Q_{i,j,\lambda}, \; E_iE_j\} \quad (i, j = 1, \cdots, m; \; i \neq j).$$

Indeed, it is proved to contain the products $T_{i,\lambda}\ T_{j,\lambda}$ and consequently (if $m \gtreqless 3$) also

$$T_{1,\lambda^2} \equiv (T_{1,\lambda}\ T_{2,\lambda})\ (T_{2,\lambda^{-1}}\ T_{3,\lambda^{-1}})\ (T_{3,\lambda}\ T_{1,\lambda}).$$

Whether or not it contains the substitution $T_{i,\nu}$, ν being a not-square, is not, as yet, determined. It requires a lengthy, detailed investigation to prove that $L''_{m,n,p}$ contains no self-conjugate subgroup other than itself and the identity, unless it be the group of order 2 generated by the substitution changing the sign of every index, viz.,

$$T_{1,-1}\ T_{2,-1} \cdots T_{m,-1}.$$

The structure of the group $L_{m,n,p}$ defined by the invariant $\xi_1\eta_1 + \cdots + \xi_m\eta_m$ is therefore fully determined aside from the lack of proof of the non-existence in the group $\{Q_{i,j,\lambda}, E_iE_j\}$ of a substitution $T_{i,\nu}$, ν being a not-square. For the case $p = 2$, there is no uncertainty, since every quantity in the $GF[2^n]$ is a square. We thus know that $L''_{m,n,2} \equiv L'_{m,n,2}$ is a *simple* group of order

$$(2^{nm}-1)\ [(2^{n(2m-2)}-1)2^{n(2m-2)}] \cdots [(2^{2n}-1)2^{2n}].$$

Judging from the proofs found for certain cases (when $p > 2$), it seems probable that the substitution $T_{i,\nu}$ does not occur in the group $L''_{m,n,p}$. If this be true, the orders of the simple quotient-groups are for $p > 2$ as follows:

$\frac{1}{2}O'_{m,n,p}$ if m be odd and -1 a not-square;

$\frac{1}{4}O'_{m,n,p}$ in all other cases.

Dr. Westlund's paper is in abstract as follows: Taking the equation for the division of the periods of the elliptic functions as starting point we are led to a class of equations of transformation whose roots are the $n + 1$ values of

$$y_{\frac{\mu}{\nu}} = \prod_{1,m}^{p} sn^{2\alpha} \cdot cn^{\beta} \cdot dn^{\gamma} (4p\bar{\omega}/k)$$

$$\bar{\omega} = \frac{\mu K + \nu i K'}{n}$$

$$\alpha, \beta, \gamma = \text{integers}, \quad n = \text{prime} = 2m + 1.$$

The object of the paper is to give a method by which all equations of this class may be computed. 1. The roots are developed into q-series $\left(q = e^{\pi i \tau}, \tau = \frac{iK'}{K}\right)$ 2. A superior limit of the degree of κ in the equation is determined by determining all the infinities (and their order) of the roots for the whole infinite κ-plane ($\kappa = k^2$). 3. Equations that can be derived from a given equation of transformation by means of linear transformations are considered. 4. The equations whose roots are

$$\prod_{1,m}^{p} \frac{dn^2}{cn}(4p\bar{\omega}/k),\ \prod_{1,m}^{p} \frac{cn}{sn^2}(4p\bar{\omega}/k),\ \prod_{1,m}^{p} sn^2 \cdot cn \cdot dn(4p\bar{\omega}/k),$$

respectively and all equations that can be derived from these by linear transformations are computed for $n = 3$.

To find the day of the week corresponding to a given date Dr. Stabler gives the following rule : add to the day of the month the index number of the month and the index number of the year, then subtract the largest multiple of seven that is less than the sum. Index numbers of months are: January, 3 (in leap years, 2); February, 6 (in leap years, 5); March, 6; April, 2 ; May, 4 ; June, 0 ; July, 2; August, 5; September, 1 ; October, 3 ; November, 6 ; December, 1. Index numbers of years 1800–1899 are found by increasing the excess of the year over 1800 by $\frac{1}{4}$ of itself (discarding fractions) and subtracting the largest multiple of seven contained in the sum. A rule is given for dates in other centuries. Example, August 19, 1898. Add index number of the month, 5, and of the year, 3, to 19. Excess over a multiple of 7 is 6, indicating Friday.

Dr. Martin's two papers will be published in the *Mathematical Magazine.* The author gives convenient processes for the extraction of roots and for determining the number corresponding to a given *natural* logarithm.

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