already well known for the hyperelliptic theta functions (depending upon two arguments).

Professor Lovett's paper, which is intended for publication in the *Transactions*, employs Lie's theory of infinitesimal transformations to construct a method for determining the singular solutions of Monge and Pfaff equations.

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THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE Sixth Semi-Annual Meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held on December 28 and 29, 1899, at the University of Chicago. The following members of the Society were in attendance:

Professor Oskar Bolza, Professor E. W. Davis, Professor Thomas F. Holgate, Dr. Kurt Laves, Professor H. Maschke, Professor John A. Miller, Professor E. H. Moore, Professor Alexander Pell, Professor D. A. Rothrock, Professor G. T. Sellew, Professor E. B. Skinner, Dr. H. E. Slaught, Dr. H. F. Stecker, Professor C. A. Waldo, Dr. J. V. Westfall, Professor H. S. White, Professor Mary F. Winston, Professor J. W. A. Young.

Professor E. H. Moore, Vice-President of the Society, occupied the chair during the first of the four sessions, after which Professor E. W. Davis presided. The Christmas meeting being the regular time for the election of officers of the Section, the Secretary was re-elected and Professors H. B. Newson and C. A. Waldo were elected members of the programme committee. The time and place of the next meeting were fixed for Saturday, April 14, 1900, at Northwestern University, Evanston, Ill.

The following papers were presented:

(1) Mr. R. E. MORITZ: "A generalization of the process of differentiation."

(2) Professor E. D. ROE: "On the transcendental form of the resultant."

(3) Dr. E. J. WILCZYNSKI: "An application of Lie's theory to hydrodynamics."

(4) Mr. F. R. MOULTON: "On the question of the stability of certain particular solutions of the problem of four bodies, together with particular solutions of the problem of n bodies of the Lagrangian type.

(5) Professor L. E. DICKSON: "The canonical form of linear homogeneous substitutions in a general Galois field."

(6) Professor L. E. DICKSON: "The cyclic subgroup of the simple group of linear fractional substitutions of determinant unity in two non-homogeneous variables with coefficients in an arbitrary Galois field."

(7) DR. J. V. WESTFALL: "On a category of transformation groups in space of four dimensions."

(8) Professor OSKAR BOLZA: "The elliptic sigma functions considered as a special case of the hyperelliptic sigma functions."

(9) Professor ALEXANDER PELL: "Calculation of the integral

$$\int_0^\infty e^{-\left(px^2+\frac{q}{x^2}\right)} \begin{matrix}\sin\\ \cos\end{matrix} \left\{ rx^2 + \frac{s}{x^2} \right\} dx."$$

(10) Professor JOHN A. MILLER: "Concerning certain modular functions of square rank."

(11) Professor R. J. ALEY: "A new collinear set of three points connected with a triangle."

(12) Professor H. MASCHKE: "Note on the unilateral surface of Moebius."

(13) Professor C. A. WALDO: "On a family of warped surfaces connected by a simple functional relation."

(14) Professor H. S. WHITE: "Plane cubics and irrational covariant cubics."

Mr. Moritz's paper was presented to the Society through Professor Davis, and was read by him in the absence of the author. Mr. Moulton's paper was presented through Dr. Laves. In the absence of the authors, Professor Roe's paper was read by Professor White, Dr. Wilczynski's by Professor Bolza, Professor Dickson's by Professor Moore, and Professor Aley's by the Secretary.

After the regular papers had been presented, a general discussion upon the topic "Limits of functions of one or more variables" was opened by Professor Moore.

Mr. Moritz's paper was a contribution to the doctrine of pure forms. Let $\underset{n}{+}$ denote the symbol of a combinatory process of the nth order, $\underset{n}{-}$ and $\underline{\,n\,}$ its two inverses, so that if $a \underset{n}{+} b = c$, then $c \underset{n}{-} a = b$, and $c \underline{\,n\,} b = a$; and let $a \underset{n}{+} M_n = a$, for all values of a, define M_n, the modulus of the process. The process of the $(n+1)$th order is built up from the process of the nth order, as multiplication is built up from addition. Three general theorems were proved: I. If M_n is the modulus of a process of the nth order, then $M_{n\,n}{}^{-}_{+1}M_n$ is indeterminate. II. If any process possesses a modulus, and is subject to the sub-associative law, viz.:

$$a^{+}_{n}b^{+}_{n}c = a^{+}_{n}(b_{n}{}^{+}_{-1}c),$$

then
$$a\,{}^{-}_{n}\,b = (a\,{}^{-}_{n}\,c)_{n}{}^{-}_{-1}(b\,{}^{-}_{n}\,c).$$

III. If in addition to the hypothesis of theorem II, the process is subject to the sub-distributive law, viz.:

$$a_{n}{}^{+}_{+1}(b_{n}{}^{+}_{-1}c) = (a_{n}{}^{+}_{+1}b)^{+}_{n}(a_{n}{}^{+}_{+1}c)$$

then
$$M_{n\,n}{}^{-}_{+1}\,a = M_{n-1}.$$

The symbol Y_n was defined by the equation

$$Y_n = \lim_{h=M\,n}\{f(x^{+}_{n}h)^{-}_{n}f(x_{n}{}^{-}_{+1}h)\};$$

for $n = 1$ this becomes $Y_1 = \dfrac{dy}{dx}$. For any process subject to the conditions of theorem I., Y_n is indeterminate. If, moreover, the conditions of theorem II. and III. exist for the process in question, then by theorem II.

$$Y_n = (M_{n\,n}{}^{-}_{+1}\,a)_{n}(M_{n\,n}{}^{-}_{+1}\,a)$$

and by theorem III. $Y_n = M_{n-1}{}^{-}_{n}M_{n-1}$, that is, it will be possible to evaluate Y_n provided Y_{n-1} can be evaluated. For $n = 2$ all the above conditions exist, and we have in fact

$$Y_2 = \lim_{h=1}\left\{\log_h\frac{f(xh)}{f(x)}\right\}.$$

The symbol $\dfrac{dy}{dx}$ was used to denote the forms thus defined, and the term quotiential coefficient was applied to them. A table of quotiential coefficients accompanied the paper.

Professor Roe's paper found its origin in two letters of Gordan, one to Hermite (*Comptes Rendus*, 127 (1898), p. 539), the other to the author, and arose from an attempt to remove certain difficulties suggested in the former of these and to work out the formulas contained in the latter. Three functions

$$f = x^m + a_1 x^{m-1} + \cdots + a_m,$$
$$\varphi = x^n + b_1 x^{n-1} + \cdots + b_n,$$
$$\psi = 1 + b_1 x + b_2 x^2 + \cdots + b_n x^n,$$

where the roots of ψ are the reciprocals of those of φ, are assumed, and the transcendental form of the resultant of f and ψ is obtained as

$$R = \sum_{\nu=0}^{\nu=\infty} \sum \frac{(-1)^{\rho_1+\rho_2+\cdots}}{\lambda_1^{\rho_1}\lambda_2^{\rho_2}\cdots\rho_1!\,\rho_2!\cdots}\{S_{\lambda_1}(a)\cdot S_{\lambda_1}(b)\}^{\rho_1}$$
$$\{S_{\lambda_2}(a)\cdot S_{\lambda_2}(b)\}^{\rho_2}\cdots$$

where $\rho_1 + \rho_2 + \cdots = \nu$ and $S_\lambda(a)$, $S_\lambda(b)$ are respectively the sums of the nth powers of the roots of f and φ. It is seen, however, that R is also the resultant of f and ψ for any other degrees than m and n and the conclusion is reached: *The transcendental form expresses all binary resultants formally in the same form and is a universal formula comprehending them all. It is the most general formula for binary resultants.*

The finite form of the resultant $R_{m,n}$ is reached by dropping all the terms of R of weight greater than mn which, when expressed in the a's and b's for given functions f and ψ, are identically zero. We have.

$$R_{m,n} = \sum\sum \frac{(-1)^{\rho_1+\rho_2+\cdots}}{\lambda_1^{\rho_1}\lambda_2^{\rho_2}\ldots\rho_1!\,\rho_2!\ldots} S_\lambda(a)S_\lambda(b),$$

with $\lambda_1\rho_1 + \lambda_2\rho_2 + \cdots \leqq mn$. This $R_{m,n}$ is still very general, formally representing all resultants $R_{\mu,\nu}$, where $\mu\nu = mn$, in the same form, until the a's and b's for given equations are substituted, when it becomes definite. It may also represent expressions which are not resultants.

A one parameter group can, as Lie has noticed, be interpreted as representing the steady motion of a fluid. In Dr. Wilczynski's paper, Lie's suggestion was followed out, a number of general theorems being thus obtained. The theory was then applied to the special cases which from this standpoint are the simplest and most important, viz., that the groups are projective, linear, or linearoid. If the forces which act upon the unit of mass have a potential, the most general ternary projective group which can represent a steady fluid motion is the general linear group. This motion was studied in detail, one of the results being that, if the coördinate planes are appropriately chosen, the projection of the orbit of every point of the fluid upon the plane of xy is a conic section. All of these conic sections are similar and similarly placed.

If the motion is periodic, and also in other cases, the orbits themselves are conics in parallel planes ; in the periodic case, of course, ellipses. A special case is the ellipsoid whose particles describe such elliptic orbits, a problem treated by Dirichlet and Dedekind. Interesting results were also found for the more difficult problem when the group is linearoid. The paper will be offered for publication to the *Transactions.*

At the Columbus meeting of the Society, August, 1899, a paper was presented by Mr. Moulton exhibiting twenty-eight particular solutions of the problem of four bodies, under the limitation that one of them should be infinitesimal. The first part of his present paper discussed the question of the stability of the motion of the infinitesimal body. Stability was defined as follows: The particular solutions have been already defined by those constant values of the coördinates and projections of the velocity of the infinitesimal body, satisfying the differential equations of motion. It is now supposed that the initial conditions are slightly different from those of the exact solutions,

$$x_0, \quad y_0, \quad z_0, \quad \frac{dx_0}{dt}, \quad \frac{dy_0}{dt}, \quad \frac{dz_0}{dt}.$$

Suppose $x = x_0 + x'$, etc., at the origin of time, where x', y', etc., are small quantities in terms of which the differential equations are now developed. All terms of degree higher than the first in the new variables are neglected, and the resulting linear system solved. If the roots of the *characteristic equation* are all pure imaginaries the generating solution is said to be *stable;* otherwise it is said to be *unstable.*

In the case where all four of the bodies lie in a line two of the roots of the characteristic equation are found to be real and two pure imaginary for all masses of the finite bodies. In the case where the infinitesimal body does not lie in the same line with the others, the roots of the characteristic equation are all pure imaginaries, or all complex, depending upon the relative masses of the finite bodies. In the case where the finite bodies are equal in mass and are at the vertices of an equilateral triangle, the roots of the characteristic equation are all pure imaginaries.

The second part of the paper took up particular solutions of the problem of n bodies of the Lagrangian type. The Lagrangian solutions of the problem of three bodies may be defined, from the standpoint of their derivation, as those solutions in which the ratios of the mutual distances of the bodies are constants; or, from the standpoint of their other properties, as those solutions in which each of the three bodies describes a conic section in the same period, and in such a manner that the center of gravity is at one of the foci of each conic, and that the law of areas holds for each body considered separately. Adopting the second definition, the necessary and sufficient conditions for solutions of this type of the problem of n bodies were written in algebraic form. It was proved that for any given n bodies there are $\frac{1}{2}n!$ solutions in which the bodies all lie in a line. There are doubtless solutions in which the bodies do not all lie in a line, but they are not proved to exist in the general case. In any particular example where the masses are given there is no difficulty in finding solutions if they exist. The associated problem, given n arbitrary points on a line, to find n masses such that, if they are placed at these points, solutions of the Lagrangian type will exist was also solved.

A more extended paper on this subject by Mr. Moulton, of which the present one will form a part, will be published in the *Transactions.*

The result due to Jordan (Traité des Substitutions, pp. 114–126) on the canonical form of a linear substitution on m variables with integral coefficients taken modulo p, a prime, may be readily generalized to substitutions in the $GF[p^n]$. Instead of following Jordan's method of proof, Professor Dickson in his first paper gave a simpler proof by induction. The theorem was supposed to hold for substitutions in the field which have the characteristic determinant (in the parameter K)

$$[F_k(K)]^{\alpha-1}[F_l(K)]^{\beta}\cdots \qquad (F_k,\ F_l,\ \cdots \text{ irreducible})$$

and was then proved to hold for every substitution in the field, which has the characteristic determinant

$$F_k^{\alpha}F_l^{\beta}\cdots \qquad (m \equiv k\alpha + l\beta + \cdots).$$

If we proceed by induction from $m-1$ to m indices, we fail to prove that part of the theorem concerning the conjugacy of the new indices. The author constructs explicitly the general substitution in the $GF[p^n]$ which is commutative with any given substitution in the field. In particular the number of the former substitutions was determined and found to agree with that derived by a different analysis by Jordan (Traité, p. 136) for the case $n = 1$.

Professor Dickson's second paper leads to a revision and a generalization to the $GF[p^n]$ of certain results due to Professor Burnside (*Proceedings of the London Mathematical Society*, vol. 26, pp. 58–106) upon groups of substitutions with integral coefficients taken modulo p, a prime. Numerous variations from Burnside's method of treatment were introduced, partly to avoid the separation of the two cases $p \equiv 1$ and $p \equiv -1 \pmod 3$ and to include the cases $p = 2$ and $p = 3$, and partly to minimize the calculations by the frequent use of known general theorems.

Burnside's results are incorrect in two places. The factor 2 should be deleted from $\frac{N}{2(p-1)^2}$ on pages 103 and 104; in fact, the statements made on page 103, lines 15–23, do not lead to the conclusion stated. Of substitutions having the canonical form

$$x' = x, \quad y' = y + x, \quad z' = z + y,$$

there exist three conjugate sets of N/p^2 substitutions, instead of one conjugate set of $3N/p^2$ substitutions as stated by Burnside, p 102, lines 1–6.

The generalization leads to an interesting result in the particular case $p^n = 2^2$. We have then a simple group of order 20160. The distribution of its cyclic subgroups in sets of conjugates is given by the following table:

960	conj.	cyclic	groups	of	order 7	with	5760	subs.	period	7
2016	"	"	"	"	5	"	8064	"	"	5
630	"	"	"	"	4	"	1260	"	"	4
630	"	"	"	"	4	"	1260	"	"	4
630	"	"	"	"	4	"	1260	"	"	4
315	"	"	"	"	2	"	315	"	"	2
1120	"	"	"	"	3	"	2240	"	"	3
							1	"	"	1

Total number of distinct substitutions 20160.

It follows at once that this group is not isomorphic with the alternating group on 8 letters having the order 20160. In fact, the latter contains substitutions of periods 6 and 15.

The two groups contain the same number of substitutions of period 7, the same number of periods 4 and 2. This non-isomorphism was first proved by Miss Schottenfels under the direction of Professor Moore. The corresponding investigation for three non-homogenous variables is now being made by Mr. T. M. Putnam.

Both of Professor Dickson's papers will appear in the *American Journal of Mathematics.*

Lie, in an article in the *Leipziger Berichte* for October, 1899, proceeding from Helmholtz's assumptions, omitting one, namely, the monodrom axiom, has determined all the groups, satisfying the given conditions under the most general interpretation of the axiom, of "unrestricted motion" (freie Beweglichkeit). He finds, besides the groups of euclidian and non-euclidian motion, five others. Kowalewski, a pupil of Lie, in his inaugural dissertation, has extended Lie's investigations in space of three dimensions to that of four and five. Besides the euclidian and non-euclidian groups, he finds in space of four dimensions three others A, B, and C. From analogy with the groups determined by Lie, he makes the statement, without proof, that group A satisfies the monodrom axiom, while B and C do not. By suitably choosing the points of general position, which are to be held fixed, and by considering the invariants of these points, Dr. Westfall proved that A satisfies the monodrom axiom, without being compelled to resort to the long integration otherwise necessary. In the case of group B, he showed that, contrary to Kowalewski's statement, the group may or may not satisfy this axiom, according to the mutual situation of the points held fixed.

To prove that A always satisfies the monodrom axiom he proceeds as follows: 1. Hold the special triplet of general points

$$P_1 = (0, 0, 0, 0); \quad P_2 = (0, 0, z_3, z_4), \quad P_3 = (0, 0, z_3', 0)$$

fixed and obtain the subgroup $(x_1 p_2 - x_2 p_1)$ which has closed path curves. 2. The three invariants of this point triplet are positive and never equal to zero. They are, further, independent with respect to z_3, z_4, and z_3'. If we choose any real positive values whatever, and set them equal to these invariants, we can solve for z_3, z_4, and z_3'. 3. From the form of the general invariant of the group we see that all triplets of points of general position have positive invariants which can never be equal to zero. If, therefore, we choose any triplet of points whatever, we can always determine z_3, z_4, and z_3' so that the invariants of the one triplet will be equal to the invariants of the other, and we can, therefore, transform one into the other by means of a transformation of the group. In the special case the path curves were closed, and we therefore know they are closed in general.

By suitable choice of the points to be held fixed and by consideration of the invariants in the case of group B, the following results were proved: 1. All groups obtained by holding fixed a triplet of general points whose invariants are all negative have closed path curves. 2. These are all the subgroups that have this characteristic.

Professor Bolza's paper gave a sketch of the theory of the elliptic sigma functions as they appear in the light of the theory of the hyperelliptic sigma functions. Section 1 contains systems of associated integrals of the first, second, and third kinds; section 2, Weierstrass's theta functions, (a) as functions of u, (b) in the Riemann surfaces; section 3, the Al-functions and the function $p(u)$; section 4, the sigma functions and the functions $\wp u$ and their invariantive properties; section 5, the partial differential equation of the sigma functions and the recursion formulæ for their expansion into power series. The paper has now been published in the *Transactions* (Vol. 1, No. 1, pp. 53–65).

Professor Miller's paper dealt with the properties of certain modular elliptic functions, the rank of which is a square number. The major part of the discussion was given to the particular cases of functions whose ranks are 9 and 4 respectively. Certain theorems used in the treatment of these special cases are true when the rank is any square number, as was proved. In the course of the discussion two groups were found; the one a quaternary group of linear substitutions of order 648, the other a ternary group of order 192. The letters of the substitutions of both these groups, the elements of the groups, and the operational character of the elements, were derived from the consideration of the X-functions, Z-functions, and $\sigma_{\lambda,\mu}$-functions, defined in many places, chiefly by Professor Klein. In case the rank $n = 9$ it is possible to define $t_\alpha (\alpha = 1, \cdots, 4)$ as linear homogeneous functions of Z_α, and thereby to find a group of monomial substitutions on t_α which is holoedrically isomorphic with the group G_{648} referred to above. It was shown that there are an infinite number of forms in t_α, which are invariant under G_{648}, and in accordance with the Hilbert law the basis of this system of forms was found. The function $\sigma_{\lambda/3,\,\mu/3}(w_1, w_2)$ was obtained as a linear homogeneous function of t_a, and by means of this the basis of the infinite system was expressed as rational functions of g_2, g_3, and $\triangle$, where g_2, g_3, and $\triangle$ have the signification usually given them in the treatment of elliptic functions.

The treatment of modular functions the rank of which is four differs in some of its details from that of the preceding, because the definition of $X_a(u)$ in case n is even differs slightly from that of $X_a(u)$ in case n is odd. However, the ends to be reached in the two cases are nearly identical. In both cases an interesting linear homogeneous connection has been found between

$$\sigma_{\frac{\lambda}{n},\,\frac{\mu}{n}}(nu, w_1, w_2) \quad \text{and} \quad X_{\frac{a}{n^2}}(u, w_1, w_2).$$

In Professor Aley's paper it was pointed out that the isogonal conjugate of the Brianchon point of any triangle, the center of the inscribed circle, and the center of the circumscribed circle are collinear.

Professor Maschke explained briefly the connection between the well known Moebius unilateral paper strip and the ruled surfaces of the third order, illustrating his demonstration by a model. A short account of the paper has been published in the *Transactions* (Volume 1, number 1, p. 39).

Professor Waldo's paper contained a demonstration of the following theorem : The equation

$$f\left(\frac{px}{p-z}, \frac{qy}{q-z}\right) = 0$$

represents a skew surface with two rectilinear directors whose equations are

$$(y = 0, \quad z = p) \quad \text{and} \quad (x = 0, \quad z = q),$$

and a plane curve director lying in the XY plane, having for its equation $f(x, y) = 0$, the common perpendicular to the two rectilinear directors being chosen for Z-axis. By this theorem the equations of the whole family of skew surfaces having two distinct rectilinear directors can be at once written down and their principal deformations shown under cartesian form. The surfaces having for curvilinear directors $x - y = 0$ and $xy - \frac{1}{2} = 0$ were studied in detail and models exhibited showing their principal types.

Professor White's paper was a further development of the topic considered in the paper presented by him at the Columbus meeting of the Society. Each mixed concomitant (2, 2) of the cubic defines (as in the paper referred to) two covariant nets of conics. These are polars of two cubics of the syzygetic sheaf; the totality of such is exactly that entire sheaf of cubics. But these concomitants (2, 2) and all the concomitants (3, 3) serve to define also *four* covariant sheaves of cubics, not in the syzygetic sheaf, intimately connected on the one hand with the four inflexional triangles, and on the other hand with the eighteen collineations of the cubic into itself. This paper will be published in the *Transactions.*

THOMAS F. HOLGATE.
Secretary of the Section.

EVANSTON, ILL.

ON CYCLICAL QUARTIC SURFACES IN SPACE OF N DIMENSIONS.

BY DR. VIRGIL SNYDER.

(Read before the American Mathematical Society, December 28, 1899.)

THE generation of the cyclide as the envelope of spheres which cut a fixed sphere orthogonally and whose centers lie on a quadric can readily be generalized to space of n dimensions.

In ordinary space it appears that the same surface is the envelope of five different systems; that the quadric loci of centers are all confocal and the associated spheres are all orthogonal; that the possibilities of the system are exactly coextensive with the ∞^{13} possible cyclides.

Let

$$(1)\quad (x_1 - ix_{n+2})\sum_{r=1}^{n} y_r^2 - 2\sum_{r=1}^{n} x_{r+1}y_i + (x_1 + ix_{n+2}) = 0$$

be the equation of a sphere in R_n; it contains $n + 2$ homo-