$$ABC = \frac{\pi}{2} - (A + A') + \frac{\pi}{2} - B - \left(\frac{\pi}{2} - A'\right)$$

$$= \frac{\pi}{2} - A - B$$

$$= \pi - (A + B + C).$$

The theorem is thus proved for a right triangle, and is readily extended to an oblique triangle by dividing it into two right triangles by a perpendicular from any vertex.

In the foregoing pages no attempt is made to give an exhaustive statement of Lobachevsky's methods and results on the plane nor to indicate his extension of his methods to space.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY, *May*, 1900.

BURKHARDT'S ELLIPTIC FUNCTIONS.

Functionentheoretische Vorlesungen. Von HEINRICH BURKHARDT. Zweiter Theil: *Elliptische Functionen.* Leipzig, Veit and Company, 1899. 8vo., x + 373 pp.

THE theory of elliptic functions has developed so rapidly and in so many different directions in recent years that an elementary treatise of moderate compass which would afford a rapid survey of its many and heterogeneous parts has been a long felt want. The admirable little treatise by Appell and Lacour is perfect in its way, but it addresses itself only to students who do not care to go very far into the theory of functions. It makes no pretentions to satisfy the needs of another large class of students, namely those who regard the theory of elliptic functions as merely one division of a greater theory and who thus study the elliptic functions not only on account of the interesting properties they offer *per se*, but also as a means of becoming more familiar with the principles and methods of the theory of functions, or as a stepping stone to the more abstruse theories of the abelian transcendents and automorphic functions.

The present volume meets the wants of this latter class most successfully. We are so impressed with its many merits that we do not hesitate to predict for it a rapid and widespread popularity.

The characteristic feature of the book is the predominance it gives to the ideas of Riemann. It is indeed remarkable, as Professor Burkhardt observes, that up to the present time no work on the elliptic functions has treated the theory from Riemann's standpoint. In several works on this subject we find reference to some of Riemann's ideas; but with the exception of Thomae's Abriss they are cursory and inadequate. We feel sure that this novel and valuable feature will be widely appreciated.

Another feature of the work is its comprehensiveness, accompanied by very moderate proportions. There is something so encouraging to the student in a text book of moderate size. The main divisions of the theory have received attention in accordance with their relative importance. By seeking everywhere the simplest form of treatment, Professor Burkhardt has succeeded in compressing a great deal into a very small compass. The student who reads this book with care will gain a very good idea of the modern theory of elliptic functions, in spite of the gigantic size this theory has assumed.

We indicate rapidly the contents. At the very outset an embarrassing question presents itself to the author of an elementary treatise on this subject: how are the elliptic functions to be introduced? Historically they arose in inverting the integral

$$u = \int_0^x \frac{dx}{\sqrt{f(x)}},$$

f being a polynomial of fourth degree.

The attempt to obtain analytic expressions of x considered as function of u led Abel and Jacobi to the theta functions, and these were used by the latter in his later university lectures as the fittest elements upon which to build up the theory. Already, then, in the infancy of this theory two lines of approach offered themselves, the one starting with an *implicit*, the other with an *explicit* definition. It was found that each had its advantages and disadvantages, and thus no traditional way of developing the theory has ever gained ground. In the last generation the two classic works were without doubt those by Briot and Bouquet and by Königsberger. The former starts with the thetas, the latter with the integral. Today we find the same lack of uniformity. Halphen's great treatise introduces the elliptic function by means of the integral definition; Weber and Krause, on the other hand, employ the thetas; finally the classic treatise of Tannery and Molk begin with Weierstrass's equivalent for the thetas, viz., the sigmas.

The advantages of starting with the integral definition seems to be chiefly these. First, it permits us to use Riemann's theories for which the elliptic integrals are merely a special, though extremely interesting, case. Secondly, in the physical applications it is as integrals that these transcendents appear. The objections are two-fold. For the student of pure mathematics who does not care to go beyond the elliptic functions, as well as for the student of physics, Riemann's theories form an unnecessary baggage. The second objection lies in the difficulty of establishing the one-valuedness of the inverse function at any early part of the course.

The advantage of beginning the elliptic functions with the thetas or sigmas is again two-fold. First, it defines them as explicit analytic expressions, which the student can see and from which he can deduce readily their principal elementary properties. Secondly, the existence theorem just mentioned falls away of itself. The disadvantage of this procedure is the unsatisfactory position it assigns the integrals.

Professor Burkhardt has followed a middle course. In chapter I. he has used the integral definition in connection with Riemann's surface. A simple proof of the uniformity of $x(u)$ for the case of real branch points is given by employing conformal representation. The general case is reserved for a later chapter.

In chapter II. an entirely new start is taken and we see no more of the elliptic functions as inverse of an integral until four chapters later, barring one or two fleeting references. The four chapters II., III., IV., V., occupying a little less than one hundred pages, give an account of double periodic functions in general, the $\wp$-function in particular, the functions σ, ϑ, ζ, and the functions of Hermite. The treatment follows the path opened up by Liouville and developed later by Hermite and Weber. There is not much chance here for an author to develop anything very novel when such masters have passed over the same route, but an attentive examination shows many minor merits.

The reviewer regrets that the historic functions of Jacobi, *sn*, *cn*, *dn*, are given an altogether inferior position. We touch here one of the serious difficulties which students encounter as soon as they begin to consult the literature of elliptic functions. They find to their dismay that there are two theories, running side by side, which, while but two aspects of the same thing, are yet so different as to make it impossible to pass from one to the other without considerable study. Now if students, when they first take up the elliptic functions, are taught almost altogether one theory, they are sure to experience a serious hindrance in their work later, since some mathematicians employ habitually the one, and some the other theory. It seems to us exclusive to maintain that either theory is the better; as well maintain that one system of coördinates always lends itself most simply to all problems. The ideal way is to have both theories equally in one's control. To this end it seems advisable, in an introductory course, to teach both simultaneously, pointing out as one goes along their interrelations. In this way the student does not acquire such a superior dexterity in one theory as to make it distasteful to him to employ the other. Such a course has not been followed here.

In chapter VI., which takes up the problem of inversion left unfinished in chapter I., we enter again the circle of ideas proper to Riemann. From now on they are our constant companions. This chapter treats, besides the problem of inversion, which is here brought to a close, various properties of integrals of the first, second, and third species; *e. g.*, Abel's theorem, the theory of Riemann-Roch, Legendre's relation, the interchange of argument and parameter in integrals of the third species, etc.

Chapters VII. to X. treat the following topics; reduction of elliptic integrals of the first species to canonical forms, the linear transformation, degeneration of the elliptic functions, and reality. In treating the linear transformation, an eighth root of unity enters as function of the coefficients of transformation. This was first determined by Hermite by making use of the sums of Gauss. Enneper and Thomae have shown how this root may be determined very simply apart from a plus or minus sign. Professor Burkhardt has been content to leave the problem at this point. For the more advanced parts of the theory this sign is simply indispensable. We take this occasion to remark that it may be determined à priori* by using no more of the theory of numbers than the simple law of reciprocity of quadratic residues.

True to his purpose to give Riemann's theory full sweep, Professor Burkhardt has not failed to show the relation be-

* Weber gives an à posteriori demonstration; but such verifications are always unsatisfactory.

tween linear transformation of the periods and alterations in the system of cuts which make Riemann's surface simply connected. Unless we are very much mistaken, this subject is treated here for the first time in any treatise on the elliptic functions. The effect of varying the branch points along certain curves, closed or not, so that the surface returns to its original shape, is also studied and the notion of the monodromy of the branch points introduced.

In chapter XI. we reach another grand division of the modern theory of elliptic functions, viz, the elliptic modular functions. The 51 pages which make up the chapter form one of the most interesting and instructive parts of the book. In the older theory the elliptic functions were studied almost uniquely as functions of a single variable, namely the argument u. But besides the argument, they involve certain parameters, the modulus $\varkappa$, the periods ω_1, ω_2, and the invariants g_2, g_3. With these, many important quantities are formed which do not contain u at all. As soon as the theory of functions became somewhat developed it must have occurred to many to investigate these quantities from a function-theoretical standpoint.

For example, what is the relation between $\varkappa$, which defines $sn(u, \varkappa)$ from the integral standpoint, and $\tau = \frac{iK'}{K}$, which we use to build the corresponding thetas. This question forces itself on one almost imperatively even in an elementary treatment.

The function $\varkappa$ of τ was found to possess the most remarkable properties. The most striking at first sight was that it had a natural boundary. Weierstrass's theory of functions, by starting with the notion of analytical continuation, had made the existence of such functions possible; and indeed isolated examples were discovered quite early. Thus it was known that the function

$$1 + 2q + 2q^4 + 2q^9 + \cdots,$$

which occurs already in the Fundamenta Nova,* possessed the unit circle as natural boundary. But here was a whole class which had this peculiarity. Another characteristic property is this; it is a one valued function remaining invariant for a subgroup of the general modular group

$$L = \left(\tau, \frac{a + b\tau}{c + d\tau}\right),$$

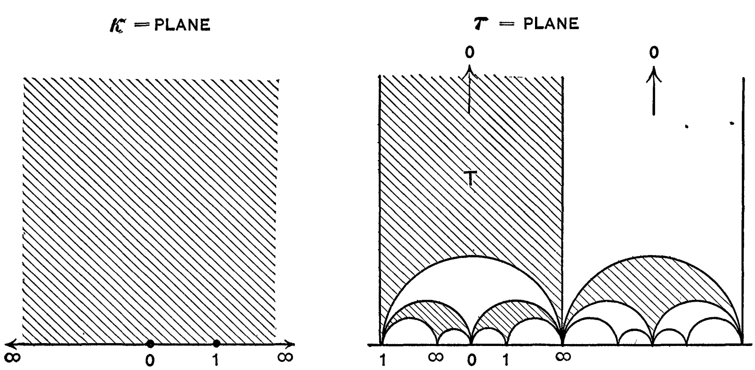
* Jacobi's Works, Vol. I., p. 235, § 65 (6).

a, b, c, d being integers and $ad - bc = 1$. The group in question, call it L_2, is, as Hermite first showed in his celebrated memoir "Sur la théorie des équations modulaires" (1859), defined by the congruences

$$a \equiv 0, \quad b \equiv 1, \quad c \equiv 1, \quad d \equiv 0 \qquad (\text{mod. } 2).$$

Such groups are called congruence groups; their order is given by the modulus, which is here 2. The conformal representation of $\varkappa$ by τ brings us to the important notion of fundamental domain. The points 0 and 1 divide the real $\varkappa$ axis into three segments $+\infty, 1$; $1, 0$; $0, -\infty$. In the τ plane these are represented respectively by the upper half of the circumference of a unit circle and the right lines yi, $1 + iy$, $(y > 0)$, which may be considered as the three sides of a circular triangle T whose vertices lie at $+1$, 0, ∞. If $\varkappa$ move now so as to cover every point once and only once in the upper half of the $\varkappa$ plane, τ passes over every point of this triangle once and only once.

Suppose now that $\varkappa$ passes into the negative half of its plane; it must cross over one of the three segments just mentioned. Then τ passes out of T crossing that side of the triangle which corresponds to the segment $\varkappa$ crossed. Let $\varkappa$ now cover all points of the negative half plane without going back into the positive half plane; we find that τ sweeps over a region which is got from T by reflection with respect to that side of T which τ crossed. This can therefore be nothing but a circular triangle, say T_1. If $\varkappa$ crosses back to the positive half plane and then covers again all its points, the same reasoning shows that τ passes out of T_1 and generates a new triangle T_2 also got from T_1 by reflection. As this process can go on indefinitely, we gradually cover the upper half of the τ plane by circular triangles as in the adjoined figure.



Any two successive triangles, as T, T_1, form a region within which $\varkappa\ (\tau)$ takes on every value once and once only. For this reason it is called a fundamental region. The triangles $T, T_1, \cdots$ we observe are of two kinds, those like the one we first considered spreading out from left to right and having all the same size, and secondly those which approach nearer and nearer the real axis, constantly diminishing in size. This shows at once that the real τ axis is a natural boundary. Suppose $\varkappa$ describes a circuit about $\varkappa = 0$ in the positive direction; we find is τ subjected to the transformation $S = (\tau, \tau + 2)$. If τ describe a circuit about $\varkappa = 1$, τ undergoes the transformation $\Sigma = \left(\tau, \dfrac{\tau}{1 - 2\tau}\right)$. As every circuit $\varkappa$ can make is made up of these two, we see that the group L_2 mentioned above is identical with the group generated by the repeated combinations of $S^{\pm 1}$, $\Sigma^{\pm 1}$.

Now the elliptic function presents an unlimited number of such functions and we are thus led to ask what are their properties and mutual relations. The attempt to answer this question has given birth to the theory of elliptic modular functions which today has become an independent branch of mathematical science. The present chapter will prove a very valuable introduction to the classic treatise by Klein and Fricke on this subject.

Closely related with the theory of modular functions are the problems of transformation and division; these occupy chapter XII. The linear transformation was discussed in an earlier chapter. Here a rapid orientation of the general transformation problem is given. The theory of this problem might alone easily fill a book. Historically it arose in trying to find commodious methods of computing the elliptic integrals of the 1° species

$$u = \int_0^x \frac{dx}{\sqrt{1 - x^2 \cdot 1 - \varkappa^2 x^2}} = \int_0^\phi \frac{d\varphi}{\sqrt{1 - \varkappa^2 \sin^2 \varphi}} \qquad (1)$$

where $x = \sin \varphi$, and $0 < \varkappa < 1$. The transformation of Landen and Gauss showed that it was possible to transform (1) into an integral of the same form, aside from a factor, viz.,

$$u = M \int_0^y \frac{dy}{\sqrt{1 - y^2 \cdot 1 - \lambda^2 y^2}} = M \int_0^\psi \frac{d\psi}{\sqrt{1 - \lambda^2 \sin^2 \psi}} \qquad (2)$$

on making a simple change of the variable x. By repeating these transformations the moduli λ, λ_1, $\lambda_2 \cdots$ may be made to approach 0 or 1 as a limit. That is, the approximate value of u is made to depend on a degenerate elliptic integral, viz., either

$$\int_0^{\psi_\kappa} d\psi = \psi_\kappa \quad \text{or} \quad \int_0^{\psi_\kappa} \frac{d\psi}{\cos\psi} = \log\tan\left(\frac{\pi}{4} + \frac{\psi_\kappa}{2}\right).$$

When κ is near $\frac{1}{2}$ it is necessary to repeat the algorithm several times before a fair degree of approximation can be obtained. In attempting to get transformations of greater rapidity Jacobi was led to his celebrated theory of transformation which he later made the basis of his Fundamenta Nova. He proposed and solved the problem: Determine the coefficients a, b, the new modulus λ, and the multiplier M so that

$$y = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_mx^m}{b_0 + b_1x + b_2x^2 + \cdots + b_nx^n}$$

converts (1) into (2).

From the standpoint of today the problem presents itself more naturally thus: Given two elliptic functions

$$f(u, \omega_1, \omega_2), \quad g(u, \bar{\omega}_1, \bar{\omega}_2),$$

constructed respectively on the networks

$$N(\omega_1, \bar{\omega}_2), \quad \bar{N}(\bar{\omega}_1, \omega_2);$$

what is their relation to one another? The most general relation we can assume between N and $\bar{N}$ is that they have a network M in common. Jacobi's problem is a special case of this, namely the case when one of the networks N, $\bar{N}$ becomes identical with M. In this case we have

$$\begin{aligned} \omega_1 &= a\bar{\omega}_1 + b\bar{\omega}_2, \\ \omega_2 &= c\bar{\omega}_1 + d\bar{\omega}_2, \end{aligned} \qquad (ad - bc = n),$$

and the transformation is said to be order n. For such a transformation the Jacobian formula (3) gives g as rational function of degree n in f. The case where $N = \bar{N}$ gives the linear transformation $n = 1$.

To treat the transformation in full requires us to pass to the thetas or their equivalents the sigmas. The most extensive and modern treatments are found in the works of Weber, Krause, and Tannery and Molk. In the elements, only the linear and quadratic transformations are useful and these are given with sufficient detail. The short sketch of the general theory which Professor Burkhardt gives will help the student to place these special transformations in their proper relation to the general theory.

The problem of division is this: The addition theorem shows that, for integral n, $\varphi(nu)$ is rational in $\varphi(u)$ and $\varphi'(u)$. This relation may however be regarded from another standpoint, viz., as defining $\varphi\left(\frac{u}{n}\right)$ as an algebraic function of $\varphi(u)$, $\varphi'(u)$. The problem is: what is the nature of this algebraic function; in particular express explicitly $\varphi\left(\frac{u}{n}\right)$ in $\varphi(u)$, $\varphi'(u)$. To solve completely either the problem of division or transformation we are led to certain algebraic equations called equations of transformation between modular functions which are of extraordinary interest. Professor Burkhardt has given an excellent account of these from a function-theoretical standpoint. They are also of equal interest from an algebraic and arithmetical point of view. For example, in algebra they define new algebraic irrationalities in terms of which the roots of a large class of algebraic equations can be expressed. Thus the equation between the old and new modulus $\varkappa$, $\bar{\varkappa}$ corresponding to a transformation of order 5 was used by Hermite (1858) to solve the general equation of fifth degree.

From the point of view of the higher arithmetic, they are chiefly interesting when complex multiplication takes place. This brings us to another great division of the modern theory of elliptic functions. What is complex multiplication? The answer lies in the following considerations: Up to the present point the periods of our elliptic functions have been left entirely free, with the sole restriction that

$$R\left(\frac{\tau}{i}\right) > 0. \tag{4}$$

But an immense field is opened up when we require τ to satisfy a quadratic equation of a certain kind. Suppose, in fact, we ask with Abel: Is it possible to determine μ other than an integer, so that $\varphi(\mu u)$ is rational in $\varphi(u)$ and $\varphi'(u)$ as in ordinary multiplication? The key to this question lies in the elementary theorem: In order that an elliptic function $\psi(u)$ may be rational in $\varphi(u, \omega_1, \omega_2)$ and its derivative, it is necessary and sufficient that ψ admits ω_1, ω_2 as periods. Applying this condition to $\psi = \varphi(\mu u)$ we have at once

$$\begin{aligned} \mu\omega_1 &= a\omega_1 + b\omega_2, \\ \mu\omega_2 &= c\omega_1 + d\omega_2, \end{aligned} \tag{5}$$

which gives a condition for τ, viz.,

$$b\tau^2 + \tau(a-d) - c = 0. \tag{6}$$

If τ is an independent variable, (6) requires that $b = c = 0$, $a = d$, which put in (5) makes μ an integer. If, however, we suppose τ is not variable, but on the contrary satisfies an equation of the type (6), equations (5) show that μ satisfies the equation

$$\mu^2 - \mu(a+d) + n = 0,$$

setting $n = ad - bc$ as usual. This gives

$$\mu = \frac{a + d \pm \sqrt{\Delta}}{2} \tag{7}$$

where $\Delta = (a-d)^2 + 4bc = (a+d)^2 - 4n$ is the discriminant of (6). The condition (4) shows that $\Delta < 0$; hence (7) shows that μ is never real, whence the name complex multiplication. We see then that, when $\varphi(u, \omega_1, \omega_2)$ admits complex multiplication, τ is root of a quadratic equation with integral coefficients and negative discriminant. The converse is obviously true. Modular functions built up on such τ's are called *singular*. The corresponding equations of transformation enjoy the most remarkable properties. The simplest case of complex multiplication arises in connection with the lemniscate. In fact, the length u of an arc is given by

$$u = \int_{\wp}^{\infty} \frac{d\wp}{\sqrt{4\wp^3 - \wp}}.$$

The value of τ is here given by $\tau^2 + 1 = 0$, hence $\mu = \pm i$. Already in the Disquisitiones Arithmeticæ (1801) Gauss calls attention to the fact that the equations of transformation for this case can be solved algebraically, a theorem which Abel generalized to all cases in the Recherches.

Let us indicate in as few words as possible the rôle complex multiplication plays in the higher arithmetic, namely, in the theory of binary quadratic forms, with negative determinant,

$$Ax^2 + 2Bxy + Cy^2 = (A, B, C).$$

Such forms Gauss always took with even middle coefficient. The modern theory shows that this was unfortunate, as most material simplifications arise when we allow the middle coefficient to be either odd or even. As no elementary account of the new theory has yet been given, we follow here the classical notation. We saw that the necessary and sufficient condition for complex multiplication was that τ be root of an equation

$$r\tau^2 + s\tau + t = 0 \tag{8}$$

with negative discriminant $\varDelta = s^2 - 4rt$. The coefficients we can suppose relative prime. According, then, as s is odd or even, we associate with (8) either the form

$$(2r,\ s,\ 2t) \quad \text{or} \quad (r,\ s/2,\ t).$$

The determinant D of this form is also negative. Conversely to every primitive form $F = (R, S, T)$ with negative determinant will correspond an equation of the type (8) which will give a singular τ.

Consider now the absolute invariant $J(\tau)$ built on the singular τ defined by (8) or the corresponding form F. Then to one form F with negative determinant corresponds one singular $J(\tau)$. Apply to τ a linear transformation, giving $\bar{\tau}$. Then $J(\bar{\tau}) = J(\tau)$. But the form $\overline{F}$ which corresponds to $\bar{\tau}$ is got from $\overline{F}$ by the same transformation; *i. e.*, F and $\overline{F}$ belong to the same class of quadratic forms. Thus not only does F give rise to a particular singular invariant $J(\tau)$ but every form in the same class as F gives rise to the same invariant. Let now $F_1, F_2, \cdots, F_H$ be properly primitive forms one from each of the $H(D)$ classes belonging to a given determinant D, and

$$J(\tau_1),\ J(\tau_2) \cdots,\ J(\tau_H) \tag{9}$$

the corresponding singular invariants. The equations of transformation between J and its transformed show that the quantities (9) are roots of an irreducible equation with integral coefficients. This equation defines thus a numerical algebraic body, which on account of this intimate relation with the classes of binary quadratic forms is called a class body. The theory of these bodies is interwoven in the most wondrous and fascinating manner with the theory of composition of quadratic forms of negative determinant and their division into genera.

The present work, being only one volume in a course on the theory of functions, quite rightly does not even touch these questions. Indeed only three pages are devoted to complex multiplication. We have however felt justified in going out of our way to speak of these questions partly because of their intrinsic interest and partly because the reviewer deplores how little the higher arithmetic is cultivated in America. The theory of complex multiplication with its intimate relation to binary quadratic forms and algebraic numerical bodies offers a promising field for young men who seek to gain distinction as original investigators.

Leaving this subject, continue with our review. The next chapter, the thirteenth, treats the question of numerical computation. This vexatious subject, so important in all practical applications, is very satisfactorily handled here. Care is taken to give limits of error, a point often neglected.

The last three chapters are devoted to applications, one being selected from each of the three broad fields of geometry, analysis, and mechanics. For geometry it is the theory of elliptic curves, *i. e.*, curves defined in homogeneous coördinates by the equations

$$\rho x_1 = T_1(u), \quad \rho x_2 = T_2(u), \cdots,$$

the T's being conjugate intermediate functions of Hermite, *i. e.*, functions of the form

$$e^{au}\sigma(u - b_1)\sigma(u - b_2) \cdots \sigma(u - b_n)\,.$$

Curves of the third and fourth orders receive especial attention. The application to analysis is the discussion of Picard's equation of the second order, in particular Lamé's equation

$$\frac{d^2y}{du^2} = \{n\,(n+1)\,\wp u + b\}y. \tag{10}$$

The application to mechanics is the spherical pendulum. In discussing the horizontal motion of the bob, the author passes to polar coördinates, thereby missing a pretty application of Lamé's equation discussed in the chapter just preceding. In fact, keeping rectangular coördinates x,y, it is easily shown that $x + iy$ and $x - iy$ are two solutions of (10) for $n = 2$.

The rapid survey we have here made shows most clearly that we have in the present volume a text book which is rich to an uncommon degree in the latest results and points of view in the subject it treats. It fills most timely a place unoccupied by any other work and will, we are sure, prove itself invaluable to the large class of readers for which it is intended.

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