The following is a summary of Dr. Porter's paper: Consider $3m - s - 1$ arbitrary fixed points P on a non-singular cubic C_3, and $u_i = \int_{ab}^{x_i y_i} du$ the integral of first kind on C_3, (ab) being a point of inflexion. If an m-ic have a $s-1$ order contact at u_1, it will cut C_3 again at u_2, $su_1 + u_2 \equiv C$ (mod. ω, ω') where $C = \Sigma u_i$ at the points P: The Schliessungsproblem thus suggested yields at once a proof of Fermat's theorem $a^n - a \equiv 0$ (mod. n (prime)) and the generalized form of the theorem $F(a, n) \equiv 0$ (mod. n). When $m = 1$, $s = 2$, we have systems of closed polygons. In case the polygon is a triangle, the equation of C_3 referred to it may be written

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + 2\eta = 0.$$

The twenty-four in-circumscribed triangles thus determined fall into four groups, each associated with an inflexion triangle, and each triangle of a group six ways perspective with its associate inflexion triangle. This configuration of inflexion triangles and in-circumscribed triangles presents numerous interesting geometrical properties.

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REPORT OF THE DISCUSSION AT THE SEVENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE final session of the Seventh Summer Meeting of the Society was devoted to an organized discussion of the following question:

What courses in mathematics shall be offered to the student who desires to devote one-half, one-third, or one-fourth of his undergraduate time to preparation for graduate work in mathematics?

The following topics were also suggested as a general basis of discussion:

How early in the course may the lecture method be used with profit?

How can the undergraduate best be trained to use the library?

How shall the history of the subject be presented?

Shall we try to place the calculus as early as possible?

Where shall we place the solid analytics? modern geometry? projective geometry?

Shall spherical trigonometry and geometric conics be given a place in the curriculum?

How much of the theory of equations may be presented with profit in the freshman year?

What courses in applied mathematics are needed as preparation for graduate work in pure mathematics?

Shall work in differential equations be merely a problem course, or shall it take up the theoretical side of the question? If the latter, to what extent?

At what points shall the work intended to fit the student for graduate work differ from that intended to fit him for secondary school teaching?

Are the best results in graduate work secured from students who have devoted most of their undergraduate time to mathematics, or from those who have combined a fair amount of mathematics with a more general culture?

Shall the undergraduate school attempt to attract the student to graduate work by offering elementary courses in the more advanced topics, or shall it confine itself to fundamental work in algebra, analytics, and calculus?

The principal papers are given in abstract below.

Professor Moore.

If the student is to be properly prepared for graduate work the teacher must be himself familiar with modern mathematics; and the fundamental modern ideas and methods must give form to his work as an instructor of undergraduates. Thus the teacher should have clearly in mind the notions of the pure analysis—of the sequence of positive integers as a complete set, of the synthetic determination of all real numbers from this sequence of integers and of the complex variable as a double real variable. Similarly he should have the (usual) idea of (a) geometry as built upon a body of basal notions and axioms.

In general, mathematics should be regarded as divided into a number of distinct but closely related sciences. Each of these sciences consists of a body of sharply defined and inter-related ideas as foundation, with a superstructure consisting of results deductively secured from these fundamental ideas. For every science the question of or the condition for its existence as a body of non-conflicting statements is of prime importance, and especially by these existence considerations are the various sciences related and organized into higher sciences.

Intuition is by no means to be excluded from mathematical work. On the contrary, in every way let intuition be kept alive and active, but when a new idea is introduced as an intuition let it be sharply defined with the understanding that in so far as subsequent work depends on that intuition a new branch of mathematics is being originated.

In illustration: It is questionable whether the undergraduate course of a student who is not specializing extremely in mathematics may advisably contain the existence theory of the real irrational number in its ultimate form as a part of pure analysis. The student, however, may be led to grant the fundamental properties of the system of all real numbers as intuitively true for the system of all points of a straight line. On the basis of these assumptions the student develops a geometric analysis which will only later in pure analysis receive a deeper foundation, the character of which however should at once be briefly indicated. Similarly, in dealing with the complex variable in the Gauss plane, the radius and the angle may be regarded as simple ideas, although they are to appear as much more complicated creations in pure analysis.

Professor Harkness.

In these days of extreme specialization half the college time is too little to serve as an adequate basis for later work, unless the student enters college well prepared in elementary mathematics. With improved educational machinery in the secondary schools, there is no reason why a fair number of our college freshmen should not have had some preliminary training in analytic geometry, differential calculus, statics, and dynamics. This is the case in England and would be the case here, if the brighter boys were not held back in order that they may not get too far in advance of the less intelligent members of their classes.

As regards college students I urge the importance of increased attention to applied mathematics. In mathematics as in everything else there are fashions. At one time the subject of invariants, at another that of the theory of functions, and later still Lie's continuous groups have engrossed the attention of pure mathematicians; today algebraic numbers, or briefly those branches of mathematics which we associate with the names of Kronecker, Hilbert, Hensel, and Weber, are in a central position. All this time applied mathematics have been out of fashion, but there are many signs that a change is coming and the intelligent college instructor will do well to anticipate this change as far as possible.

However this may be, it is of great importance for the pure mathematician of today that he should be well grounded in the fundamental principles of statics, dynamics, and the theory of the potential. Riemann's programme cannot be appreciated at its proper value without a firm grasp of the fundamental principles of the theory of the potential so far as it relates to Dirichlet's Problem. It would be easy to multiply examples of this kind ; for instance Schottky's memoir in the 83d volume of Crelle contains Weierstrass's Lückensatz, and is therefore of the greatest interest to the mathematician concerned with algebraic functions and Abelian integrals, but papers of this kind are apt to be passed by as unintelligible unless the reader has had some preliminary training in the discussion of conjugate functions, conform representation, and the behavior of the potential in multiply connected regions.

A good early training in statics and dynamics (including dynamics of a particle and the elements of rigid dynamics) would go far to remove many of the difficulties experienced by graduate students in the reading of modern memoirs, and would widen their mental horizon in a variety of ways. In the college instruction itself such a training would throw much needed light on many of the problems of the differential and integral calculus; partial differential equations would be connected with physical problems, and the undergraduate would feel, as he hardly feels at present, that the formulæ he meets with are capable of concrete representations.

I urge the necessity of a thorough revision of college courses in differential calculus. The subject is too often taught along traditional lines ; the lecturer should ask himself whether certain parts of his subject possess the same importance today that they had in Euler's time. Much might be rejected as of secondary importance in comparison with the newer developments. Our text books abound, too, with proofs that are no proofs and with extremely misleading statements. It is very desirable that such proofs should be rejected, or given only with clear indications of their limitations. A graduate student in the theory of functions often finds it impossible to rid his mind of erroneous notions derived from the days when he first began the study of the differential calculus.

While due emphasis should be laid on the fundamental concepts of function, limit, continuity, etc., the work of the undergraduate should be lightened as far as possible. He should not be asked to differentiate $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ or to work out examples which illustrate no general theory and are unlikely to be of use to him later on. Moreover care should be exercised when he comes to applications; the simple cases are usually discussed correctly, but the writers of our text books are too apt to go further and attempt a partial and inaccurate discussion of more difficult cases. Here again it would be easy to multiply examples; it will be sufficient to refer to the criteria for maximum and minimum values and to the accounts usually given of the higher singular points on an algebraic curve.

In conclusion I desire to emphasize the great importance of thoroughness; the greatest service that can be rendered to a prospective graduate student is to give him a complete mastery over those parts of the technique of mathematics which he is certain to need at a later stage of his work.

Professor Osgood.

In school instruction in geometry the idea is emphasized from the beginning that certain definitions and axioms are laid down, on which the whole subject of geometry is built up by means of logical reasoning; and the pupil's reasoning power and appreciation of geometric truth are developed by means of original exercises that are given him to work out by himself. That instruction in algebra is at present far less highly developed as regards effectiveness for the purposes both of general training and of giving the pupil a correct conception of the subject of algebra itself, is due doubtless in part to the formal side of the subject. Dexterity in the manipulation of formulas is indeed important for later work; but the foundation of algebra is arithmetic, and the arithmetic source of algebraic principles ought not to be lost sight of to any such extent as is at present the case. How far improvement here in school instruction is possible I will not undertake to say, but of this I am sure, that this difficulty must be met and can be met from the start in instruction in calculus. The foundation of the calculus as we have the science to-day is arithmetic. Now it is neither feasible nor desirable to begin a college course in calculus by first developing the number concept. I find it useful to assume the conception of a scale ; more precisely, to make the assumption that, two points having been chosen at pleasure on a straight line to represent the numbers 0 and 1, every other (real) number can then be represented by a point of the line; and conversely, to every point of the line there will correspond a number. Similarly the conception of a function of a single variable or of two variables, is coextensive with the intuitive picture of a plane curve or of a surface. The fundamental principles of the calculus must be taught in a manner wholly different from that set forth in the text books if they are to become flesh and blood to the student. It is an easy method for the teacher to expound the notion of the limit in the opening lectures of the course and from there on to appeal to this conception, this mode of thinking, as if it had become a part of the mental machinery of the beginner. No greater mistake can be made. The student will indeed learn to perform differentiations and integrations ; but he will not make progress in mastering the central ideas of the calculus. The notion must be set forth repeatedly throughout the whole course. A valuable aid in making this notion clear is furnished by the application of the calculus not merely to differential geometry, but (the integral as the limit of a sum having been introduced early in the first course in calculus) to problems in volumes, centers of gravity, moments of inertia, fluid pressures, gravitational attractions, etc.* It is the deduction of the formula that brings out anew each time the fundamental principle of the limit, while the problems themselves are typical applications of the calculus to physics, and interest and train alike the student who is preparing for work in pure mathematics and him who is to concern himself primarily with its applications.

I have thought it necessary to state with some fulness some of the leading features of a course in calculus, for there are still colleges in this country in which calculus means a systematic study of Williamson's text book—formal differentiations, formal integrations, and a total lack of the ideas that are central in the calculus of today. A course in calculus along the lines that I have indicated is effective in introducing the student to modern mathematics, and its notions and methods are in themselves interesting

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* This part of an introductory course in calculus is well set forth by the problems in Chap. V. of Professor Byerly's Problems in Differential Calculus.

and valuable for the purpose of general training. It is best placed in the sophomore year for students who have entered college on the ordinary requirements.

As to how early the lecture method may be used with profit I may say that at Harvard the lecture method has been employed with success for over twenty years in practically all of the courses in mathematics, except in the freshman course in solid geometry. Written exercises, chiefly problems, are handed in by the student at each meeting of the course, and the instructor frequently spends some minutes on points that may have given general difficulty or require special comment.

It has been the practice at Harvard, ever since the graduate school came into existence, not to separate the graduate and the undergraduate departments as regards methods of instruction or the unrestrained admission of properly qualified students registered in the one department to the courses of the other department. Since there is free election from the freshman year on and the choice of courses is large after a student has passed beyond the elementary courses in calculus, geometry, and mechanics, it is the rule rather than the exception that the bright student, following his inclination to a special line of work in mathematics, will proceed rapidly to higher courses in this particular line; and thus a course ordinarily taken by first year graduates, say, will have seniors or even juniors in it, while on the other hand students that have paid more attention to other lines of work may not take it till a later graduate year. The advantage that is gained from this arrangement is obvious. The older students bring with them greater maturity for the work of the course and contribute to setting a higher standard, and the younger ones derive from the work they see their fellow students doing profit which they could not get from the lecture alone. In this respect the German university is especially strong. There, a student has free scientific intercourse with more advanced fellow students from the time he enters the university till he becomes himself a professor. At Göttingen, for example, there is a mathematical society composed of the university teachers, students whose attainments are measured approximately by the doctor's degree, and a goodly number of advanced students whose attainments are intermediate between those of these two classes. At its weekly meetings the current literature is discussed and lectures on current topics are given by its members. The formal meeting is followed by an informal one, usually held at an inn or in a summer garden, at which there is abundant opportunity for free discussion of scientific topics. The advantages that accrue to the undergraduate courses in one of our colleges from the presence of a graduate school are not unlike the benefits that the younger members of this society derive from their intercourse with its older members.

Professor Morley.

As to the socalled geometrical conics, a little of this might well be tacked on to elementary plane geometry. The beginner is apt to suppose that Euclid's methods in plane geometry are good only for the line and circle. Spherical trigonometry should form not a separate course but a chapter in solid analytics. The latter is somewhat neglected in college work, owing to the difficulty which poor students find in it. The plan of joining both two and three dimensions in one book is a good one, but the books in use are too old fashioned for the class of men we are considering. As to the theory of equations in the old sense (that of Burnside and Panton), it can be and I think should be attached to elementary analytic geometry. For instance when the curve

$$y = a + bx + cx^2$$

is mastered and its slope can be found, then the curve

$$y = a + bx + cx^2 + dx^3$$

should be fully discussed. The points where the curve meets $y = 0$ would lead to the consideration of Horner's method. The general parabolic curve would then be mentioned. Taylor's theorem would make its first appearance here, as a convenient formula for changing the origin and discussing the curve near a point other than $x = 0$. Similarly the notion of covariants should grow out of geometry.

In general the notions of arithmetic and geometry with which a student enters college should be developed by interaction, say through plane and solid analytics and a course in calculus, before either the theory of arithmetic or that of geometry is considered philosophically. This is in effect a plea for the existing order of subjects, if they are presented in a way which leads up to the subjects as they stand, and satisfies both the refined views of the teacher and the common sense of the student. Thus, in the case of the calculus, the books used to begin with some tremendous generalities.

We have either to justify these, or to throw them aside, or to do both. We should do both—first have a calculus proper, of the kind dear to physicists, accurate but not abstract, carefully weeded of difficult generalities; and at a later stage a philosophical theory. To begin with the latter would to my mind be a mistake; the appeal would be to too limited a class.

Thus in my view the first half of a college course would be the same for the future mathematician as for the future physicist. And in fact there would be direct gain for the mathematician if the work were better adapted for the physicist than at present. Thus we ought not only to connect arithmetic with geometry, but also with dynamics, and perhaps with other physical subjects such as electricity, taking care that the student has the definite physical concepts to which we are appealing.

Lastly we must take hints from the technical schools, for instance with regard to drawing diagrams and the use of integrators or of abaci.

Professor Young.

Considering the fundamental topic of this discussion to be:—*The work in mathematics in college: its purpose, scope, methods and relation to graduate work in mathematics*, my remarks are to be made from a special point of view, viz., when the purpose is to prepare teachers of mathematics for the secondary schools.

I. *The need of such preparation.*—It is a truism to say to this audience that secondary school teachers of mathematics should have had a collegiate education with special attention to mathematics either by electives in the college course or by some graduate study. It is equally well known to this audience that this standard is by no means generally reached. The time seems to be ripe, however, for setting up the general culture of a full college course and the special preparation of a certain minimum of mathematical attainment as norm for the educational qualifications of new appointees to the work of teaching mathematics in our secondary schools.*

II. *The preparation, when to be attained.*—However desirable some graduate work may be, the fact must be recognized that at present the norm mentioned above represents a decided advance, and that for some time to come the majority of the secondary school teachers will have no preparation

* Cf. Resolution III of the Committee on the College Entrance Requirements, p. 30 of report submitted July, 1899.

beyond a college course. The college program should consequently include all the work needed for the minimum of preparation.

III. *The subject matter of the minimum course.*—As the minimum of mathematical attainment the following courses are suggested:—Thorough courses in plane trigonometry and college algebra, and good first courses in analytic geometry of the conic sections, differential and integral calculus, theory of algebraic equations, determinants, modern synthetic geometry, and, if possible, analytic mechanics.

IV. *Pedagogic preparation.*—While experience has shown that special training in the art of teaching mathematics is of high value, it is desirable that the recipient of this training be as mature as is feasible. The best place for the pedagogic training and instruction is therefore *after* the college course; either in the university or by some form of apprenticeship in the secondary schools. (Under favorable circumstances, helpful work of this sort could doubtless also be done in the senior year of the college.)

V. *Mathematical Independence.*—By mathematical independence I mean a degree of mastery of subject matter and methods such that the belief in the truth of results is based solely on the authority of one's own reason, and such that one is conscious of the power to apply these results appropriately and correctly. Within a narrow range, this independence may be attained very early; even the young pupil may have it, widening its scope as his attainments increase. Indeed, good teaching strives to cultivate this independence from the outset, and the teacher who is to cultivate it must himself evidently possess it and in a distinctly higher grade. The pupil, the teacher, the scholar, each should be in his degree master of his work. The teacher's mastery requires that he teach on his own authority, and not on that of others (though the mastery of the teacher need not be in the same degree productive as the mastery of the scholar). Unfortunately, our teachers have only too often not attained this independence. Too frequently a text is taught rather than the subject, the teacher having in reality not gone beyond the pupil's stage of mastery.

It is easier to point out the difficulties than to specify the remedy. The attainment of some measure of scientific independence suggests itself. This is the German plan, but it is not now feasible here. It seems to me, however, that somewhat can be done even within the limits of the college course. Our present methods and programmes tend in one respect to encourage diffidence rather than independence.

Mathematics requires to be digested and assimilated as well as acquired. The trend of our work is to cover as many subjects and as much ground as possible, and this continual advance with its successive generalizations and its revelations of incompleteness in what has been accepted as complete tends to cultivate a lack of confidence, accentuated if the student is embarrassed by difficulties of detail. This feeling of uncertainty takes shape in reliance upon others (book or person) to pronounce final judgment upon whatever is done. Perhaps something could be done to give the student the more secure grip on the secondary school subjects which the teacher must have by taking up these subjects again at the close of the college course and treating their principles and methods (not new details, as a rule), with the greater thoroughness and breadth of view which are made possible by the collegiate mathematics as outlined in the minimum course above.

VI. *Difference between the work of the prospective teacher and the prospective graduate student of mathematics.*—In the college this difference may be very slight.

VII. The teachers needed (in college) to carry out the above programme fully must have the wide horizon of a university training.

W. H. Maltbie.

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ON A MEMOIR BY RICCARDO DE PAOLIS.

BY PROFESSOR CHARLOTTE ANGAS SCOTT.

About twenty years ago de Paolis published a series of memoirs* dealing with the (2, 1) transformation of the plane; of these the second and third are concerned with applications of the theory to non-euclidean geometry and to

* "Le trasformazioni piane doppie"; *Atti d. r. Accad. d. Lincei*, series 3ª, vol. 1 (1877); pp. 511–544.

"La trasformazione piana doppia di secondo ordine, e la sua applicazione alla geometria non euclidea"; *Atti d. r. Accad. d. Lincei*, series 3ª, vol. 2 (1878); pp. 31–50.

"La trasformazione piana doppia di terzo ordine, primo genere, e la sua applicazione alle curve del quarto ordine"; *Atti d. r. Accad. d. Lincei*, series 3ª, vol. 2 (1878); pp. 851–878.