THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE eighth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Thursday and Friday, December 27 and 28, 1900. There were two sessions each day, opening at 10 o'clock A. M. and 2.30 o'clock P. M. Thirty-eight persons were in attendance among whom were the following twenty-six members of the Society :

Professor Henry Benner, Dr. G. A. Bliss, Professor Oskar Bolza, Professor D. F. Campbell, Professor L. E. Dickson, Professor L. W. Dowling, Dr. J. C. Fields, Professor A. S. Hathaway, Professor Thomas F. Holgate, Dr. Kurt Laves, Professor Heinrich Maschke, Professor E. H. Moore, Dr. F. R. Moulton, Professor H. B. Newson, Mrs. H. B. Newson, Professor D. A. Rothrock, Dr. F. H. Safford, Professor Oscar Schmiedel, Miss Ida M. Schottenfels, Professor J. B. Shaw, Professor E. B. Skinner, Mr. Burke Smith, Professor E. J. Townsend, Professor C. A. Waldo, Dr. Jacob Westlund, Professor H. S. White.

Professor E. H. Moore, Vice-President of the Society, occupied the chair. The Christmas meeting being the regular time for the election of officers of the section, Professor Thomas F. Holgate was reëlected Secretary and Professors E. J. Townsend and J. B. Shaw were made members of the programme committee.

The following papers were read :

(1) Professor E. H. MOORE : "On the uniformity of continuity."

(2) Professor A. S. HATHAWAY : "Quaternions and fourfold space."

(3) Professor IRVING STRINGHAM : "On the geometry of planes in a parabolic space of four dimensions."

(4) DR. F. H. SAFFORD : "Flow of heat in two dimensions."

(5) MR. A. C. LUNN : "Certain mathematical aspects of experimental science."

(6) MR. E. A. HOOK : "Some properties of circulating decimals."

(7) Professor ARNOLD EMCH : "Note on the congruences of twisted curves."

(8) Professor H. B. NEWSON : "A generalization of the

Wessel-Gauss-Argand diagram" (preliminary communication).

(9) DR. F. R. MOULTON: "On straight line solutions of the problem of n bodies."

(10) DR. G. A. BLISS: "Geodesic lines on an anchor ring."

(11) MR. FRANZ A. LA MOTTE: "On the determination of the algebraic equations invariant under Tschirnhausen transformations, with the parameter representation of all such irreducible equations, with rational coefficients, of the third and fourth degrees."

(12) Professor E. J. TOWNSEND: "Functions of two real variables which are continuous with respect to each variable."

(13) Professor L. E. DICKSON: "The group of the equation for the twenty-seven lines on a general cubic surface."

(14) Professor OSKAR BOLZA: "Concerning the expression of abelian integrals in terms of a fundamental set of integral functions."

(15) Dr. J. C. FIELDS: "Proof of the Riemann-Roch theorem and of the independence of the conditions for adjointness."

(16) Professor OSCAR SCHMIEDEL: "Two reduction formulas applicable to certain particular integrals."

(17) Professor E. B. SKINNER: "Some forms which remain invariant with respect to certain ternary monomial substitution groups."

(18) Professor JAMES B. SHAW: "Note indicating a new development of a determinant."

(19) Professor E. H. MOORE: "On double limits."

(20) Professor E. H. MOORE: "Concerning the Harnack theory of improper definite integrals."

(21) Professor L. E. DICKSON: "Canonical forms of quaternary abelian substitutions in an arbitrary Galois field."

(22) Miss IDA M. SCHOTTENFELS: "Proof of the existence of a particular substitution group of degree twenty-one and order 20160."

Professor Hathaway also read a paper on "Pure mathematics for engineering students," which was followed by a very interesting discussion. This paper is printed in the present number of the BULLETIN.

Mr. Lunn was introduced to the Society by Dr. Moulton, Mr. Hook by Professor Skinner, Mr. La Motte by Professor Moore. Professor Stringham's paper was presented to the Society through Professor Moore and in the author's absence was read by Professor Hathaway ; Professor Emch's paper and Professor Dickson's paper (No. 21) were read by title. Abstracts of the papers follow below.

In Professor Moore's paper on the uniformity of continuity, the well known theorem and a generalization of it which occurs in the theory of discontinuous functions are proved by a very simple method involving the partition of a fundamental interval (a, b) into three halves (a, d), (c, e), (d, b), where the points $a\ c\ d\ e\ b$ are the extremities of the four quarters of (a, b).

The first part of Professor Hathaway's paper consisted of a development of quaternions from the definition : Quaternions is the most general linear associative algebra in which division is determinate. The merit claimed is in starting from this known property as a definition, and developing smoothly and briefly a working knowledge of quaternions, without reference to the units 1, i, j, k.

The characteristic equation of a number p is the equation of least degree with numerical coefficients that is satisfied by p, say,

$$p^{r+1} + c_1 p^r + \cdots + c_r p = 0.$$

There is one and only one such equation for a given number ; it is not factorable into expressions of the same form ; and c_r is not zero. This determines a unit

$$I = -(p^r + c_1 p^{r-1} + \cdots + c_{r-1} p)/c_r,$$

with the properties

$$I^2 = I, \quad Iq = qI = q,$$

where q is any quaternion. I is the scalar unit, and xI, or simply x, where x is a numerical coefficient, will denote a scalar number. A scalar number may also be defined as a quaternion whose square is a positive scalar ; and a vector number is defined as a quaternion whose square is a negative scalar. Scalars are seen to be commutative factors.

Using scalar coefficients, the characteristic equation becomes, in irreducible form, $p^r + c_1 p^{r-1} + \cdots + c_r = 0$; and as this factors in the same way as ordinary equations with *real* coefficients, we must have $r = 1$ or $r = 2$, according as p is a scalar or a non-scalar. When p is a scalar, we take the square of its characteristic, and thus have a characteristic equation for any number p, of the form $(p-p_1)^2+p_2^2=0$, where p_1, p_2 are scalars depending upon p. We define :

Scalar of $p = Sp = p_1$; vector of $p = Vp = p - p_1 = p - Sp$.

Tensor of $p = Tp = \sqrt{(p_1^2+p_2^2)}$; unit or versor of $p = Up = p/Tp$.

Conjugate of $p = Kp = Sp - Vp$; angle of $p = \angle p = \cos^{-1} SUp$.

The characteristic equation of p is, in this notation,

$$(a) \qquad p^2 - 2Sp \cdot p + Tp^2 = 0.$$

Other forms of the same characteristic are

$$Tp^2 = pKp = Kp \cdot p = Sp^2 - Vp^2 = Sp^2 + TVp^2,$$
$$p^{-1} = Kp/Tp^2, \text{ etc.}$$

The uniqueness of the characteristic (a), determines the uniqueness of the resolutions

$$p = Sp + Vp = Tp \cdot Up = Up \cdot Tp, \text{ etc.}$$

From the characteristic equation of pq, we find by operating with $p^{-1}(\quad)p$, the characteristic of qp, and hence, $T(pq) = T(qp)$, $S(pq) = S(qp)$. In particular, if α, β be any two vectors we find

$$(\alpha\beta + \beta\alpha)(\alpha\beta - \beta\alpha) = (\alpha\beta)^2 - (\beta\alpha)^2 = 2S\alpha\beta(\alpha\beta - \beta\alpha) ;$$

and thence, whether $\alpha\beta = \beta\alpha$ or not, we find $\alpha\beta + \beta\alpha = 2S\alpha\beta$. Thus $(\alpha + \beta)^2$ is a scalar, whence $\alpha + \beta$ is either zero or a vector. Thence follow the usual distributive properties of the functional symbols S, V, K over a sum. Also, the conjugate of a product equals the product of the conjugates of its factors in reverse order ; the tensor of a product equals the product of the tensors of its factors, etc. The usual relations of products of vectors are found as in Tait, §§ 89–91 ; and in particular

$$\delta \cdot S(\alpha\beta\gamma) = \alpha S(\beta\gamma\delta) + \beta S(\gamma\alpha\delta) + \gamma S(\alpha\beta\delta);$$

or, given three independent vectors, any fourth vector can be expressed in terms of them. If there is only one independent vector, the system is the ordinary imaginary system. If there are two unit vectors α, β that are independent, then $\alpha - \beta$, $\alpha + \beta$, $(\alpha - \beta)(\alpha + \beta) = 2V\alpha\beta$ are three independent vectors, whose units are, say i, j, k, with the relations $i^2 = j^2 = k^2 = -1 = ijk$.

The second part of the paper is an application of quaternions to fourfold space, first given in the BULLETIN, November, 1897, and afterwards extended in the papers on Alternate processes (*Proceedings of the Indiana Academy of Sciences*, 1897) and Linear transformations in four dimensions (abstract in BULLETIN, November, 1898, no. 5, p. 93). The application is founded on the definitions

(1) Line $(w + xi + yj + zk) =$ line whose components along the four mutually perpendicular axes of reference are w, x, y, z;

(2) $\quad$ Line $p +$ line $q =$ line $(p + q)$,

(3) $\quad$ Line $p \cdot$ line $q =$ line pq.

With p as multiplier and q as multiplicand there are two products pq and qp. Such multiplications *with* or *by* p, are the two kinds of multiplication referred to as direct and contra multiplication in the first paper, and are now called in multiplication and by multiplication. A given in- or by-multiplication is shown to possess a system of invariant planes, one and only one through each line; and the angular displacement in each plane is constant in magnitude and therefore in sense, and equal to the angle of the multiplier. The ratio of elongation is the tensor of the multiplier. The invariant planes of an in-multiplication are called in-parallel planes; and by-parallel planes are invariant planes of a by-multiplication. A given angularly-directed plane is shown to be a plane of an in-parallel system of one and only one unit vector α, and a plane of a by-parallel system of one and only one unit vector β; and these vector units (corresponding to directions in ordinary or vector space) are called the in direction and by direction of the given plane. In the case of two planes, the angle between these in-directions is called their in-angle of inclination, and the angle between their by-directions their by-angle of inclination. It is shown that these angles remain unaltered by rigid displacement, the effect of such rigid displacement being to give all in-directions of planes a revolution about one vector axis, and all by-directions a revolution about another vector axis. It is shown that two planes have in general two and only two common perpendiculars (a perpendicular to a plane being a plane that "transverses" it in a right diedral angle). These two common perpendiculars are "orthogonal" to each other, *i. e.*, every line of one is perpendicular to every line of the other. In-parallel or by-parallel planes are shown to have a continuous turn-parallel system of orthogonal common perpendiculars, and all "plane" angles between the two lines of intersection with a common perpendicular are equal. For any two planes the two plane angles of intersection by the orthogonal common perpendiculars are the half sum and half difference of the in-angle and by-angle between the planes. The area-projecting factor between two planes is in sign and magnitude the product of the cosines of their plane angles. The second part of this paper will appear in the *Transactions*.

Professor Stringham adopts essentially definitions (1) and (2) of Professor Hathaway's paper, but not definition (3). The subject is analytically treated by equations of loci, with the interpretation and manipulation of their parameters. The fundamental equation of a plane through the origin is $\alpha r + r\beta = 0$ where α, β are unit vectors and r is the director line of any point R of the plane. This is the plane which Professor Hathaway fixes as of in-direction $\pm \alpha$ and by-direction $\mp \beta$. After developing the analytical material of distances between points, and points and planes, and of angles between lines, the author discusses the maximum angle between two variable lines lying in given planes, applying calculus methods to the varying parameters. He finds in general two such angles called the plane angles of the two planes. He finds the projective factor between the two planes $\{\alpha, \beta\}$, $\{\alpha', \beta'\}$ to be $-\frac{1}{2}(S\alpha\alpha' + S\beta\beta')$, which is also the product of the cosines of the plane angles. This expression is Professor Hathaway's

$$\tfrac{1}{2}(\cos\theta + \cos\varphi) = \cos\tfrac{1}{2}(\theta + \varphi)\cos\tfrac{1}{2}(\theta - \varphi).$$

Professor Stringham defines this projective factor as the cosine of the angle between the planes. This paper is to be published in the *Transactions*.

Dr. Safford inserts a term expressing surface leakage in the usual differential equation for the flow of heat in two dimensions, considers the possibility of obtaining solutions of a particular form for this equation, and deduces the isothermal curves and curves of flow. The methods used involve the discussion of a solution of a partial differential equation in the form λ, R_1, R_2, where R_1 and R_2 are functions of the two independent variables respectively, and are the general solutions of two ordinary differential equations. The factor λ is a function of both variables but does not contain the integration constants appearing in R_1 and R_2.

Mr. Lunn showed that if a physical hypothesis is given by implication in the form of an equation defining the value of a variable quantity as a function of the time, there exists at least a double infinity of functions which are not distinguishable from it by measurement and so far to be considered as equally verified by observation. Also if there exists one satisfactory hypothesis, there exists an infinity. Hence an induction using only the observed values and no hypothetical elements can never give a definite result.

Mr. Hook gave a method by which the number of digits in the repetend of any circulating decimal may be found when the prime factors of the denominator are known. His result differs from similar results found in text books in that it gives the exact number of digits while most such formulas yield only a number of which this is a divisor.

In Professor Emch's paper a study is made of the congruence of curves of intersection of the surfaces $F(x, y, z, a, b) = 0$ and $\varphi(x, y, z, a, b) = 0$. If $b = f(a)$ the system of curves lies upon a surface which has an envelope when

$$\frac{\partial F}{\partial a} + \frac{\partial F}{\partial b} \cdot \frac{db}{da} = 0 \text{ and } \frac{\partial \varphi}{\partial a} + \frac{\partial \varphi}{\partial b} \cdot \frac{db}{da} = 0.$$

Eliminating x, y, z from these conditions and the equations of the surfaces, a new relation

$$\psi\left(a, b, \frac{db}{da}\right) = 0$$

is found. If this last equation has a singular solution, the corresponding system of curves forms the singular surface of the congruence and all surfaces of the congruence are tangent to it.

In Dr. Moulton's paper those solutions of the problem of n bodies are sought in which the bodies always lie in a straight line and describe conic sections with respect to their center of gravity. The problem was solved by Lagrange for three bodies; this paper reduces the problem in the general case of n bodies to the solution of simultaneous algebraic equations. The solutions, which can be obtained in each case for a particular value of one of the masses, are studied as functions of this mass regarded as variable; it is shown that these functions have no poles and, by the non-vanishing of the Jacobian of the functions with respect to the coördinates, that they have no branch points. Therefore there are $\frac{1}{2}n!$ distinct straight line solutions of the problem of n bodies for all values of the masses. The method applies with only slight modifications for laws of force varying as any power of the distance.

Dr. Bliss discussed the forms of the geodesic lines on the anchor ring. The curves of minimum length were found by the methods of the calculus of variations and the resulting equations, which define a doubly infinite system of geodesic lines, were expressed in terms of Weierstrassian σ, ζ, and $\wp$ functions. By applying the properties of these functions the shapes of the geodesic lines are determined. In the paper all points of the surface of this ring are classified. The paper is intended for publication in the *Transactions.*

Mr. La Motte's paper is in abstract as follows: Given a realm of rationality Ω and in Ω an equation $f(x) = 0$ of the nth degree. This latter is invariant under Tschirnhausen transformations of the same realm if, and only if, its Galois group in Ω is a subgroup of one or more of certain groups which as to type are fully determined for each degree n. By means of resolvents a general method is developed to find parameter representations of all *irreducible* equations whose Galois groups are subgroups of one of the groups mentioned. In the cases of the third and fourth degrees these parameter representations are actually found They represent for every set of rational parameters a required equation of the given degree, and also every required equation is represented by one of them and a set of rational parameters. These formulæ for the fourth degree are

$$x^4 + px^3 + qx^2 + rx + s =$$

$$(1) \qquad x^4 + \frac{4\vartheta(1-\xi)+\eta^2}{1-\xi+\varepsilon(\eta-\varepsilon\vartheta)}(\varepsilon x^3 + \xi x^2 + \eta x + \vartheta) = 0$$

and

$$(2) \qquad x^4 + \frac{4p_2 - p_3^2}{p_1}\left(x^3 + \frac{p_2}{p_1}x^2 + \frac{p_3}{p_1}x + \frac{1}{p_1}\right) = 0,$$

with the added negative conditions 1° that the discriminant of

$$(3) \qquad t^3 - qt^2 + (pr - 4s)t - (r^2 - 4qs + p^2 s) = 0$$

does not vanish, and 2° that none of the three numbers

$$\sqrt{p^2 - 4q + 4\sigma_i} \qquad (i = 1, 2, 3)$$

is rational, where σ_i $(i = 1, 2, 3)$ are the roots of (3). For the third degree we have the simple formula

$$x^3 + qx + r = x^3 - \frac{1 + 27\varepsilon^2}{4\vartheta^2}\left(x + \frac{\varepsilon}{\vartheta}\right) = 0,$$

with the negative condition that

$$4q^3 + 27r^2 \neq 0.$$

In these formulas the Greek letters denote parameters which are, as the case may be, arbitrary or definite quantities of the realm Ω.

Professor Townsend's paper dealt with the applications of the double limit to the investigation of certain properties of a function defined as follows: Let $f(x, y)$ be a one-valued function of two real independent variables x and y and continuous with respect to each variable separately within the region, $\alpha \leqq x \leqq \beta$, $y_0 < y \leqq y_n$. Then it follows, that within this region the regular points (*i. e.*, points where $\underset{\substack{x=a\\y=b}}{L} f(x, y) = f(a, b)$) must be everywhere dense, but at the same time the irregular points (*i. e.*, where either $\underset{\substack{x=a\\y=b}}{L} f(x, y)$ does not exist or is different from $f(a, b)$) may be also everywhere dense. When further $f(x, y)$ is defined on the boundary $y = y_0$ for each value of $x = x_0$ by the limit $\underset{y=y_0}{L} f(x_0, y) = f(x_0, y_0)$, then it follows that $f(x, y_0)$ can be discontinuous in x at a set of points everywhere dense in the interval (α, β), but not at every point of this interval. By aid of this result it was shown that the points on the boundary $y = y_0$ at which $f(x, y)$ is continuous with respect to both variables together must lie everywhere dense. When all of the points of the boundary, including the end points of the interval, are regular points, then we have the necessary and sufficient condition for the uniform convergence of the function $f(x, y)$ toward the boundary function $f(x, y_0)$. These same results hold concerning the convergence of an infinite series of continuous functions of x, since the convergence of such a series is but a special case of the convergence of the function defined above toward the boundary function $f(x, y_0)$. The necessary and sufficient condition that $f(x, y_0)$ and hence that a function defined by an infinite series of continuous functions be continuous at $x = x_0$ is that there exists a set of values of y, say $y = \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_k, \cdots$, dense at y_0, such that for each such value of α_k we have $|f(x, y_0) - f(x, \alpha_k)| < \sigma$ for every value of x within a definitely defined interval $(x_0 - \delta_{\alpha_k}, x_0 + \delta_{\alpha_k}')$, which may, however, vary with α_k and may have, in fact, an under limit equal to zero, thus differing from the necessary and sufficient condition for uniform convergence. The full paper from which this is taken constituted the author's dissertation for the doctor's degree and has already been published.

Professor Dickson's paper (No. 13), outlined a chapter of his work on linear groups which is soon to appear in Teubner's Sammlung. An orthogonal group O of order 25920 on 5 indices modulo 3 is readily shown to be isomorphic with the abelian linear group for the trisection of the periods of a hyperelliptic function of four periods. A rectangular table for O having 27 rows serves to define an isomorphic substitution group which is recognized as a subgroup of index two of the group of the equation for the 27 lines on a general cubic surface. This proof of the identity of the two problems avoids the lengthy calculations of M. Jordan, the discoverer of the relation.

The object of Professor Bolza's paper* is to prove by methods of the theory of functions the converse of the propositions given in Baker's work on abelian functions, chapter IV, concerning the expression of abelian integrals in terms of a fundamental set of integral functions.

Let $f(x, y) = 0$ be an irreducible equation of degree n in y; suppose that for a finite value $x = a$ there are m branch places $(a, b_1), (a, b_2), \cdots, (a, b_m)$ of orders $w_1, w_2, \cdots, w_m$ respectively, where $(w_1 + 1) + (w_2 + 1) + \cdots + (w_m + 1) = n$, and put, in the vicinity of (a, b_k), $x - a = t_k^{w_k+1}$.

(a) If then $g_0, g_1, \cdots, g_{n-1}$ constitute a fundamental set of integral functions for $f(x, y) = 0$, and if the expansion of g_i in (a, b_k) is

$$g_i = \sum_{\nu=0}^{\infty} c_{k\nu}^{(i)} t_k^{\nu}$$

the determinant

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* After the paper was finished, I found that I had been anticipated in essential points, by Landsberg, Zur Algebra des Riemann-Roch'schen Satzes, *Math. Annalen*, vol. 50, p. 333. O. BOLZA.

$$D = |c_{k\nu}^{(0)} c_{k\nu}^{(1)} \cdots c_{k\nu}^{(n-1)}|, \qquad (\nu = 0, \cdots, w_k ; k = 1, 2, \cdots, m),$$

is different from zero.

(*b*) Let now $\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$ denote the complementary functions associated with $g_0, g_1, \cdots, g_{n-1}$; then it follows from the characteristic property* of the complementary functions, that in the vicinity of (a, b_j)

$$\sum_{i=0}^{n-1} \sum_{\mu=0}^{\infty} c_{k, \rho+\mu(w_k+1)}^{(i)} (x-a)^{\mu} \gamma_i = \frac{\delta_{jk}}{(w_j+1) t_j^{\rho}}$$

where $\delta_{jk} = 1$ or 0 according as $j = k$ or $\neq k$, for $\rho = 0, 1, \cdots, w_k$; $k = 1, 2, \cdots, n$. The determinant of these n equations with $\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$ as unknown quantities is an ordinary power series in $(x - a)$ which for $x = a$ reduces to the non-vanishing determinant D. Thence follows

1. In the vicinity of (a, b_j)

$$\gamma_i = \frac{P(t_j)}{t_j^{w_j}} \qquad (j = 1, 2, \cdots, m),$$

P being the general symbol for an ordinary power series.

2. If $\nu = \rho + q(w_k + 1)$, where $0 \leqq \rho \leqq w_k$, and if we define

$$\gamma_{k\nu} = \sum_{\mu=0}^{q} \sum_{i=0}^{n-1} c_{k, \rho+\mu(w_k+1)}^{(i)} (x-a)^{\mu} \gamma_i,$$

this function γ_k has in the vicinity of (a, b_j) the expansion

$$\frac{\gamma_{k\nu}}{(x-a)^q} = \frac{\delta_{jk} t_j^{-\nu}}{w_j + 1} + t_j P(t_j).$$

(*c*) If, in particular, $g_0 = 1, g_1, \cdots, g_{n-1}$ form a *normal* fundamental set ("normal basis") of dimensions $d_0 = 1, d_1, \cdots, d_{n-1}$ respectively, the expansions of the functions γ_i and $\gamma_{k\nu}$ at infinity can be obtained by combining the above results with Weber's theorem according to which in this case

$$h_0 = \frac{g_0}{x^{d_0}}, \; h_1 = \frac{g_1}{x^{d_1}}, \cdots, h_{n-1} = \frac{g_{n-1}}{x^{d_{n-1}}}$$

constitute a normal fundamental set for the algebraic equation

$$f\left(\frac{1}{\xi}, y\right) = 0.$$

* Baker, l. c., p. 63.

The result is that

1. The integrals

$$\int x^{d_i-2}\gamma_i\,dx \qquad (i=1, 2, \cdots, n-1)$$

are integrals of the first kind.

2. The integral

$$\int \frac{-\nu\gamma_{k\nu}\,dx}{(x-a)^{\nu+1}}$$

is an integral of the second kind with the one pole (a, b_k) and the expansion

$$\frac{1}{t_k^{\nu}} + P(t_k)$$

in its vicinity.

3. The integral

$$\int\left[\frac{\sum_{i=0}^{n-1} g_i(\xi, \eta)\gamma_i(x, y)}{x-\xi} - \frac{\sum_{i=0}^{n-1} g_i(\xi', \eta')\gamma_i(x, y)}{x-\xi'}\right]dx$$

is an elementary integral of the third kind with the logarithmic points (ξ, η) and (ξ', η').

Dr. Field's proof of the Riemann-Roch theorem is as follows: Let $F(z, u) = 0$ be the equation to an irreducible algebraic curve of degree n. The form of this equation is supposed to be such that the only multiple points are double points which are not at the same time branch points. Regard u as the dependent variable and further assume that no line parallel to the axis of u passes through more than one double point, or is tangent at a double point, or is an asymptote, and also that the asymptotes are all distinct from one another and no two parallel to each other. To an equation of the character in question, the equation of any irreducible algebraic curve can be reduced by a birational transformation.

By $(a_1, b_1), \cdots, (a_d, b_d)$ indicate the d double points, and by $(a_{d+1}, b_{d+1}), \cdots, (a_{d+q}, b_{d+q})$ any other q points of the curve, and consider the form of the function

$$(1) \qquad \sum_{\lambda=1}^{d+q} \frac{\gamma_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} + T(z, u),$$

where $T(z, u)$ is an arbitrary polynomial in (z, u) and the quantities γ_λ are arbitrary constants. This form represents the most general rational function of (z, u) which, apart from infinities at ∞, only becomes infinite for points among the q points

$$(a_{d+1}, b_{d+1}), \cdots, (a_{d+q}, b_{d+q})$$

and for these to the first order only.

By a purely algebraic process seek to determine the limitations which must be imposed upon the form (1) in order that it may represent a function which is finite at ∞. We find that the most general rational function whose infinities are all included under the q infinities here in question is given by the form

$$(2) \qquad \sum_{\lambda=1}^{d+q} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + c,$$

where c is an arbitrary constant and the coefficients γ_λ satisfy the system of $\frac{1}{2}(n-1)(n-2)$ equations

$$(3) \qquad \sum_{\lambda=1}^{d+q} \gamma_\lambda a_\lambda^i b_\lambda^k = 0, \qquad (i + k = 0, 1, 2, \cdots, n-3).$$

In the case $q = 0$ the form (1), for arbitrary values of the coefficients γ_λ, represents the most general rational function of (z, u) which becomes infinite only at ∞, and the form (2), for conditioned values of the coefficients γ_λ furnished by the equations (3), represents the most general rational function which is nowhere infinite.

Now, an algebraic function which is nowhere infinite can only be a constant, and in the case in question—namely $q = 0$—it follows that the coefficients γ_λ in (2) must all have the value 0. The system of equations

$$(4) \qquad \sum_{\lambda=1}^{d} \gamma_\lambda a_\lambda^i b_\lambda^k, \quad (i + k = 0, 1, 2, \cdots, n-3)$$

can therefore only be satisfied when the quantities γ_λ all have the value 0.

From the interpretation of the equations (4), it immediately follows that the d conditions for adjointness in the case of a curve of degree $n - 3$, and there ore also in the case of a curve of higher degree, are independent of one another.

In the general case $q \neq 0$, an examination of the system of equations (3) shows that the number of arbitrary quantities γ_λ involved in their solutions is $q - s$, where s is the "*strength*" of the system of q points $(a_{d+1}, b_{d+1}), \cdots, (a_{d+q}, b_{d+q})$ in determining an adjoint polynomial of degree $n - 3$. From (2) it is seen that $q - s + 1$ is the number of arbitrary constants involved in the expression of the most general rational function of (z, u), whose infinities, all of the first order, correspond to points among the q points here in question.

Professor Schmiedel presented the results of an investigation into the general formula of reduction for the function

$$\int x^m y^n dx, \quad \text{where} \quad y = \sum_0^i t a_t x^t,$$

and deduced two formulas by which the reduction of the exponents m and n may be effected.

The forms studied by Professor Skinner are such that they are invariant with respect to substitutions of the form $z_1' = a_1 z_i$, $z_2' = a_2 z_j$, $z_3' = a_3 z_k$, $(i, j, k, = 1, 2, 3$ in some order), where a_1, a_2, a_3 are roots of unity and $a_1 a_2 a_3 = \pm 1$. The groups for which i, j, k have the order 1, 2, 3 are abelian and all such ternary monomial groups can be generated by at most two independent generators. If T_1, T_2 denote the two generators of the group and τ a multiplicative substitution of order 2 with determinant -1, all ternary monomial groups may be found by combining the substitutions T_1, T_2, τ with the generators of the symmetric group of three elements. The systems of invariant forms are then found by setting up the forms which are invariant with respect to the generators of the symmetric group, taken singly or together and subjecting these forms to the substitutions T_1, T_2, τ. The conditions for invariance appear in the form of certain linear homogeneous congruences, the modulus being either the order of T_1, or of T_2. The computation of the reduced systems and the complete systems depends largely upon auxiliary quantities which appear in the solution of the congruences.

Professor Shaw exhibited a method for the development of any determinant in terms of partial determinants of the general form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ a_{41} & 0 & 0 & a_{44} & \cdots & a_{4n} \\ a_{51} & 0 & 0 & 0 & \cdots & a_{5n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & 0 & 0 & 0 & & a_{nn} \end{vmatrix}$$

The central theorem of Professor Moore's paper (No. 19) on double limits, which will appear in the *Transactions*, will be sufficiently indicated by the following particular case. On the supposition that a function $f(x, y)$ is defined for

$$-1 < x < 1, \quad -1 < y < 1, \quad x \neq 0, \quad y \neq 0$$

and that the limits

$$\underset{x=0}{\mathrm{L}} f(x, y), \quad \underset{y=0}{\mathrm{L}} f(x, y),$$

exist and are respectively denoted by

$$h(y); \quad g(x)$$

for respectively

$$-1 < y < 1, \quad y \neq 0; \quad -1 < x < 1, \quad x \neq 0,$$

the necessary and sufficient condition that the limits

$$\underset{y=0}{\mathrm{L}}\ h(y), \quad \underset{x=0}{\mathrm{L}}\ g(x)$$

exist and are equal is that $f(x, y)$ approaches its limit $h(y)$ on the y-set $-1 < y < 1$, $y \neq 0$, subuniformly near $y = 0$, that is, that for every positive ε there exists a positive δ_ϵ ($\delta_\epsilon \leqq 1$) and for every x, ($x \neq 0$, $|x| < \delta_\epsilon$) there is a positive $\delta_{\epsilon x}{}'(\delta_{\epsilon x}{}' \leqq 1)$ such that

$$|f(x, y) - h(y)| < \varepsilon$$

for every (x, y) satisfying the conditions

$$|x| < \delta_\epsilon, \quad |y| < \delta_\epsilon{}', \quad x \neq 0, \quad y \neq 0.$$

An equivalent condition is that $f(x, y)$ approaches its limit $g(x)$ in an analogous way.

The definition and the exposition of the elements of the theory of improper definite integrals as given by Harnack in volumes 21 and 24 (1883, 1884) of the *Mathematische Annalen*, leave much to be desired. In recent memoirs in the *Wiener Sitzungsberichte* (1898, 1899), and in the appendix to the third volume of his treatise on the calculus (1899) Stolz has given a new development of the theory, in which the absolutely convergent integrals play the leading rôle. Professor Moore's paper (No. 20), written in the spirit of the original Harnack memoirs, contains a development of the theory throwing additional light on the varying properties of the absolutely and the conditionally convergent integrals. This paper will be published in the *Transactions*.

Professor Dickson's second paper (No. 21) is also intended for publication in the *Transactions*. A set of canonical forms for the substitutions of a given group should possess the property that two of its substitutions are conjugate within the group, if, and only if, they are reducible to the same canonical form according to a definite scheme of reduction. For linear groups in a Galois field of order p^n, reduction to a canonical form belonging to a field of order p^{nr}, $r > 1$, must be possible by the introduction of new indices conjugate with respect to the initial $GF[p^n]$. The memoir gives a set of canonical forms belonging to the abelian group itself and deduces a set of ultimate canonical forms, the former depending upon the coefficients of the characteristic equation, the latter upon its roots. By these results tables are derived which give a classification into conjugate sets of all quaternary abelian substitutions and also of the operators of a single quotient group. For $p^n = 3$, the latter is the group of order 25920 occurring in the problem of the 27 lines on a general cubic surface. Its operators fall into 20 sets of conjugates: the identity, one set of period 5, 4 sets of period 3, 2 sets of each of the periods 2, 4, 9 and 12, and 6 sets of period 6.

Miss Schottenfels presented a proof of the existence of a substitution group of degree 21 and order 20160, having cyclical subgroups identical with those enumerated by Professor Dickson in the *American Journal of Mathematics*, vol. 22, p. 252, in the ternary linear fractional group of order 20160 in the Galois field $[2^2]$.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILL