SURFACES WHOSE FIRST AND SECOND FUNDAMENTAL FORMS ARE THE SECOND AND FIRST RESPECTIVELY OF ANOTHER SURFACE.

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THE fundamental theorem in the theory of surfaces is :* *Given two quadratic differential forms*

$$(1)\qquad \begin{aligned} f &= Edu^2 + 2Fdudv + Gdv^2 \\ \varphi &= Ddu^2 + 2D'dudv + D''dv^2, \end{aligned}$$

of which the first is definite, in order that there exist a surface which shall have f and φ for its first and second fundamental forms respectively, it is necessary and sufficient that the coefficients of these forms satisfy the Codazzi equations

$$(2)\qquad \begin{aligned} &\frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} D + \left(\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \right) D' + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} D'' = 0 \\ &\frac{\partial D''}{\partial u} - \frac{\partial D'}{\partial v} + \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} D + \left(\begin{Bmatrix} 22 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \right) D' - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} D'' = 0 \end{aligned}$$

and the equation of Gauss

$$(3)\qquad \begin{aligned} \frac{1}{\sqrt{EG - F^2}} \Bigg[&\frac{\partial}{\partial v} \left(\frac{\sqrt{EG - F^2}}{E} \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \right) \\ &- \frac{\partial}{\partial u} \left(\frac{\sqrt{EG - F^2}}{E} \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \right) \Bigg] = \frac{DD'' - D'^2}{EG - F^2}; \end{aligned}$$

where the Christoffel symbols $\begin{Bmatrix} rs \\ t \end{Bmatrix}$ are formed with respect to the form f. And, moreover, *when these conditions are satisfied the corresponding surface is unique and determinate.*

We propose now to determine those surfaces S with fundamental forms (1) which have associated with them surfaces S_1, whose first and second fundamental forms are, respectively,

* Bianchi, Lezioni, p. 92.

$$Ddu^2 + 2D'dudv + D''dv^2,$$
$$Edu^2 + 2Fdudv + Gdv^2.$$

Denoting by subscript $_1$ functions belonging to S_1, we have

$$(4) \qquad \begin{aligned} f_1 &= ds_1^2 = E_1du^2 + 2F_1dudv + G_1dv^2, \\ \varphi_1 &= D_1du^2 + 2D_1'dudv + D_1''dv^2. \end{aligned}$$

Comparing these with the above, we remark that for a surface S_1 satisfying the conditions of the problem,

$$(5) \quad E_1 = D,\ F_1 = D',\ G_1 = D'';\ D_1 = E,\ D_1' = F,\ D_1'' = G.$$

It is evident that the double system of lines which is conjugate for S and S_1 is composed of the lines of curvature on each surface. We will refer both surfaces to these lines; then

$$(6) \qquad F = 0, \qquad D' = 0.$$

With this special choice of parametric lines the Codazzi equations become

$$(7) \qquad \begin{aligned} &\frac{\partial D}{\partial v} - \tfrac{1}{2}\frac{\partial E}{\partial v}\left(\frac{D}{E} + \frac{D''}{G}\right) = 0, \\ &\frac{\partial D''}{\partial u} - \tfrac{1}{2}\frac{\partial G}{\partial u}\left(\frac{D}{E} + \frac{D''}{G}\right) = 0; \end{aligned}$$

and the Gauss equation

$$(8) \quad \frac{DD''}{\sqrt{EG}} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) = 0.$$

In order that S_1 may be a surface the coefficients of its fundamental forms must satisfy these equations. Hence in order that S may be the kind of surface sought, it is necessary that its fundamental coefficients satisfy the conditions

$$(9) \qquad \begin{aligned} &\frac{\partial E}{\partial v} - \tfrac{1}{2}\frac{\partial D}{\partial v}\left(\frac{E}{D} + \frac{G}{D''}\right) = 0, \\ &\frac{\partial G}{\partial u} - \tfrac{1}{2}\frac{\partial D''}{\partial u}\left(\frac{E}{D} + \frac{G}{D''}\right) = 0, \end{aligned}$$

and

$$(10) \quad \frac{EG}{\sqrt{DD''}} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{D}}\frac{\partial\sqrt{D''}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{D''}}\frac{\partial\sqrt{D}}{\partial v}\right) = 0.$$

Substituting in equations (7) the expressions for $\frac{\partial E}{\partial v}$ and $\frac{\partial G}{\partial u}$, as given by equations (9), we get in each case

$$\left(\frac{D}{E}+\frac{D''}{G}\right)\left(\frac{E}{D}+\frac{G}{D''}\right)=4. \tag{11}$$

Denote by ρ_1 and ρ_2 the principal radii of curvature at the point (u, v) on S; then

$$\frac{1}{\rho_1}=\frac{D}{E}, \quad \frac{1}{\rho_2}=\frac{D''}{G}.$$

When these expressions for ρ_1 and ρ_2 are replaced by the latter in the equation of condition (11), it becomes

$$\frac{1}{\rho_1\rho_2}=\tfrac{1}{2}\left(\frac{1}{\rho_1^{\,2}}+\frac{1}{\rho_2^{\,2}}\right). \tag{12}$$

Hence S must be a surface for which the total curvature at a point and the Casorati * curvature are equal. Developing equation (12) we remark that it is necessary that

$$\rho_1=\rho_2.$$

Hence, in order that the first and second fundamental forms of a surface S may determine a surface for which these forms shall be respectively the second and first forms it is necessary that S have its two radii equal and of the same sign.

This class of surfaces has been the object of a study by Monge,† and later by Forsyth.‡ The former has shown that the sphere is the only *real* surface belonging to the class. His discussion was with reference to the differential equation of the second order, characteristic of such surfaces, viz.

$$\begin{aligned}&[(1+q^2)r-2pqs+(1+p^2)t]^2\\ &\quad=4(rt-s^2)(1+p^2+q^2),\end{aligned} \tag{13}$$

for the determination of which the surface was given by the equation

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* "Mésure de la courbure des surfaces suivant l'idée commune", *Acta Mathematica*, 14, 2.

† Application de l'analyse à la géométrie, 5th ed., pp. 196–211.

‡ "Note on surfaces whose radii of curvature are equal and of same sign," *Messenger of Mathematics*, new series, no. 321.

(14) $$z = f(x, y),$$

and where, as originally denoted by Monge,

(15) $$p,\ q,\ r,\ s,\ t = \frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y},\ \frac{\partial^2 z}{\partial x^2},\ \frac{\partial^2 z}{\partial x \partial y},\ \frac{\partial^2 z}{\partial y^2}.$$

Forsyth integrated the same equation (13) by a process differing from that used by Monge and found that all such surfaces are given by eliminating μ between the equations

16) $$y + \mu x = \varphi(\mu),\ z - ix(1 + \mu^2)^{\frac{1}{2}} = \psi(\mu)$$

where φ and ψ are arbitrary functions. We shall now determine the forms of the functions, which correspond to surfaces furnishing a solution to our problem.

To this end we remark that the notations of Gauss and of Monge have the following relations :

$$E = 1 + p^2, \qquad F = pq, \qquad G = 1 + q^2,$$

(17) $$D = \frac{r}{\sqrt{1 + p^2 + q^2}},$$

$$D' = \frac{s}{\sqrt{1 + p^2 + q^2}},\ D'' = \frac{t}{\sqrt{1 + p^2 + q^2}}.$$

It is evident then that our problem reduces to the finding of φ and ψ such that

(18) $$\frac{r,\ s,\ t}{\sqrt{1 + p^2 + q^2}} = 1 + p_1^2,\ p_1 q_1,\ 1 + q_1^2,$$

and

(19) $$1 + p^2,\ pq,\ 1 + q^2 = \frac{r_1,\ s_1,\ t_1}{\sqrt{1 + p_1^2 + q_1^2}},$$

where the subscript $_1$ denotes that the functions belong to the associated surface S_1.

Substituting for the functions in (18) their expressions as derived from equations (16) and from the similar equations

$$y + \mu_1 x = \varphi_1(\mu_1),$$

$$z - ix(1 + \mu_1^2)^{\frac{1}{2}} = \psi_1(\mu_1),$$

corresponding to the surface S_1, we find on equating to zero the coefficients of x^2, x, and 1, the following relations :

$$(20)\quad \begin{cases} \dfrac{-i\mu^2 A}{(1+\mu^2)^{\frac{3}{2}}} = \dfrac{\mu_1^2}{1+\mu_1^2}, \\ -\dfrac{2\mu}{(1+\mu^2)^{\frac{1}{2}}} - \mu^2 AB = 2\Big[\varphi_1'\mu_1^2 \\ \quad + i\mu_1(i\varphi_1'\mu_1 - \sqrt{1+\mu_1^2}\psi_1') + \dfrac{i\mu_1^3\psi_1'}{\sqrt{1+\mu_1^2}}\Big], \\ \dfrac{2\varphi'\mu}{\sqrt{1+\mu^2}} + A\mu^2(\varphi'\psi'' - \varphi''\psi') \\ \quad = -\varphi_1'^2\mu_1^2 + 2i\mu_1\sqrt{1+\mu_1^2}\varphi_1'\psi_1' + \mu_1^2\psi_1'^2; \end{cases}$$

$$(21)\quad \begin{cases} \dfrac{-i\mu A}{(1+\mu^2)^{\frac{3}{2}}} = \dfrac{\mu_1}{1+\mu_1^2}, \\ -\dfrac{1}{\sqrt{1+\mu^2}} - \mu AB \\ \quad = i(i\varphi_1'\mu_1 - \sqrt{1+\mu_1^2}\psi_1') + \dfrac{2i\mu_1^2\psi_1'}{\sqrt{1+\mu_1^2}}, \\ \dfrac{\varphi'}{\sqrt{1+\mu^2}} + A\mu(\varphi'\psi'' - \varphi''\psi') \\ \quad = i\sqrt{1+\mu_1^2}\varphi_1'\psi_1' + \mu_1\psi_1'^2; \end{cases}$$

$$(22)\quad \begin{cases} \dfrac{-iA}{(1+\mu^2)^{\frac{3}{2}}} = \dfrac{1}{1+\mu_1^2}, \\ -AB = -2\varphi_1' + \dfrac{2i\mu_1}{\sqrt{1+\mu_1^2}}, \\ A(\varphi'\psi'' - \psi'\varphi'') = \varphi_1'^2 + \psi_1'^2, \end{cases}$$

where, for the sake of brevity, we have written

$$(23)\quad \begin{aligned} A &= \frac{1}{i\mu\varphi' + \sqrt{1+\mu^2}\psi'}, \\ B &= \psi'' - \varphi' i(1+\mu^2)^{-\frac{3}{2}} + i\mu(1+\mu^2)^{-\frac{1}{2}}\varphi''. \end{aligned}$$

The similar relations arising from equations (19) can be gotten at once from the above by giving subscripts to the terms not having them and removing them from those which have; we shall refer to these relations as (20′), (21′) and (22′), respectively.

From the first relations of (20), (21), and (22) we see that

$$\mu_1 = \mu,$$

hence these three relations and the corresponding ones in (20′), (21′) (22′) may be replaced, respectively, by

$$(24)\qquad \begin{aligned} \mu\varphi' - i\sqrt{1+\mu^2}\psi' &= -\frac{1}{\sqrt{1+\mu^2}},\\ \mu\varphi_1' - i\sqrt{1+\mu^2}\psi_1' &= -\frac{1}{\sqrt{1+\mu^2}}. \end{aligned}$$

Combining these with the second relations of (20), (21), (22) and (20′) (21′), (22′), we find that the latter may be replaced by

$$(25)\qquad \begin{aligned} i\sqrt{1+\mu^2}[\psi'' - i(1+\mu^2)^{-\frac{3}{2}}\varphi' + i\mu\,(1+\mu^2)^{-\frac{1}{2}}\varphi''] &\\ = +2\varphi_1' - \frac{2i\mu}{\sqrt{1+\mu^2}}&,\\ i\sqrt{1+\mu^2}[\psi_1'' - i(1+\mu^2)^{-\frac{3}{2}}\varphi_1' + i\mu(1+\mu^2)^{-\frac{1}{2}}\varphi_1''] &\\ = +2\varphi' - 2i\mu(1+\mu^2)^{-\frac{1}{2}}&. \end{aligned}$$

Differentiating the second of equations (24) and combining the result with the second of equations (25) and both of (24), we find

$$(26)\qquad \varphi_1' = \varphi', \qquad \psi_1' = \psi',$$

hence φ_1 and ψ_1 can differ at most from φ and ψ, respectively, by an additive constant. Hence *S and S_1 are the same surface, to a translation près.*

Making use of the preceding results, we find that the third equations of (20), (21), (22), (20′), (21′), and (22′) are equivalent to the single one

$$(27)\qquad i\sqrt{1+\mu^2}(\varphi'\psi'' - \psi'\varphi'') = \varphi'^2 + \psi'^1.$$

Expressing this relation in terms of φ and its derivatives, we find

$$(28)\qquad \varphi'' = \frac{1}{(1+\mu^2)^{\frac{3}{2}}},$$

from which we get

$$(29)\qquad \varphi = \sqrt{1+\mu^2} + C_1\mu + C_2,$$

where C_1 and C_2 are arbitrary constants. On substituting this expression in (24), we find

$$(30) \qquad i\psi = C_1\sqrt{1+\mu^2} + \mu + C_3,$$

C_3 being a third arbitrary constant. From (29) and (30) we get

$$(31) \qquad \begin{aligned} \Phi = \varphi + i\psi &= (C_1+1)(\mu + \sqrt{1+\mu^2}) + C_2 + C_3, \\ \Psi = \varphi - i\psi &= (C_1-1)(\mu - \sqrt{1+\mu^2}) + C_2 - C_3. \end{aligned}$$

It is readily seen that equations (16) can be replaced by

$$(32) \qquad \begin{aligned} y + iz + (\mu + \sqrt{1+\mu^2})x &= \Phi, \\ y - iz + (\mu - \sqrt{1+\mu^2})x &= \Psi, \end{aligned}$$

where Φ and Ψ have the expressions (31). Writing in these expressions

$$C_1 = a, \qquad C_2 = b, \qquad C_3 = ic,$$

and eliminating μ from the equations (32) thus obtained we get

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = 1;$$

that is, a sphere of center (a, b, c) and radius unity. Moreover, only when the radius is unity are the conditions satisfied.

We have then the following results:

The ruled surfaces, defined by the equations

$$\begin{aligned} y + \mu x &= \sqrt{1+\mu^2} + C_1\mu + C_1 \\ z - ix\sqrt{1+\mu^2} &= \mu + C_1\sqrt{1+\mu^2} + C_3, \end{aligned}$$

are the only surfaces whose first and second fundamental forms can be taken for the second and first fundamental forms of a surface. Further the second surface is only the first to a translation près. And of these surfaces the only real one is the sphere of radius unity.

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