continuously. This is of fundamental importance, as hereon rests the proof that the same quadrilateral-hypothesis must hold everywhere in the same plane, but it should not be left without proof, merely because it is plausible, or because we ardently wish it to be true.

In the historical note at the end are serious errors. The author states (p. 92) that Saccheri, after developing elliptic, hyperbolic, and parabolic geometries side by side, states dogmatically that the two first are false. Now this is not only incorrect, but it conveys an unfair impression of the Jesuit mathematician. He does not state dogmatically that these geometries are false, but gives an elaborate and ingenious sequence of propositions in each case, to show that the system is self-contradictory.* Again the author states on the following page, that elliptic geometry was discovered by Riemann. This seems to us rash. It is by no means clear whether Riemann looked upon the geodesics of a surface of constant positive curvature as cutting in one or two points: the probability being that he held the latter view, so that it is customary to credit Klein and Newcomb with the discovery of elliptic geometry.†

We may say, in conclusion, that the book is a contribution to educational rather than to mathematical literature. It is neither a scholarly nor a profound work, but comes in answer to a real need, and marks a step in advance.

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BIANCHI'S DIFFERENTIAL GEOMETRY.

Vorlesungen über Differentialgeometrie. Von LUIGI BIANCHI. Autorisierte deutsche Uebersetzung von MAX LUKAT. Leipzig, B. G. Teubner, 1896–1899. Pp. xvi + 659.

IN 1886 Bianchi published a lithographed edition of his lectures on differential geometry given at the University of Pisa during the winter 1885–86. This publication, on which the book now before us is based, consisted of only fourteen chapters. The *Vorlesungen* contains twenty-two. Professor Loria, of Genoa, says of the lithographed edition :‡

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* Conf. Engel und Staeckel, Die Theorie der Parallellinien von Euklid bis auf Gauss, pp. 67 and 109.

† Killing, loc. cit., p. 70. Russel, The Foundations of Geometry, pp. 39, 40.

‡ *Jahrbuch über die Fortschritte der Mathematik*, 1886, p. 648.

"In our opinion it fills a lamentable gap in mathematical literature * * * We know no book which treated *ex professo* infinitesimal geometry. * * * In these noteworthy lectures we have to admire not only the clearness of presentation, the elegance of the calculations, and the great number of interesting examples, but also the selection and the coördination of the material." It will perhaps be of interest to give the headings of the fourteen chapters of this first edition. I., Space curves ; II., Curvilinear coördinates on surfaces, conformal representation ; III., Geodesic lines, geodesic curvature ; IV., Curvature of surfaces ; V., Surfaces defined by their linear element ; VI., Developable surfaces ; VII., Surfaces one of whose radii of curvature is a function of the other ; VIII., Surfaces of constant curvature ; IX., Surfaces whose mean curvature is constant ; X., Surfaces defined by properties of their lines of curvature ; XI., Ruled surfaces ; XII., Curvilinear coördinates in space, triply orthogonal systems of surfaces ; XIII., Elliptic coördinates, geodesic lines on the ellipsoid ; XIV., Triply orthogonal Weingarten systems ; Appendix, Codazzi's formulas.

In 1894 appeared the enlarged and improved Geometria differenziale, published by Spoerri in Pisa, containing twenty chapters. The mathematical situation was then different, for during these eight years had appeared the first three volumes of Darboux's Lecons sur la théorie générale des surfaces,* a work which in the words of Bianchi "gives a complete presentation of all the results obtained in the field of infinitesimal geometry." Bianchi's book has none the less its place, and that an honorable one. His aim as stated in his preface is entirely different from Darboux's. Bianchi's purpose is "to confine himself to an explanation of the principles of the theory, and to its chief applications ; to collect in a volume of moderate compass everything necessary to enable beginners * * * themselves to read the original memoirs." That Bianchi has succeeded in this task is evident. That this purpose is indeed different from Darboux's will appear to all who read the latter's book.

The Geometria differenziale is not merely the lithographed edition with six additional chapters. Apparently the whole book has been re-written, and the subjects are taken up in somewhat different order and in different connections. I cannot compare this book in any detail with

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* Referred to in this review as "Darboux."

the original lithographed edition, for I have not been able to obtain a copy of the latter.

Shortly after the appearance of the Geometria differenziale, a German translation this work was undertaken by Max Lukat. The translation was published under the title of Vorlesungen über Differentialgeometrie. This edition appeared in three parts, of which the first was published in 1896, the last in 1899. All references in the following pages are to the German edition.

I shall now consider in some detail the contents of the Geometria differenziale. The first chapter on "Curves of double curvature" is a brief and complete exposition of the principal properties of space curves. In it are discussed much the same questions that are taken up by Picard in Chapter XI. and in the following chapters of the first volume of his Traité d'analyse. Bianchi's treatment is usually concise and clear. It is noticeable, however, that he uses for the direction cosines of the principal directions of a curve at any point the expressions cos α, cos β, etc., instead of employing, as is customary with French writers, single letters for these quantities. It seems to me that the French practice is to be preferred.

In this chapter there is no explanation of the term "edge of regression," in the treatment of developable surfaces. Such an explanation as is to be found for example in Picard, l. c., I., p. 298, would add to the value of the discussion. It is perhaps worth mentioning that Bianchi does not suppose his coördinate axes disposed as French and German writers generally do, but adopts the so-called "English disposition."

In the second chapter we have the analytic foundation of the book, a study of quadratic differential forms. Bianchi defines the first, second, and mixed differential parameters, and states in a remark (p. 47) that all the differential parameters of two arbitrary functions U and V, formed from a binary quadratic differential form, may be obtained by repeated applications of the symbols $\triangle_1$, $\triangle_2$, and ∇, which indicate the first, second, and mixed differential parameters, respectively. For a proof he refers to Darboux, III., p. 260. The reference should be to page 203. This mistake in the Italian is not corrected in the translation. The proof given by Darboux is simple and short, and might with advantage have been reproduced by Bianchi. The several symbols of Christoffel are then defined. These symbols are used throughout the book. As the author says in his preface, they have the advantage of giving the formulas a simple and elegant appearance which makes them easier to remember. While I agree fully with this statement, I am not yet convinced that the use of these symbols is entirely desirable. To one not accustomed to them, they have always a strange look, and surely many an American reader, at least, must turn back to the definitions if ever he wants to know what a formula really means. It is questionable too if it is desirable to memorize the complicated formulas of differential geometry. On the other hand it may be urged that the formulas would be of great complexity and of ungainly appearance without the use of Christoffel's symbols. Often that is the case. It would be easy to mention books where it is so. But Darboux does not use them, and I think no book is more elegant in style and appearance than his. It is finally a matter of taste. The chapter contains also a study of the equivalence of two quadratic forms. One other criticism of this very important chapter must be made. The whole chapter would be much more readily understood by the beginner, and so more in accord with the purpose of the book, if the theory had been developed for binary forms instead of for forms in n variables. The binary form is the only one used in the study of surfaces. Let those who will study surfaces in n dimensions generalize for themselves, and let the student of surfaces in space not be forced to swallow what he cannot digest.

The next seven chapters (III.–IX.) contain an admirable development of the general theory of surfaces. Their headings are as follows: III., Curvilinear coördinates on surfaces, conformal representation; IV., The fundamental equations of the theory of surfaces; V., Spherical representation after Gauss, plane coördinates; VI., Geodesic curvature, geodesic lines; VII., Applicable surfaces; VIII., Deformation of ruled surfaces; IX., Evolute surfaces and the theorem of Weingarten.

Bianchi's consistent point of view in these chapters, and indeed throughout the book, is that the study of surfaces is the study of two simultaneous quadratic differential forms; the "first fundamental form" being the square of the linear element

$$ds^2 = dx^2 + dy^2 + dz^2 = Edu^2 + 2Fdudv + Gdv^2;$$

the "second fundamental form" is

$$\begin{aligned} Xd^2x + Yd^2y + Zd^2z &= -(dXdx + dYdy + dZdz) \\ &= Ddu^2 + 2D'dudv + D''dv^2. \end{aligned}$$

The letters X, Y, Z denote the direction cosines of the normal to the surface at the point (x, y, z). These seven chapters are admirably done, and contain thoroughly clear discussions of many most interesting and fundamental problems. I feel that the author is to be taken to task, however, for advising, in his preface, the beginner to omit the eighth chapter (on the deformation of ruled surfaces). It is not an especially difficult chapter, but it is an especially interesting one, and no one should omit it. Chapter III. deals with the fundamental properties of curvilinear coördinates on a surface. The conformal representation of a surface upon a plane or upon another surface is then studied by means of isothermal systems of coördinates. In the study of isothermal systems, the author omits to prove that it is necessary, as well as sufficient, in order that the curves (u, v) form an isothermal system, that the linear element in terms of these coördinates be

$$ds^2 = \lambda(Udu^2 + Vdv^2).$$

A simple proof of the necessity of this condition is given by Darboux, Volume I., p. 146. Bianchi makes no use of "symmetric coördinates," which are the parameters of the minimal curves of the surface. These coördinates seem to me to furnish the simplest method of treating the problems of conformal representation. Bianchi fails also to refer to the subject of the ordinary maps of a sphere, such as Mercator's projection. This subject furnishes one of the most direct and interesting applications of conformal representation. The chapter closes with a study of stereographic projection, where the author shows, following Cayley, that the movements of a sphere on itself may be represented by a linear substitution with constant coefficients.

The fourth chapter begins the discussion of the two fundamental forms of the surface. The formulas connecting the coefficients E, F, G, D, D', D'' of the two forms, due to Mainardi and Codazzi, are deduced. There follows a clear and simple discussion of the problem of determining the surface, having given the coefficients of the fundamental forms. This discussion seems to me one of the best things in the book. One footnote in this chapter will hardly please the reader. On page 95 we are referred for a proof of a simple theorem in differential equations, "for example, [to] C. Jordan, Cour d'analyse, vol. III." Such an indefinite reference is of no value, and is even irritating. Moreover, I cannot find that Jordan gives explicitly the theorem referred to.

The last part of the chapter contains applications of the results of the first part to proofs of the theorems of Euler and Meusnier on the curvature of the curves of a surface passing through any point of the surface. The celebrated theorem of Gauss on the total curvature of a surface at any point is then proved. Finally, the indicatrix of Dupin is discussed. Bianchi departs from what is, I believe, the general usage, in calling the indicatrix at a hyperbolic point of a surface the *two* conjugate hyperbolas. By so doing there is a certain loss of generality in the definition of the indicatrix. In the *Vorlesungen* are inserted in the proper place (p. 114) the differential equations of the lines of curvature and of the asymptotic lines of a surface having its equation in the form, $z = f(xy)$. In the Italian these equations were accidentally omitted, and were printed at the end of the book.

Chapter V. treats Gauss's spherical representation of the surface, and tangential or plane coördinates. Especially to be noticed in this chapter are the formulas of Bonnet and of Enneper relative to the torsion of asymptotic lines, the study of a surface referred to its asymptotic lines, the formulas of Lelieuvre, and the discussion of some particular problems (p. 144). Bianchi defines a moulding surface (*surface moulure, Gesimsfläche*) as a surface generated by a plane curve whose plane rolls without slipping on any developable surface. By Darboux and some other writers the term is applied only when the developable surface is cylindrical.

It is not desirable to discuss in detail all the chapters of Bianchi's book, pointing out what the author has omitted that might have been put in, and describing and criticizing his methods. It seems to me that it will give a good idea of the scope and completeness of the book to discuss in considerable detail one chapter of the book, and to compare that with the writings of other authors on the same subject. I have chosen for that purpose the sixth chapter, on geodesic curvature and geodesic lines, because it is both interesting and typical. I shall compare this chapter with the part of Darboux's work which treats the same subject. It is to be remembered that the purposes of the authors are utterly different, and that the standard of comparison is the most difficult that could be set. The comparison is not meant to be in any way disparaging to Bianchi. It is intended merely to show how much about any subject we may expect to find in his book, and to illustrate the choice and coördination of material which he has made.

The sixth chapter of Bianchi's *Vorlesungen* consists of thirty-two pages. Darboux devotes to the same subject about two hundred and forty pages, printed somewhat more closely, besides a discussion of mechanics, based on the treatment of geodesic lines, covering seventy-three pages. Bianchi defines first the geodesic or tangential curvature, and obtains the analytic expression for the tangential curvature of the coördinate lines supposed orthogonal. Following Bonnet, he then obtains by the use of differential parameters the geodesic curvature of any line of the surface, $\varphi(u, v) = 0$. Darboux defines geodesic curvature at first in the same way in a chapter (Livre V., Ch. I.) on "Formules générales," and notes that the center of geodesic curvature is the intersection of the axis of the osculating circle of the curve with the plane tangent to the surface at the point considered. Darboux obtains his expression for the geodesic curvature from kinematical considerations. Later (Livre VI., Ch. VI.) he gives various other definitions and properties of the geodesic curvature, and finally in a chapter on differential parameters (Livre VII., Ch. I.) we find Bonnet's expression for the geodesic curvature of any curve. Bianchi gives next Liouville's expression for the total curvature of a surface at any point expressed in terms of the geodesic curvature of the coördinate lines supposed orthogonal. From this expression may at once be proved the theorem that only on surfaces of constant negative curvature can there exist two orthogonal families of curves having all the same constant geodesic curvature.

Bianchi then defines geodesic lines on a surface as lines whose principal normals coincide always with the normals to the surface. The differential equation of the geodesics is next obtained in several forms. Darboux obtains practically the same equations (Livre V., Ch. IV.), starting again from kinematical considerations. Darboux obtains also the differential equation of the geodesic lines, in a form which is frequently useful, by defining them as the shortest lines of the surface between two points of the surface, and by solving the simple problem in the calculus of variations posed by this definition. I regret that Bianchi does not study at all this problem of the calculus of variations. He merely states its result (p. 154). Darboux next calls attention to the fact that the minimal curves of the surface are geodesic lines in the sense that they satisfy the differential equation of the geodesics. He shows also how they satisfy the first definition given for geodesics. We may note here that Bianchi makes no mention of the minimal curves of a surface, so far as I know, throughout the book. This seems to me unfortunate. For, though these curves are always imaginary, they are of considerable interest, and sometimes of use in solving problems concerning the real elements of surfaces. They have many curious properties which deserve mention in such a book as Bianchi's. Besides being geodesic lines, they are all lines of curvature, and furnish the only exception to the theorem proved by Bianchi (p. 166) that all geodesic lines which are at the same time lines of curvature are plane curves. Another property of these curves is the following, proved, for example, by Raffy, Applications géométriques de l'analyse, p. 147 : The only surfaces, not spheres, all of whose points are umbilics are developable surfaces whose edges of regression are minimal curves. All these theorems are omitted by Bianchi.*

A considerable rôle in the theory of geodesic lines is played by the problem of determining in how far the geodesic lines are the shortest lines of the surface between two of their points. Fundamental for this problem is the system of "polar coördinates" consisting of the geodesics radiating from a point of the surface and their orthogonal trajectories. Before such a system can be used as a coördinate system it is necessary to know that any point of a surface can be surrounded by a region in which the geodesics issuing from the point do not again intersect. This is a point assumed by most writers, among them Bianchi. Darboux proves the theorem very carefully (Volume II., p. 408). This theorem assumed, it is easy to prove that the geodesic line is actually the shortest line from the origin of the polar coördinates to any point of the surrounding region. This fact is proved by Bianchi (p. 163). The minimum properties of the geodesics are not further considered by him. Darboux devotes a very interesting chapter of twenty-six pages (Livre VI., Ch. V.) to this subject. This chapter is to a considerable extent the application of Jacobi's work in the calculus of variations to the special problem of the geodesic lines. In §80 Bianchi returns to the differential equation of the geodesics and obtains for it a new form, due to Gauss, in terms of the angle which a geodesic makes with one of the coördinate curves. This equation he uses to give a new definition of geodesic curvature, which is the natural generalization of the curvature of a plane curve. The author next discusses systems of geodesically parallel

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* See p. 93.

curves, defining them and obtaining the necessary and sufficient condition that a given family of curves shall be geodesically parallel. In the next section he discusses the subject of geodesic circles. Bianchi defines a geodesic circle as the locus of a point whose geodesic distance from a fixed point is constant; Darboux defines a geodesic circle as a curve of constant geodesic curvature. Both authors call attention to the fact that these definitions are not in general coincident. Neither discusses the question. Bianchi states in a footnote (p. 162), that the two definitions are equivalent for surfaces of constant curvature, but gives no proof. The statement may be proved in a couple of lines, and might perhaps well have been proved. The consideration of this problem leads to the more general theorem that the necessary and sufficient condition that each curve of one family of concentric geodesic circles (in Bianchi's sense) should have constant geodesic curvature is that the total curvature of the surface should be constant along each curve. This may be otherwise stated: The necessary and sufficient condition that there exist on a surface a single family of concentric geodesic circles, such that the geodesic curvature of each circle is constant, is that the surface shall be applicable to a surface of revolution so that the geodesic circles fall on the latitude circles of the surface of revolution.

It is noticeable that Bianchi has very little to say about curves of constant geodesic curvature. In the last section of the chapter (§91) he proves the theorem: A doubly orthogonal system of curves of constant geodesic curvature is always isothermal, and conversely, if one family of curves of an isothermal system be curves of constant geodesic curvature, the curves of the other family will also be of constant geodesic curvature. We have no discussion of the isoperimetrical problem on a surface. The solution of this problem gives, as is known, always a curve of constant geodesic curvature. This and analogous problems are discussed at length by Darboux in a chapter on "Les cercles géodésiques" (Livre VI., Ch. VI.). It seems to me that the subject of curves of constant geodesic curvature is worthy of even fuller discussion than that given by Darboux. It would be interesting, for example, to find what surfaces admit such curves which are in general closed; for on such surfaces the isoperimetrical problem is of especial interest. That these curves are not generally closed appears from the fact that a surface may be passed through any curve without singularities, so that, for that surface, the curve is of constant geodesic curvature. In §83, Bianchi takes up the subject of geodesic conics, and gives a concise treatment of them. He omits the discussion of surfaces on which there exist families of curves which may be regarded in more than one way as geodesic conics. These surfaces are surfaces for which the linear element may be written in the form known by the name of Liouville, as is proved by Darboux (Volume 3, p. 17). Next are taken up the subjects of the torsion of the geodesic line, and of the geodesic torsion of any curve of the surface. In a footnote is added the interesting expression for the torsion of a geodesic line, when the lines of curvature are chosen as coördinate curves. In the next two sections are taken up, briefly, methods of finding the geodesic lines of a surface from an integral of the partial differential equation $\Delta_1\theta = 1$. Though these sections are naturally much less complete than Darboux's treatment of the same subject, we find in them the important results of the theory. These methods are then applied to Liouville's surfaces and to surfaces of revolution. The equation of any geodesic line on a Liouville's surface is $\partial\theta/\partial a = b$, where

$$\theta = \int\sqrt{\alpha(u) + a}\; du \pm \int\sqrt{\beta(v) - a}\cdot dv,$$

a and b being arbitrary constants. As is proved by Bianchi, the curves (u, v) are at the same time isothermal coördinates and the parameters of families of a system of confocal geodesic conics. It is easily seen that the "ground curves" of the families of conics are any two curves θ = constant, with opposite connecting signs, where the same value is given to a in both cases. From this it follows, as stated above, that the system of geodesic conics (u, v) may be regarded as conics in an infinite number of ways, *i. e.*, for all values of a. In his discussion of the geodesics on a surface of rotation, Bianchi obtains "Clairaut's equation," $r \sin\psi = k$. In the last paragraph of the section (p. 174) is the obvious slip, for the case where the surface has a maximum circle of latitude: "die Curve verläuft ganz innerhalb der Zone, in der die Radien der Parallelkreise nicht *grösser* sind als k." For *grösser* read *kleiner*. This mistake occurs in the Italian, and was not corrected in the translation. Darboux discusses (Volume 3, pp. 2–9) the determination of surfaces of revolution, all of whose geodesics are closed curves; Bianchi omits this discussion. Finally (§90) we have a very neat and simple proof of Gauss's theorem, that the total curvature of a geodesic triangle is equal to the excess of the sum of its angles over two right angles. Darboux gives a very elaborate discussion of the properties of geodesic triangles based on Gauss's memoir of 1828 (Livre VI., Ch. VIII.).

There are various problems connected with geodesic lines to be found in other parts of the volume. For example, on page 436 is discussed very simply the problem of finding the surfaces which can be represented in such a way on the plane that the geodesics of the surface are represented by the straight lines of the plane. Darboux gives an entirely different proof, longer but simpler in some respects, and discusses the more general problem of the "geodesic representation" of one surface on another.

In closing my account of this chapter I must say that it appears to me a most admirable one. All the principal results of the theory, with almost insignificant exceptions, are given in very small compass and in perfectly clear form.

Of the remaining chapters I can give but a brief summary. In writing this summary I have followed a short review of Bianchi's book published by Darboux in the *Bulletin des Sciences mathématiques*, Volume 32 (1897), p. 253. Chapter VII. contains the fundamental theorems relative to applicable surfaces. Chapter VIII. discusses ruled surfaces, especially the problem of the deformation of such surfaces. In it are given the chief results of Chasles, Bonnet, Minding, and Beltrami. Chapter IX. discusses evolute surfaces, and begins the study of surfaces known as W or Weingarten surfaces, for which there exists a relation between the two principal radii of curvature of the surface at every point. Chapter X. gives the infinitesimal theory of rectilinear congruences. Chapters XI. and XII. treat the infinitesimal deformation of surfaces and the correspondence of surfaces with orthogonal elements. Chapter XIII. discusses "cyclical systems," that is to say congruences of circles normal to a family of surfaces. Chapter XIV. discusses the elements of the theory of minimal surfaces. In Chapter XV. is given a very detailed study of Schwarz's minimal surface bounded by four contiguous edges of a regular tetrahedron. Chapter XVI. studies the conformal representation of a pseudo-spherical surface on the half plane. Chapter XVII. contains a study of the transformations of surfaces of constant curvature. Chapters XVIII., XIX., and XX., the last three chapters of the Italian edition, contain a summary of the theory of orthogonal curvilinear coördinates in space or of triply orthogonal systems of surfaces. There are to be found the theorems of Dupin and Darboux, the theorem of Liouville on the conformal representation of space, the theorem of Combescure on triple systems having a given spherical representation, and the application of the general theory to homofocal surfaces of the second degree, together with a detailed study of the geodesic lines of these surfaces.

Finally is given the investigation, due to Bianchi, of triple systems containing a family of surfaces of constant curvature. In chapters XXI. and XXII., which are published for the first time in the German translation, are given the principal formulas of n-dimensional differential geometry, with especial reference to spaces of constant Riemann's curvature. An appendix to chapter XVII., printed at the end of the *Vorlesungen*, contains a short report of Bianchi's recent researches on the transformation of surfaces of constant positive curvature.

It remains only to say a few words of the book as a whole. It is certainly a magnificent production; the treatment is throughout clear and usually exhaustive. There is not a page that is dull, though there are many that are difficult. If the book has any fault, it is that it is in many places too difficult for the beginner. This fact is recognized by the author who advises the student to omit, in his first reading, certain chapters. A good testimony to the excellence of the book is that Darboux finds in it nothing to criticise but its title.* It is, I think, to be regretted that the author gives so few direct references. A table of authors consulted, printed at the end of the volume, hardly fills their place. A very great convenience is the giving of page references to formulas previously obtained. This very excellent plan is adopted in no other book that I know.

Strange as it may seem, the translation is in many respects an improvement on the original. It is, so far as I can see, an excellent translation. It is printed with better type and on better paper than the original. A heading is printed at the beginning of each section. This makes the page more attractive than that of the original, besides being a considerable convenience. Moreover to the translation has been added a general index. Most of the errors and misprints of the Italian have been corrected, but not all, as I have already noted. A few new errors and misprints have crept in, but none would, I think, inconvenience the reader. It seems not worth while to give a list of these. Certainly Bianchi and his translator, Lukat, have rendered the mathematical public no inconsiderable service in the production of the "Differentialgeometrie."

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* *Bulletin des sciences mathématiques*, vol. 34 (1899), p. 323.