$$(22) \quad z_i P(y) = \frac{d}{dx}\left[(-1)^{i-1}\frac{W(y \cdot y_1, \cdots, y_{i-1} \cdot y_{i+1}, \cdots, y_n)}{W(y_1, \cdots, y_n)}\right],$$

where $y_1, \cdots, y_n$ are any set of n linearly independent solutions of (19) and $z_1, \cdots, z_n$ are the functions adjoint to them. Thus we have proved the theorem

XVIII. *A necessary and sufficient condition that z be a multiplier of* (19) *is that it be a member of the linear family adjoint to the family which consists of the solutions of* (19).

GRUND IM HARZ,
July 20, 1901.

THE CONFIGURATIONS OF THE 27 LINES ON A CUBIC SURFACE AND THE 28 BITANGENTS TO A QUARTIC CURVE.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, August 20, 1901.)

Introduction.

AFTER determining * four systems of simple groups in an arbitrary domain of rationality which include the four systems of simple continuous groups of Lie, the writer was led to consider the analogous problem for the five isolated simple continuous groups of 14, 52, 78, 133, and 248 parameters. The groups of 78 and 133 parameters are related to certain interesting forms of the third and fourth degrees respectively.† They suggested the forms C (§ 1) and Q (§ 3).

It is shown in § 1 that the cubic form C defines the configuration of the 27 straight lines on a cubic surface in or-

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the functions $y_1, \cdots, y_k$ being supposed to be any functions of x which throughout (J) have continuous derivatives of the first $k-1$ orders. This establishes the truth of (F) at all points of (J) except where

$$W(y_1, \cdots, y_{m-1}) = 0.$$

If c is a point where this last equality holds two cases are possible: 1° there may be points in every neighborhood of c where the equality does not hold and where therefore (F) holds. In this case, on account of the continuity of both sides of (F), this formula holds also at c 2°, $W(y_1, \cdots, y_{m-1})$ may vanish identically throughout the neighborhood of c. In this case (F) also holds at c since all the Wronskians which occur in it vanish at c; cf. *Transactions*, vol. 2, p. 148.

* Abstract presented to the Society, Aug. 20, 1901, to appear in extenso in the *Transactions*. A note on the subject appeared in *Comptes rendus*, CXXXII. (1901), pp. 1547-8 .

† Cartan, Thèses, Paris, 1894.

dinary space. After proving this result, the writer observed that the formulæ remained unaltered if the notation for the variables was chosen to be $x_i, y_i, z_{ij} \equiv z_{ji}$ $(i, j = 1, \cdots, 6;$ $j \neq i)$, a notation given by Burkhardt. * The notation (1) has been retained in view of the relation with the later sections and to retain uniformity with the notation of a paper † on the transformation group defined by the invariant C for an arbitrary domain of rationality.

The group of the configuration of the 27 lines on a cubic surface is exhibited in § 2. A study of the quartic form Q and the group of the configuration defined by it is made in §§ 3–6.

§ 1. *The* 27 *Lines on a Cubic Surface.*

A general cubic surface contains 27 straight lines such that

1°. Any one of the 27 lines A meets 10 other lines which intersect by pairs, forming with A 5 triangles. The total number of such triangles on the surface is therefore 45.

2°. Any two triangles ABC and $A'B'C'$ having no side in common determine uniquely a third triangle $A''B''C''$, such that the corresponding sides of the three triangles intersect and form three new triangles $AA'A''$, $BB'B''$, $CC'C''$. The former set of three triangles is said to constitute a *trieder*. Each triangle lies in exactly 16 trieders.

These two properties completely define the configuration of the 45 triangles formed by the 27 lines on the cubic surface.

We employ the 27 variables

$$(1) \qquad x_i,\ y_i,\ z_{ij} \equiv -z_{ji} \quad (i, j = 1, \cdots, 6;\ j \neq i)$$

and consider the cubic form with 45 terms ‡

$$C \equiv \sum_{i \neq j}^{i,j=1,\ldots,6} x_i y_j z_{ij} + \sum z_{\lambda\mu} z_{\nu\rho} z_{\sigma\tau},$$

the second sum comprising the 15 terms of the Pfaffian [1 2 3 4 5 6], so that the subscripts have the following values:

12 34 56,	13 24 65,	14 23 56,
12 35 64,	13 25 46,	14 25 63,
12 36 45,	13 26 54,	14 26 35,

* *Math. Annalen*, vol. 41, p. 339.

† Offered by the writer, July 18, 1901, to the *Quar. Jour. of Math.*

‡ Derived from the function J of Cartan (l. c., p. 143) upon replacing each y_j by $-y_j$. The character of J was not considered by Cartan.

$$\begin{array}{ll} 15\ 23\ 64, & 16\ 23\ 45, \\ 15\ 24\ 36, & 16\ 24\ 53, \\ 15\ 26\ 43, & 16\ 25\ 34. \end{array}$$

Let the 27 lines on the cubic surface be represented by the variables (1), so that $z_{ji} \equiv - z_{ij}$ represents the same line that z_{ij} represents. We proceed to prove that *the arrangement of the 27 variables into the 45 triples given by the terms of C furnishes a suitable notation for the configuration of the 45 triangles formed by the 27 straight lines on a general cubic surface in ordinary space.*

Of the triples exhibited by the terms of C, exactly 5 contain x_i; exactly 5 contain y_j; exactly 5 contain z_{ij}; viz.,

$$x_iy_jz_{ij}, \quad x_jy_iz_{ji}, \quad z_{ij}z_{mn}z_{pq}, \quad z_{ij}z_{mp}z_{qn}, \quad z_{ij}z_{mq}z_{np},$$

where i, j, m, n, p, q form a permutation of 1, 2, 3, 4, 5, 6.

For i and j fixed, the triple $x_iy_jz_{ij}$ lies in exactly 16 trieders

$$\left\{\begin{matrix} x_i & y_j & z_{ij} \\ z_{ki} & x_k & y_i \\ y_k & z_{jk} & x_j \end{matrix}\right\} \text{(four)}, \quad \left\{\begin{matrix} x_i & y_j & z_{ij} \\ y_k & x_l & z_{lk} \\ z_{ik} & z_{lj} & z_{rs} \end{matrix}\right\} \text{(twelve)},$$

where i, j, k, l, r, s form an even permutation of 1, 2, 3, 4, 5, 6.

The triple $z_{12}z_{34}z_{56}$ lies in 12 trieders of the type just given and in exactly four additional trieders

$$\left\{\begin{matrix} 12 & 34 & 56 \\ 35 & 26 & 14 \\ 64 & 15 & 23 \end{matrix}\right\}, \quad \left\{\begin{matrix} 12 & 34 & 56 \\ 53 & 16 & 24 \\ 46 & 25 & 13 \end{matrix}\right\}, \quad \left\{\begin{matrix} 12 & 34 & 56 \\ 36 & 15 & 24 \\ 54 & 26 & 13 \end{matrix}\right\}, \quad \left\{\begin{matrix} 12 & 34 & 56 \\ 63 & 25 & 14 \\ 45 & 16 & 23 \end{matrix}\right\}.$$

Similarly, $z_{ik}z_{lj}z_{rs}$ lies in exactly 16 trieders. Hence the notation is suitable to exhibit the configuration.

§ 2. *Group G of the Equation for the 27 Lines.*

The group G of the configuration of the 45 triangles formed by the 27 lines on a general cubic surfaces is composed of the literal substitutions on the variables (1) which leave the function C invariant. The determination of G has been effected by Jordan (Traité, pp. 316–329) and by the writer (Linear groups, Chapter XIV.). The notations of the latter treatment may be identified with the present notations as follows:

$$\begin{array}{llllll} x_1 = R_0, & x_2 = R_{212}, & x_3 = R_{111}, & x_4 = R_{121}, & x_5 = R_{131}, & x_6 = R_{141}, \\ y_1 = R_{112}, & y_2 = R_1, & y_3 = R_{210}, & y_4 = R_{120}, & y_5 = R_{130}, & y_6 = R_{140}, \\ z_{12} = R_2, & z_{13} = R_{110}, & z_{14} = R_{220}, & z_{15} = R_{230}, & z_{16} = R_{240}, & \\ z_{23} = R_{211}, & z_{24} = R_{221}, & z_{25} = R_{231}, & z_{26} = R_{241}, & z_{34} = R_{222}, & \\ z_{35} = R_{232}, & z_{36} = R_{242}, & z_{45} = R_{142}, & z_{46} = R_{132}, & z_{56} = R_{122}. & \end{array}$$

It follows that G is generated by the substitutions

W: $(x_1y_2z_{12})(x_2y_3z_{23})(x_3y_1z_{13})(y_4x_4z_{56})(y_5x_5z_{46})(y_6x_6z_{45})$
$(z_{14}z_{24}z_{34})(z_{15}z_{25}z_{35})(z_{16}z_{26}z_{36})$;

E_1: $(x_4y_5z_{26})(y_6y_4z_{13})(x_6z_{24}z_{25})(x_2z_{46}z_{56})(y_1z_{35}z_{34})(y_3z_{15}z_{14})$;

E_2: $(y_5y_6)(y_4z_{13})(y_3z_{14})(y_1z_{34})(x_2z_{56})(x_5z_{25})(x_6z_{26})$
$(z_{15}z_{16})(z_{35}z_{36})(z_{45}z_{46})$;

E_3: $(y_2y_6)(y_4z_{13})(y_3z_{14})(y_1z_{34})(x_2z_{25})(x_5z_{26})(x_6z_{56})$
$(z_{12}z_{16})(z_{23}z_{36})(z_{24}z_{46})$;

T: $(y_4y_6)(x_4x_6)(z_{14}z_{16})(z_{24}z_{26})(z_{34}z_{36})(z_{54}z_{56})$.

Of these, W, E_1, E_2, and E_3 give rise to even substitutions on the 45 triangles, while T gives rise to an odd substitution. The group G of order 51840 therefore has a subgroup G_{25920} generated by W, E_1, E_2, E_3. A suitable product of the latter replaces x_1 by an arbitrary one of the 27 variables. Hence G_{25920} is transitive.

§ 3. *Character of the Quartic Form* Q.

We employ the 56 variables

(2) $\quad x_i,\ y_i,\ x_{ik} = -x_{ki},\ y_{ik} = -y_{ki} \quad (i, k = 1, \cdots, 7;\ k \neq i)$

and consider the quartic form with 630 terms*

$$Q \equiv \sum x_i y_j x_{ik} y_{jk} + \sum x_{\lambda\mu} x_{\nu\rho} y_{\mu\nu} x_{\lambda\rho} + \sum (x_i y_{\lambda\mu} y_{\nu\rho} y_{\sigma\tau} + y_i x_{\lambda\mu} x_{\nu\rho} x_{\sigma\tau}),$$

where in the first sum $i, j, k = 1, \cdots, 7$; $i \neq j$, $i \neq k$, $j \neq k$; in the second sum λ, μ, ν, ρ run through the various permutations of $1, 2, \cdots, 7$ four at a time, but taking only one of the four equal terms

$$x_{\lambda\mu} x_{\nu\rho} y_{\mu\nu} y_{\lambda\rho} = x_{\mu\lambda} x_{\rho\nu} y_{\lambda\rho} y_{\mu\nu} = x_{\nu\rho} x_{\lambda\mu} y_{\rho\lambda} y_{\nu\mu} = x_{\rho\nu} x_{\mu\lambda} y_{\nu\mu} y_{\rho\lambda};$$

in the third sum i, λ, μ, ν, ρ, σ, τ denotes an even permutation of $1, 2, \cdots, 7$, so that the coefficient of x_i is the Pfaffian $[1\ 2 \cdots i-1\ i+1 \cdots 7]$ defined in § 1.

Each of the three sums in Q extends over 210 terms. The fact that there are 630 terms in Q also follows from the lemma next proved, since $\frac{1}{4}\, 45 \times 56 = 630$.

Lemma.—*Each variable lies in exactly* 45 *terms of* Q. *The camplementary cubic factor defines the configuration of the* 45 *triangles formed by the* 27 *straight lines on a cubic surface.*

* Suggested by the essentially different function J of Cartan (p. 144).

There are 30 and 15 terms respectively in the sums

$$\frac{\partial Q}{\partial x_i} \equiv \sum y_j x_{ik} y_{jk} + \sum y_{\lambda\mu} y_{\nu\rho} y_{\sigma\tau},$$

where $j, k = 1, \cdots, 7$; $j \neq i$, $k \neq i$, $j \neq k$; and λ, μ; ν, ρ; σ, τ run through the subscripts of the 15 terms of the Pfaffian $[1, 2, \cdots, i-1, i+1, \cdots, 7]$. To identify $\frac{\partial Q}{\partial x_i}$ with the function C of § 1, we have only to set $x_{ik} = -x_k$, $y_{jk} = z_{jk}$. From $\frac{\partial Q}{\partial x_i}$ we obtain $\frac{\partial Q}{\partial y_i}$ by interchanging y_j with x_j, y_{jk} with x_{jk}, an operation defining a substitution A which leaves Q unaltered. Similarly (§ 4) there exist substitutions which leave Q invariant and replace x_1 by any one of the 56 variables. Hence the lemma holds for each x_{ij} and y_{ij}. For a direct proof, consider the case

$$\frac{\partial Q}{\partial y_{67}} \equiv \sum_{i=1}^{5} (x_i y_6 x_{i7} + x_i y_7 x_{6i}) + \sum x_{6\mu} x_{\nu 7} y_{\mu\nu} + \sum x_i y_{\nu\rho} y_{\sigma\tau},$$

where in the second sum $\mu, \nu = 1, \cdots, 5$; $\mu \neq \nu$; in the third sum $i = 1, \cdots, 5$, while ν, ρ, σ, τ form a cyclic permutation of 1, 2, 3, 4, 5, with i excluded. To identify this expression of 45 terms with C, we may take

$$x_i = z_{6i},\ x_{i7} = y_i,\ x_{6i} = x_i,\ y_6 = x_6,\ y_7 = -y_6,\ y_{ij} = z_{ij}$$
$$(i, j = 1, \cdots, 5;\ j \neq i).$$

§ 4. *Substitution Group H with the Invariant Q.*

Among the literal substitutions on the 56 letters which leave Q invariant occur the following types:

$$A:\quad (x_i y_i)(x_{ij} y_{ij}) \qquad [i, j = 1, \cdots, 7;\ j \neq i];$$

$$B_{ij}:\quad (x_i y_j)(x_j y_i)(x_{ik} y_{kj})(x_{jk} y_{ki})(x_{ij} y_{ij})(x_k y_k)(x_{lk} y_{kl})$$
$$[k,\ l = 1, \cdots, 7, \text{ excluding } i, j];$$

$$D_1:\quad (x_1 y_1)(x_i x_{i1})(y_i y_{1i})(x_{jk} y_{jk}) \qquad [i, j, k = 2, \cdots, 6].$$

Each is the product of 28 transpositions. A suitable product of A, B_{ij}, D_1 throws x_1 to any one of the 56 letters. *The group H leaving Q invariant is, therefore, transitive.*

The order Ω of H is therefore $56\,\Omega_1$, where Ω_1 is the order of the subgroup H_1 which leaves x_1 fixed. The substitutions of H_1 permute amongst themselves the 45 terms of $\frac{\partial Q}{\partial x_1}$. In view of § 2, Ω_1 is *at most* equal to 51840 Ω', where Ω' is the order of the subgroup of H which leaves fixed the 28 variables x_1, y_j, x_{1j}, y_{jk} $(j, k = 2, \cdots, 7;\ j \neq k)$. We readily verify that the identity is the only substitution S of H which leaves these 28 variables fixed, so that $\Omega' = 1$ and therefore

$$(3) \qquad \Omega \leqq 56 \times 51840.$$

In fact, S must permute amongst themselves the terms of Q which involve both y_7 and x_{1j} and hence leave invariant

$$\frac{\partial^2 Q}{\partial y_7 \partial x_{1j}} \equiv x_1 y_{7j} + x_j y_{17} + x_{\nu\rho} x_{\sigma\tau} + x_{\nu\sigma} x_{\tau\rho} + x_{\nu\tau} x_{\rho\sigma},$$

where $j, \nu, \rho, \sigma, \tau$ is an even permutation of 2, 3, 4, 5, 6. Hence the letters common to two such quadratic forms must be permuted amongst themselves. It follows that S leaves invariant

$$y_{17},\ x_{46},\ x_{56},\ x_{45},\ x_{24},\ x_{25},\ x_{26},\ x_{23},\ x_{34},\ x_{35},\ x_{36},$$

and therefore also x_2, x_3, x_4, x_5, x_6. Hence each term $x_i y_{1i} y_{\nu\rho} y_{\sigma\tau}$ is invariant and therefore x_7, y_{1j} $(j = 2, \cdots, 7)$. Similarly, $x_2 y_1 x_{23} y_{13}$ and $x_7 y_1 x_{7k} y_{1k}$ $(k = 2, \cdots, 6)$ require that y_1, x_{7k} be fixed. Hence S leaves all 56 letters fixed and is the identity.

A shorter proof follows from the important lemma

If a substitution of H leaves x_i fixed, it leaves y_i fixed, and inversely. If it leaves x_{ij} fixed, it leaves y_{ij} fixed and, inversely.

In view of the form of the substitutions A, B_{ij}, D_1 and the transitivity of H, it suffices to consider the case of a substitution Σ of H which leaves x_1 fixed. Then Σ permutes the 27 variables y_j, x_{1j}, y_{jk} $(j, k = 2, \cdots, 7)$ amongst themselves and therefore permutes the remaining 28 variables y_1 and

$$(4) \qquad x_j,\ y_{1j},\ x_{jk} \qquad (j, k = 2, \cdots, 7;\ j \neq k).$$

Hence Σ permutes the 45 terms of Q which involve y_1, these being the only terms containing exclusively letters of the set (4). Since each of the letters (4) occurs in terms of Q which do not involve y_1, no one of the letters (4) is replaced by y_1 under Σ. Hence y_1 is not altered by Σ.

In view of § 5, the group H_1 contains a subgroup holoedrically isomorphic with the group G_{51840}^{27} of the equation for the 27 lines on a cubic surface. It follows that

$$\Omega \geqq 56 \times 51840.$$

Combining this result with the result (3), we conclude that *the order of group H is* 56×51840.

§ 5. *Generators of the Group H_1.*

In investigating the subgroup H_7 of H which leaves x_7 fixed, we consider certain substitutions of H_7 which leave $\frac{\partial Q}{\partial x_7}$ fixed. By § 3, the latter function is identical with the cubic form C of § 1 if we let $x_{7k} = -x_k$, $y_{jk} = z_{jk}$, for $k, j = 1, \cdots, 6$; $k \neq j$. We obtain from the substitutions W, E_i, T of § 2 the following substitutions leaving $\frac{\partial Q}{\partial x_7}$ invariant:

$$[E_2]: \quad (y_5y_6)(y_4y_{13})(y_3y_{14})(y_1y_{34})(x_{27}y_{56})(x_{57}y_{25})$$
$$(x_{67}y_{26})(y_{15}y_{16})(y_{35}y_{36})(y_{45}y_{46}),$$
$$[E_3]: \quad (y_2y_6)(y_4y_{13})(y_3y_{14})(y_1y_{34})(x_{27}y_{25})(x_{57}y_{26})$$
$$(x_{67}y_{56})(y_{12}y_{16})(y_{23}y_{36})(y_{24}y_{46}),$$
$$[T]: \quad (y_4y_6)(x_{47}x_{67})(y_{14}y_{16})(y_{24}y_{26})(y_{34}y_{36})(y_{54}y_{56}),$$

together with $[W]$ and $[E_1]$, whose forms are equally evident. These substitutions do not leave Q invariant; but leave invariant $\frac{\partial Q}{\partial x_7}$ and $\frac{\partial Q}{\partial y_7}$. By interchanging x_i with y_i and x_{ij} with y_{ij} $(i, j = 1, \cdots, 7)$, we obtain the substitutions

$$A^{-1}[W]A, \; A^{-1}[E_i]A, \; A^{-1}[T]A \quad (i = 1, 2, 3)$$

affecting only the 27 variables x_j, y_{7j}, x_{ij} $(i, j = 1, \cdots, 6)$. Hence they are commutative with $[W]$, $[E_i]$, $[T]$. The products

$$(5) \qquad \begin{gathered}[W]A^{-1}[W]A, \; [E_i]A^{-1}[E_i]A, \quad (i = 1, 2, 3) \\ [T]A^{-1}[T]A\end{gathered}$$

therefore generate a group K which is holoedrically isomorphic with the group G^{27}_{51840} of the configuration defined by C, and hence with the group of the equation for the 27 lines on a cubic surface. Without proof,* I will state the theorem *that the substitutions* (5) *leave invariant the function* Q.

It follows from § 4 that the group K generated by the substitutions (5) is identical with the subgroup H_7 of the substitutions of H which leave x_7 invariant. Hence every substitution S of H_7 can be expressed as a product $S_1A^{-1}S_1A$, where S_1 affects only x_{7j}, y_j, y_{jk} $(j = 1, \cdots, 6)$, whereas $A^{-1}S_1A$ affects only y_{7j}, x_j, x_{jk}, being similar to S_1. To give a direct

* I first verified by direct computation that the theorem is true for $[E_3]A^{-1}[E_3]A$ and that $[E_2]A^{-1}[E_2]A$ is the transform of $[E_3]A^{-1}[E_3]A$ by B_{52}.

proof, let S replace y_j by $y_{\lambda\mu}$, for example. There exists a substitution $R = R_1 A^{-1} R_1 A$, where R_1 leaves x_7 fixed and affects the same variables as does S_1, which replaces y_j by $y_{\lambda\mu}$ and therefore x_j by $x_{\lambda\mu}$. Then SR^{-1} leaves y_j fixed and therefore also x_j. Hence S replaces x_j by $x_{\lambda\mu}$.

§ 6. *Isomorphic Substitution Group on* 28 *Letters.*

THEOREM.—*The group H of all substitutions on the* 56 *variables* (2) *which leave Q invariant is imprimitive, possessing the* 28 *systems of imprimitivity*

$$(6) \qquad S_i \equiv [x_i, y_i], \quad S_{ij} \equiv [x_{ij}, y_{ij}] \qquad (i, j = 1, \cdots, 7;\ j \neq i).$$

Since A transforms each B_{ij} and D_1 into themselves, they preserve the imprimitive systems (6). We find that A corresponds to the identical substitution on S_i, S_{ij}; while

$$B_{ij} \sim (S_i S_j)(S_{ik} S_{kj}) \qquad (k = 1, \cdots, 7; k \neq i, j),$$
$$D_1 \sim (S_k S_{k1}) \qquad (k = 2, \cdots, 7),$$

each substitution on the systems being composed of 6 cycles. A suitable product of them replaces S_1 by any given one of the 28 systems. To the substitutions (5) correspond substitutions on the systems analogous to $[W]$, $[E_i]$, $[T]$, respectively. For example, to $[E_2]A^{-1}[E_2]A$ corresponds the substitution

$$(S_5 S_6)(S_4 S_{13})(S_3 S_{14})(S_1 S_{34})(S_{27} S_{56})(S_{57} S_{25})$$
$$(S_{67} S_{26})(S_{15} S_{16})(S_{35} S_{36})(S_{45} S_{46}).$$

The group H is hemiedrically isomorphic with a transitive substitution group H' of order 28×51840 *on the* 28 *letters S_i, S_{ij}.*

There is a known transitive substitution group of the same order and degree, viz., the group Γ of the equation for the 28 bitangents to a quartic curve without double points. By the adjunction of a root,* the group reduces to a group holoedrically isomorphic with the group G of the 27 lines on a cubic surface (§ 2). By an earlier result, the subgroup of H' which leaves one letter fixed is holoedrically isomorphic with G. It would seem quite probable that Γ and H' are isomorphic, and indeed identical groups. A formal investigation has not been attempted by the author. Granting the truth of this conjecture, the invariant Q would define the configuration of the 56 points of contact of the 28 bitangents to a quartic curve without double points.

THE UNIVERSITY OF CHICAGO,
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* Jordan, Traité des substitutions, p. 330.