THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 26, 1901, extending through the usual morning and afternoon sessions. Vice-President Professor Thomas S. Fiske occupied the chair. The American Physical Society met on the same day, and the mutual interests of the two societies were renewed by a joint session, presided over by President A. A. Michelson of the Physical Society, at which Professor J. Hadamard, the official delegate of the University of Paris at the Yale bicentennial celebration, read a paper entitled "On the theory of elastic plates."

The total attendance at both sessions exceeded forty persons, including the following thirty-three members of the Society :

Professor E. W. Brown, Professor F. N. Cole, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Dr. William Findlay, Professor H. B. Fine, Professor T. S. Fiske, Miss Ida Griffiths, Miss Carrie Hammerslough, Dr. G. W. Hill, Dr. A. A. Himowich, Professor Harold Jacoby, Dr. Edward Kasner, Mr. C. J. Keyser, Dr. G. H. Ling, Dr. Emory McClintock, Professor James Maclay, Mr. H. B. Mitchell, Dr. I. E. Rabinovitch, Professor J. K. Rees, Mr. C. H. Rockwell, Miss I. M. Schottenfels, Professor C. A. Scott, Professor D. E. Smith, Dr. H. F. Stecker, Professor J. H. Van Amringe, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor L. A. Wait, Professor A. G. Webster, Miss E. C. Williams, Dr. Ruth G. Wood, Professor R. S. Woodward.

The Council announced the election of the following twelve persons to membership in the Society : Mr. C. H. Ashton, Harvard University ; Professor H. Y. Benedict, University of Texas ; Dr. William Findlay, Columbia University ; Dr. W. B. Fite, Cornell University ; Professor G. W. Greenwood, McKendree College ; Professor F. W. Hanawalt, Iowa Wesleyan University ; Dr. E. V. Huntington, Harvard University ; Professor H. W. Kuhn, Ohio State University ; Dr. I. E. Rabinovitch, New York, N. Y. ; Professor W. D. Tallman, Montana State College of Agriculture and Mechanic Arts ; Mr. H. M. Tory, McGill University; Mr. A. H. Wilson, Princeton University. Seven applications for membership were received.

Upon recommendation of the Council, By-Law VII was amended to provide that the presidential address shall hereafter be delivered at the annual meeting at which the presidential term expires. As this amendment goes into effect immediately, the next presidential address will occur at the annual meeting in December, 1902.

The following papers were presented at the meeting:

(1) Professor G. A. MILLER: "On the abelian groups which are conformal with non-abelian groups."

(2) Dr. H. F. STECKER: "Concerning the elliptic $\wp(g_2; g_3; z)$-functions as coördinates in a line complex, and certain related theorems."

(3) Miss I. M. SCHOTTENFELS: "Generational definitions of certain groups of order 960."

(4) Professor OTTO STOLZ: "Zur Erklärung der Bogenlänge und des Inhaltes einer krümmen Fläche."

(5) Dr. L. P. EISENHART: "Conjugate rectilinear congruences."

(6) Dr. S. E. SLOCUM: "The symbols of the infinitesimal transformations which generate the parameter groups corresponding to all possible types of structure of two, three, and four parameter complex groups."

(7) Dr. E. V. HUNTINGTON and Mr. J. K. WHITTEMORE: "Some curious properties of conics touching the line infinity at one of the circular points."

(8) Professor J. HADAMARD: "On the theory of elastic plates."

(9) Professor E. B. VAN VLECK: "On the zeros of fundamental integrals of regular linear differential equations of the second order, with a determination of the number of imaginary roots of the hypergeometric series."

(10) Dr. E. J. WILCZYNSKI: "Reciprocal systems of linear differential equations."

(11) Dr. I. E. RABINOVITCH: "On some contradictions involved in the elliptic geometry in a point space."

(12) Dr. EDWARD KASNER: "Determination of the integrals in the calculus of variations leading to an assigned system of extremals."

Professor Hadamard was introduced by Vice-President Fiske. Professor Stolz's paper was presented to the Society through Professor Moore. In the absence of the authors, Professor Miller's paper was read by Dr. Ling, and the papers of Professor Stolz, Dr. Slocum, Dr. Huntington and

Mr. Whittemore, and Dr. Wilczynski were read by title. The papers of Dr. Miller, Dr. Stecker, and Dr. Huntington and Mr. Whittemore will appear in the BULLETIN. Abstracts of the papers are given below.

Two distinct groups are said to be conformal when they contain the same number of operators of each order. Professor Miller's paper is devoted to the determination of all the abelian groups which are conformal with non-abelian groups. The results are as follows: The necessary and sufficient conditions that any abelian group of order $2^{a_0}p_1^{a_1}p_2^{a_2}\cdots$ ($p_1, p_2, \cdots$ being distinct odd primes) is conformal with at least one non-abelian group are: 1° At least one of its subgroups of orders $2^{a_0}, p_1^{a_1}, p_2^{a_2}, \cdots$ is noncyclic; 2° if the order $p_\beta^{a_\beta}$ of this subgroup is odd then $a_\beta > 2$, if the order is even (2^{a_0}) then the subgroup must include operators of order 4 and $a_0 > 3$.

Miss Schottenfels's paper treats of the generational definitions of certain abstract groups of order 960 containing an invariant abelian subgroup of order 16 and an icosahedral subgroup of order 60, with factors of composition 60, 2, 2, 2, 2, together with the determination of the generators of the substitution and Galois field groups holoedrically isomorphic with the above abstract groups.

Professor Stolz's paper, which will appear in the *Transactions*, gives a rigorous demonstration of the theorem that the integral

$$\int_a^\beta \sqrt{\varphi'(t)^2 + \psi'(t)^2}\, dt$$

satisfies the definition which he has established for the length of the curve

$$x = \varphi(t), \quad y = \psi(t) \qquad (a \leqq t \leqq \beta)$$

also in the case that the interval (a, β) may be divided into a finite number of parts on every one of which the derivatives $\varphi'(t)$, $\psi'(t)$ are continuous. He further defines the area or content of a surface and indicates the course of the proof of the theorem analogous to that just mentioned.

Certain general formulæ, established by Cifarelli, showing the relations existing between the Kummer functions of a rectilinear congruence referred to a general system of parameters, are applied by Dr. Eisenhart to several important cases, afforded by taking for the double family of parametric ruled surfaces in turn the principal ruled surfaces, the mean ruled surfaces, and one family of developables and their orthogonal trajectories. This application is made for the determination of all congruences having a given spherical representation of each one of these three double families of ruled surfaces. In each case it is found that the abscissa, measured from the surface of reference, of the point where the line of shortest distance between the lines (u, v) and $(u + du, v)$ meets the former satisfies a partial differential equation of the second order, whose coefficients are functions of the coefficients, and their derivatives, of the fundamental quadratic differential form of the sphere. And when this abscissa has been found, the further determination of the congruence reduces to the solution of a Riccati equation and quadratures.

These general results are then applied to the particular case where the given spherical representation of the ruled surfaces $u =$ const., $v =$ const. consists of a family of great circles and their orthogonal trajectories; this particular choice is made throughout the remainder of the paper. After it is shown that the line of shortest distance between the lines (u, v) and $(u + du, v)$ is different from the line between the latter and its consecutive one in the ruled surface $v =$ const. of which all three are generatrices, it follows that these lines of shortest distance form a congruence, which is called the *conjugate* of the original. It is evident from this definition that, with the choice of another system of great circles and their orthogonal trajectories for the representation of the ruled surfaces $u =$ const., $v =$ const. of the original congruence, there is a different conjugate congruence. Hence to each congruence there corresponds an indefinite number of conjugate congruences. Further, it can be shown that reciprocally the original congruence is a conjugate of each of its conjugates.

The above definition fails where the ruled surfaces $v =$ const. are developable; but another definition consistent with the results in the general case permits the consideration of this exceptional case.

Certain simple relations are readily found to exist between the Kummer functions for a congruence and a conjugate, from which can be determined the equation of condition that a conjugate congruence of a normal congruence be itself normal. It is found that the only surfaces whose normals form a congruence which has for a conjugate the aggregate of the tangents to the v lines on the surface are those for which the spherical representation of one system of asymptotic lines, $v = \text{const.}$, is a family of geodesic parallels. When the original congruence is defined as the intersection of two planes, the corresponding conjugate congruence can be found by algebraic operations.

The problem of finding the direction cosines of a congruence for which $u = \text{const.}$, $v = \text{const.}$ is a double family of great circles and their orthogonal trajectories is the same as the determination of a skew curve from its intrinsic equation, which in general reduces to the integration of a Riccati equation and quadratures.

A further investigation of congruences and particular ones of their conjugates shows that where a congruence and its conjugate corresponding to the developables of the former are normal, their developable surfaces correspond, and further that all such congruences can be determined by quadratures; the complete determination of normal congruences whose mean ruled surfaces $u = \text{const.}$, $v = \text{const.}$ correspond to the mean ruled surfaces of the corresponding conjugate congruence reduces to quadratures, and further this conjugate is not a normal congruence.

Professor Hadamard presented his views on the theory of thin elastic plates, the manner in which the equation of the problem is to be obtained, and the different methods which have been proposed for this purpose. Especially as to the boundary conditions, different methods have been employed which lead to apparently opposite results. The explanation of the discrepancy lies in the properties of the integrals of the equation $\Delta u = k^2 u$, when k is a large number. It is easily recognized that in this case the function u, defined within a given area by the preceding equation and by boundary values given once for all, is indefinitely small at any sensible distance from the boundary. More precisely, if δ is the distance of any interior point M from the boundary and A the maximum of the absolute values of the given values of u on the boundary, we have

$$|u_M| < \frac{A}{J(ik\delta)},$$

J being the well-known Bessel function, which increases very rapidly when the argument increases by purely imaginary values, so that the second member of the above inequality is very small, when k is very large, as soon as δ has a finite value.

The author is thus led to consider the formula for the asymptotic representation of the Bessel function $J(x)$ for large values of x. This formula furnishes a *formal* expression for J by means of a divergent series. It would be of interest to apply to this case Borel's methods relative to such series. The following method might also be tried: Given a sequence of numbers $a_0, a_1, \cdots, a_n, \cdots$, such that the series $a_0 + a_1 x + \cdots + a_n x^n + \cdots$ is divergent for any value of x, consider the series

$$a_0[1 - e^{-1/x} P_0(1/x)] + a_1 x[1 - e^{-1/x} P_1(1/x)] + \cdots + a_n x^n[1-e^{-1/x} P_n(1/x)] + \cdots,$$

where $P_1, P_2, \cdots, P_n, \cdots$ are polynomials, viz., the beginnings of the development of $e^{1/x}$ in powers of $1/x$. As in Weierstrass's proof of Mittag-Leffler's theorem, it will be easy to prove that the degrees of these successive polynomials can be so chosen that the above series shall be convergent for every sufficiently small value of x and represent a function whose asymptotic development in powers of x is precisely $a_0 + a_1 x + \cdots + a_n x^n + \cdots$.

Professor Van Vleck's paper will appear in the January number of the *Transactions*.

Dr. Wilczynski's paper is in abstract as follows: Let y_k and z_k for $k = 1, 2, 3, 4$ be four pairs of simultaneous solutions forming a fundamental system of the differential equations

$$(1) \qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0, \end{aligned}$$

with the independent variable x.

If y_k and z_k are interpreted as homogeneous coördinates of two points, the integration of (1) gives two curves in space. If corresponding points, *i. e.*, points corresponding to the same value of x, of the two curves are joined by straight lines, the ruled surface formed by these lines remains invariant for all transformations of the form

$$(2) \quad x = f(\xi), \quad y = \alpha(\xi)\eta + \beta(\xi)\zeta, \quad z = \gamma(\xi)\eta + \delta(\xi)\zeta,$$

where $f, \alpha, \beta, \gamma, \delta$ are arbitrary functions.

At corresponding points of the two curves, construct planes tangent to the ruled surface. Let their coördinates be u_k and v_k respectively. The author sets up a system of the same form as (1) satisfied by u_k and v_k and calls it the system adjoined to (1). This system is set up in several different forms, the most convenient being

$$(3)\quad \begin{aligned} &U'' + p_{11}U' + p_{12}V' + [q_{11} + \tfrac{1}{4}(u_{11} - u_{22})]\,U \\ &\qquad\qquad + (q_{12} + \tfrac{1}{2}u_{12})\,V = 0, \\ &V'' + p_{21}U' + p_{22}V' + (q_{21} + \tfrac{1}{2}u_{21})\,U \\ &\qquad\qquad + [q_{22} + \tfrac{1}{4}(u_{22} - u_{11})]\,V = 0. \end{aligned}$$

The relation between (1) and (3) is a reciprocal one. The two systems have the same invariants. Their monodromic and transformation groups stand in a simple relation to each other. They are equivalent if, and only if, the integrating ruled surface of (1) is of the second order.

It is well known that the Riemannian geometry, assuming space to possess constant positive curvature, postulates the finiteness of the plane and the straight line and leaves room for only two assumptions concerning two straight lines. The first is that two straight lines always meet in two points (the antipodal points), at a distance πk from each other ($k \equiv$ constant of curvature); the second makes the plane a unilateral surface and postulates that the two antipodal points coincide in one point on opposite sides of the plane. The first assumption corresponds to the so-called spherical space, in which the geometry for two dimensions coincides completely with the spherical geometry in euclidean space and the geometry for three dimensions, according to Beltrami, necessarily presupposes a space of four dimensions, in which the spherical three dimensional space is the analogon of a sphere in three dimensional euclidean space. The second assumption belongs to the elliptic geometry proper, as expounded by Clifford, Klein, and Newcomb. The complete length of a straight line is πk, and we can pass from the part of the plane on one side of the straight line to the other, without crossing the line itself, by moving in a direction parallel to it. All the curves parallel to a straight line, that is, whose points are at equal perpendicular distances from the line, form a system of concentric circles, whose center is at a distance $\frac{1}{2}\pi k$ from the straight line. This distance is the greatest possible absolute distance and the straight line and the center are in relation of pole and polar. All rays at an angle $\frac{1}{2}\pi$ to the polar pass through the pole and all points at a distance $\frac{1}{2}\pi k$ from the pole lie on the polar.

It is this last form of the Riemannian geometry which Dr. Rabinovitch, in his paper, finds to involve internal contradictions. The procedure followed is both synthetic and analytic. The first considers two concentric small sectors, between the same two rays, of lengths r for one and $\pi k - r$ for the other sector ($r < \frac{1}{2}\pi k$), the radii being measured upon the same side of the plane. He shows that, as a direct consequence of the above postulates of the elliptic geometry, when the figure, as a rigid whole, is rotated about the pole through an angle π, the two sectors, measured on the same side of the plane as originally, interchange in magnitude, the larger taking up the position of the trace of the smaller and *vice versa.* The only escape from this contradiction is that the two antipodal points must be distinct, which was originally Beltrami's contention.

Further, a comparison of two formulæ from spherical geometry, which must hold for all surfaces of constant positive curvature, shows that the assumption of the period πk for a geodesic leads to the equation $+1 = -1$.

The paper intimates the possibility of a concrete and true interpretation of the analytical formulæ of the elliptic geometry in a space whose element is a line or some other geometrical figure instead of a point, with corresponding alteration of meaning of metrical terms, like angle, distance, etc.

The simplest type of problem in the calculus of variations,

$$I = \int F(x, y, y')\, dx = \text{minimum},$$

leads to the differential equation

$$y'' F_{y'y'} + y' F_{y'y} + F_{y'x} - F_y = 0.$$

The doubly infinite system of functions or curves satisfying this equation are called the extremals connected with the integral. The question considered in Dr. Kasner's paper,* stated for this simplest type of problem, is an inverse one: Given an arbitrary doubly infinite system of curves, is it always possible to regard it as the system of extremals connected with an integral I? This question is answered in the affirmative: there is an infinite number of integrals I, *i. e.*, of integrands F, which have the assigned system of curves for extremals. The determination of F depends upon

* After this paper had been presented, my attention was called by Professor Osgood to the dissertation of G. Hamel, Göttingen, 1901, from which I learn that the inverse problem of the calculus of variations has been discussed, and most of my results anticipated, by Hirsch, *Math. Annalen*, 1897, and Boehm, *Crelle*, 1900. E. KASNER.

the solution of a certain partial differential equation of the second order which is reduced by quadratures to a linear partial differential equation of the first order.

The more complicated types of problems in the calculus of variations are examined in a similar manner. But in no type except the simplest are the differential equations which present themselves as necessary conditions of general character, *i. e.*, the system of extremals is no longer an arbitrary system of curves or surfaces (in space of any number of dimensions). Thus an integral of type

$$\int F(x, y, y', y'')\,dx$$

leads to a four parameter system of extremals; but only a restricted class of four parameter systems of curves can be obtained in this way. The characterization of this class depends upon the consistency of a certain set of partial differential equations.

The results obtained are of interest in connection with Hilbert's recent investigations, as serving to characterize the types of differential equations which can be studied by means of corresponding problems in the calculus of variations. All ordinary equations of the second order, but only a restricted class of linear partial differential equations of the second order, can be so treated.

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MODERN METHODS OF TREATING DYNAMICAL PROBLEMS AND IN PARTICULAR THE PROBLEM OF THREE BODIES.

PRÉCIS OF FOUR LECTURES DELIVERED BEFORE THE ITHACA COLLOQUIUM, AUGUST 21-24, 1901.

BY PROFESSOR E. W. BROWN.

IN this course an attempt was made to give some idea of the methods which have been used during the last twenty-five years to obtain information about certain classes of dynamical problems and in particular about those problems which are ordinarily considered by physicists to be insoluble in all their generality. One of the most important of these problems and one specially considered in the lectures is the *Problem of Three Bodies*, which at