which the notes of Dr. Schröder, secretary of the meeting, have been to me.

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GÖTTINGEN,
October, 1901.

SOME CURIOUS PROPERTIES OF CONICS TOUCHING THE LINE INFINITY AT ONE OF THE CIRCULAR POINTS.

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(Read before the American Mathematical Society, October 26, 1901.)

THE subject of the present note was suggested by the apparent contradiction between the following well known theorems of modern analytic geometry :

1. Through any four given points of the plane there pass two parabolas.

2. Through the vertices and the orthocenter of a triangle there pass no conics but equilateral hyperbolas.

The difficulty here presented may be removed as follows : We know that conics through four points determine an involution on any line of the plane. Since the conics of theorem 2 are equilateral hyperbolas the involution which they determine on the line infinity will have the circular points I and J as double points. Two of the hyperbolas are therefore tangent to the line infinity, one at I and the other at J. But any conic tangent to the line infinity is a parabola ; therefore these two hyperbolas are parabolas, and the contradiction with Theorem 1 disappears.

We shall now investigate these curves more closely.

Let S be any conic touching the line infinity at one of the circular points, I.

The conic S is a parabola, being tangent to the line infinity. But the straight lines joining any finite point with the infinite points of the curve are perpendicular, since any line through I is perpendicular to itself ; hence S is also an equilateral hyperbola.

The center of S, *i. e.*, the pole of the line infinity, is the point I.

The four foci, *i. e.*, the points of intersection of tangents from I and J, coincide with J.

The four directrices coincide with the polar of J, and pass therefore through I.

The eccentricity m/n is wholly indeterminate ;* for while the distance n of any point of the conic from the directrix is in general infinite (since the directrix is a line through I), the distance m from the focus is always indeterminate (since the focus is at J).

The two perpendicular axes, joining the center with the foci, coincide with the line infinity, which, like every line through I or J, is perpendicular to itself.

The four vertices, where the conic intersects the axes, coincide with I.

The asymptotes, being tangents to the conic drawn from its center, coincide with the line infinity.

Any conic S' touching the line infinity at the other circular point J will of course have corresponding properties, obtained by interchanging I and J.

Through any three finite points A, B, C of the plane there pass one conic S and one conic S'.

For the three given points, and a fourth point (I or J) with its tangent, determine a conic.

The two conics S, S' through A, B, C intersect also in the orthocenter, O, of the triangle ABC.

For, let A, B, C have the coördinates $(a, 0)$, $(b, 0)$, $(0, 1)$, whence O will be $(0, -ab)$; and let I and J be the infinitely distant points on the lines $x + yi = 0$ and $x - yi = 0$ respectively.

Then the two parabolas S and S' through A, B, C and I or J will have the equations

$$x^2 \pm 2xyi - y^2 - (a + b)x + (1 - ab)y + ab = 0,$$

where the upper sign goes with S and the lower with S'; and both of these curves contain the point O.

The directrices of the two parabolas S, S' through A, B, C and O intersect in the center of gravity G of these four points.

For the directrices of S and S', being the polars of J and I respectively, are found to have the equations

$$4x \pm 4yi = (a + b) \pm (1 - ab)i,$$

and both of these lines evidently contain the real point $\left(\frac{a+b}{4}, \frac{1-ab}{4}\right)$, which is the center of gravity G of the four given points.

* This is consistent with the fact that the eccentricity of a parabola is 1, and that of an equilateral hyperbola $\sqrt{2}$.

Further : *the directrices of S and S' are the asymptotes of the nine-points circle of the triangle ABC.*

For, the nine-points circle (passing through the middle points of the sides) has the equation

$$2x^2 + 2y^2 - (a + b)x - (1 - ab)y = 0\,;$$

hence its center is the point G already found, and the directrices, joining G with I and J, are the asymptotes of the circle.

The barycentric equation of the conic S referred to the triangle ABC which it circumscribes is

$$\beta\gamma(z_2 - z_3)^2 + \gamma\alpha(z_3 - z_1)^2 + \alpha\beta(z_1 - z_2)^2 = 0, \qquad (1)$$

where (x_1, y_1), (x_2, y_2), (x_3, y_3) are the rectangular coördinates of the points A, B, C, and $z = x + yi$.

This equation may be found by expressing the fact that the tangent to S at the point I (whose barycentric coördinates are $\alpha : \beta : \gamma = z_2 - z_3 : z_3 - z_1 : z_1 - z_2$) coincides with the line infinity (whose barycentric equation is $\alpha + \beta + \gamma = 0$).

The equation of the conic S' through the same three points is

$$\beta\gamma(\bar{z}_2 - \bar{z}_3)^2 + \gamma\alpha(\bar{z}_3 - \bar{z}_1)^2 + \alpha\beta(\bar{z}_1 - \bar{z}_2)^2 = 0, \qquad (2)$$

where $\bar{z} = x - yi$.

When the triangle of reference ABC is *equilateral*, these equations assume the simple forms

$$\beta\gamma + \omega\,\gamma\alpha + \omega^2\alpha\beta = 0, \qquad (1')$$

$$\beta\gamma + \omega^2\,\gamma\alpha + \omega\alpha\beta = 0, \qquad (2')$$

where ω is one of the imaginary cube roots of unity.

PICARD'S TRAITÉ D'ANALYSE.

Traité d'Analyse. Par ÉMILE PICARD. Deuxième édition revue et corrigée. Tome I. Paris, Gauthier-Villars, 1901. 8vo, xvi + 483 pp.

IN looking back over the last half of the nineteenth century, it is interesting to notice how slowly the great ideas in analysis which we associate with the names of Riemann