Further : *the directrices of S and S' are the asymptotes of the nine-points circle of the triangle ABC.*

For, the nine-points circle (passing through the middle points of the sides) has the equation

$$2x^2 + 2y^2 - (a + b)x - (1 - ab)y = 0 ;$$

hence its center is the point G already found, and the directrices, joining G with I and J, are the asymptotes of the circle.

The barycentric equation of the conic S referred to the triangle ABC which it circumscribes is

$$\beta\gamma(z_2 - z_3)^2 + \gamma\alpha(z_3 - z_1)^2 + \alpha\beta(z_1 - z_2)^2 = 0, \qquad (1)$$

where (x_1, y_1), (x_2, y_2), (x_3, y_3) are the rectangular coördinates of the points A, B, C, and $z = x + yi$.

This equation may be found by expressing the fact that the tangent to S at the point I (whose barycentric coördinates are $\alpha : \beta : \gamma = z_2 - z_3 : z_3 - z_1 : z_1 - z_2$) coincides with the line infinity (whose barycentric equation is $\alpha + \beta + \gamma = 0$).

The equation of the conic S' through the same three points is

$$\beta\gamma(\bar{z}_2 - \bar{z}_3)^2 + \gamma\alpha(\bar{z}_3 - \bar{z}_1)^2 + \alpha\beta(\bar{z}_1 - \bar{z}_2)^2 = 0, \qquad (2)$$

where $\bar{z} = x - yi$.

When the triangle of reference ABC is *equilateral*, these equations assume the simple forms

$$\beta\gamma + \omega\,\gamma\alpha + \omega^2\alpha\beta = 0, \qquad (1')$$

$$\beta\gamma + \omega^2\,\gamma\alpha + \omega\alpha\beta = 0, \qquad (2')$$

where ω is one of the imaginary cube roots of unity.

PICARD'S TRAITÉ D'ANALYSE.

Traité d'Analyse. Par ÉMILE PICARD. Deuxième édition revue et corrigée. Tome I. Paris, Gauthier-Villars, 1901. 8vo, xvi + 483 pp.

IN looking back over the last half of the nineteenth century, it is interesting to notice how slowly the great ideas in analysis which we associate with the names of Riemann and Weierstrass gained general currency in the mathematical world. Thirty years ago these ideas were only just beginning to be really understood by a small circle of mathematicians in Germany. Ten years later the great mathematical revival in France gave evidence that the same had become true in that country. Even ten years ago, however, the number of persons to whom these ideas were familiar in their various aspects was still small, and it was hardly possible to become initiated into them except through personal contact with a master. That this is no longer the case is due to a number of excellent text-books which have appeared during the last ten years, and to none so much, perhaps, as to the Traité d'Analyse by M. Picard, with which it is to be hoped that every reader of the BULLETIN is familiar. The first volume of this treatise which was published in 1891 has been followed by two others;* and now before the fourth and final volume appears, we are pleasantly surprised by a new edition of volume one.

The changes which have been made in this new edition are chiefly in the nature of small additions at various points, the total increase in the size of the volume being only twenty-six pages. The number of bibliographical references scattered through the volume has been increased, and certain sections have been shifted to a new position; thus, for instance, Bonnet's second law of the mean (often attributed to DuBois-Reymond) has now received a place (page 9) alongside of the first law of the mean. Among the additions and changes, the more important are the following:

In the first chapter § III and § IV are entirely new. The first of them relates to integrable and non-integrable functions according to Riemann. In the second Jordan's necessary and sufficient condition that a plane curve be rectifiable is given without proof, and then follows an account of the Peano-Hilbert curve which fills a square.

In the first note on page 27 Pringsheim's necessary and sufficient condition that a function of a real variable be developable by Taylor's theorem is stated.

A different method from that used in the first edition has been adopted, on pages 71–76, for the reduction of abelian integrals of the second kind.

The formula for the area of a curved surface is reduced on page 123 from the form involving Jacobians to the form $\int\int \sqrt{EG - F^2}\, du\, dv$.

* A review of the first two volumes of the Traité, by the late Professor Thomas Craig, appeared in the BULLETIN for November, 1893 (1st ser., vol. 3, no. 2, pp. 39–65).

On pages 202–203 a paragraph on doublets has been added.

§ II of Chapter VIII is entirely new and treats of uniformly convergent series whose terms are discontinuous functions. Both here and in connection with the Peano-Hilbert curve it is shown how continuous functions can fail to have a derivative.

Mertens' proof of the theorem that two convergent series may be multiplied together term by term provided one of them is absolutely convergent is reproduced on page 224.

On pages 283–287 two additional proofs are given of Weierstrass's theorem that any function $f(x)$ continuous throughout the interval $\alpha \leqq x \leqq \beta$ can be represented by a series of polynomials absolutely and uniformly convergent throughout this interval. The first of these proofs is essentially Weierstrass's * and depends, as does Picard's, upon the possibility of making a divergent Fourier's series convergent by the introduction of suitable "convergence factors" † into the terms of the series. The second proof, due to Volterra, is extremely simple, but assumes the ordinary theory of Fourier's series.

In Chapter XII, paragraph 25, which in the first edition related only to twisted cubics, has been extended by two pages so as to cover twisted quartics as well.

On the last two pages of the volume a paragraph has been added concerning maps in which *areas* instead of angles remain unchanged.

The moderation and good judgment displayed by the author in making these additions is at once obvious and in fact I think it was these two qualities, moderation and good judgment, which, together with the proverbial lucidity of the French style, more than anything else, have made the Traité what it is—a classic. Few books have done so much as this one to make accessible those parts of mathematics in which no real progress can be made by one who has not a firm grasp of the rigorous methods of modern analysis, and yet the ε's and the δ's, with which most writers who strive for rigor surfeit their readers, are most sparingly used here. The intelligent reader of these volumes will learn what are the real difficulties to be overcome and what is the nature of the methods to be used, far better than he could have done if these salient points were obscured by a mass of detail in the form of inequalities.

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* Or rather a special case, particularly emphasized by Weierstrass himself, of the very general treatment given by him.

† Cf. Sommerfeld, "Die willkürlichen Functionen in der mathematischen Physik." Dissertation, Königsberg, 1891.

The first two-thirds of the present volume forms essentially an introduction, though a somewhat unsystematic one, to the theory of functions of one or more real variables. It is true that the term "theory of functions of a real variable" calls to mind the idea of functions bristling with discontinuities or, if continuous, strewn with points where they have no derivative. The study of such functions has rendered a real service to mathematics during the last half century. There is, however, no more reason why an introduction to the study of functions of a real variable should plunge at once into such questions than there is in the theory of functions of a complex variable for turning at once to functions having non-analytic curves (or worse) as natural boundaries. M. Picard carefully avoided all questions of this nature in his first edition by making such assumptions as seem desirable in each case concerning continuity and existence of derivatives, even though by so doing he often failed to give his theorems the greatest possible generality. In this new edition he has still, in the main, pursued the same course; for the additions mentioned above which refer to discontinuous functions and the like are so brief as not seriously to divert the reader's attention from what may be called the cases of everyday life, while they still serve the useful purpose of calling his attention to the interesting possibilities which lie beyond.

In this volume the author makes no attempt at a systematic presentation of all parts of the subject and has thus been enabled to make it possible for the reader to open it at almost any chapter without being obliged to turn back too frequently to what has gone before. This is certainly a great advantage. I think, however, that in the treatment of uniform convergence, and of other questions which the word "uniform" suggests, the author has gone too far in this direction. The conception here involved is one of considerable difficulty in spite of its great simplicity, but it is one of such fundamental importance and constant application that it deserves to hold a central position in a treatise of this sort and to be used freely whenever occasion presents itself. The subject of the uniform convergence of series is admirably set forth in Chapter VIII, although it is to be regretted that Weierstrass's extremely simple and useful sufficient condition for uniform convergence is not explicitly mentioned.* It is, however, not pointed out that this is

* Namely that $\Sigma u_n(x)$ converges uniformly if a convergent series ΣM_n exists where $|u_n(x)| \leqq M_n$, the M's being positive constants. Weierstrass, Werke, vol. 2, p. 202.

merely a special case of the uniform convergence of a function involving several variables to a limit as some of these variables approach limits. Thus for instance we have in the question of the convergence of definite integrals precisely the same questions concerning uniform convergence as in the case of series.* All these questions, as well as the question of uniform continuity,† are not explicitly considered. Even in the case of series, however, M. Picard often seems to shrink from using the theorems he has established in Chapter VIII, and prefers to go over again in each special case what is practically the same proof he has already given in the general case. Thus on page 19 of Volume II, a mere glance at the series in line 6, written without a remainder, shows that it is uniformly convergent for all values of ψ and can therefore be integrated term by term. A precisely similar remark applies to page 57 of Volume II. Again on page 303–304 of Volume II. (the proof by successive approximations of the existence theorem for ordinary differential equations) the words uniform convergence do not appear, although this is the central idea involved.

There is no book on analysis better suited to the needs of graduate students in our universities than the treatise we are considering, avoiding as it does on the one hand the superficiality of almost all books on the subject in the English language, and on the other the heaviness and remoteness from other parts of mathematics characteristic of too many continental books on analysis which aim at giving a rigorous treatment.

MAXIME BÔCHER.

HARVARD UNIVERSITY,
November 2, 1901.

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* Cf. a paper by de la Vallée-Poussin : *Annales de la Société Scientifique de Bruxelles*, vol. 17 (1892-93), p. 323.

† The fundamental fact concerning uniform continuity for functions of two real variables is practically proved on p. 102, and states that if such a function is continuous within and on the boundary of a region it is uniformly continuous there. From this it follows that if r and ϑ are polar coördinates, and if $F(r, \vartheta)$ is continuous within and on the circumference of the circle $r=1$ then $F(r_1, \vartheta)$ $(r_1<1)$ approaches its limit $F(1, \vartheta)$ uniformly as r_1 approaches the value 1. This is the theorem proved in a special case by means of two very special inequalities on p. 276. It is also essentially identical with the fact referred to at the bottom of p. 42.