The conditions for the invariance of f under a linear transformation are bilinear in the coefficients and therefore may be treated by rational processes. Similar remarks hold for a treatment of the remaining two forms by induction from $m-1$ to m. The second set of canonical forms defines groups in an arbitrary realm whose structures were determined in the papers cited. Moreover, the second set proves more advantageous than the first set in the question of the representation of the linear groups as transitive substitution groups. The first set was employed in the *American Journal of Mathematics*, October, 1901. For the second set, the investigation becomes simpler and the results appear in a simpler and more natural classification.

Not only for these groups, but for other classes, it appears that the simplest methods are those applicable to an arbitrary realm, viz., methods which depend essentially upon rational operations only.

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VECTOR ANALYSIS.

Vector Analysis. A text-book for the use of students of mathematics and physics, founded upon the lectures of J. WILLARD GIBBS, Ph.D., LL.D., Professor of Mathematical Physics in Yale University. By EDWIN BIDWELL WILSON, Ph.D., Instructor in Mathematics in Yale University. (Yale Bicentennial Publication.) New York, Charles Scribner's Sons, 1901. 8vo. xx + 436 pp.

IT is well known that Professor Gibbs's "Elements of vector analysis," a pamphlet of 83 pages, printed in 1881–84 for the use of his students, although not published for general circulation, attracted somewhat wide attention. Thus, in particular, Mr. Oliver Heaviside adopted Professor Gibbs's system with but slight modifications and expounded it very fully in his "Electromagnetic Theory" (1893); this again formed the basis, to a large extent, of Professor Föppl's "Einführung in die Maxwell'sche Theorie der Elektricität, mit einem einleitenden Abschnitte über das Rechnen mit Vectorgrössen in der Physik" (1894). But vector analysis, as conceived by Professor Gibbs, is not merely a method for studying electricity and magnetism; it finds its application in all branches of mathematical physics and has, besides, a strong claim to the attention of the pure mathematician. Dr. Wilson's volume, which is based on a course of lectures delivered by Professor Gibbs in 1899–1900, is therefore exceedingly welcome, not only as the first generally accessible authentic record of Professor Gibbs's admirable system, but also as an easy and simple introduction to vector analysis, sufficiently complete to satisfy the needs of applied science and sufficiently exact and rigorous to satisfy the student of pure mathematics.

The additions to the theory found in this volume as compared with the pamphlet of 1881–84 are perhaps not very extensive, though some are quite important. The student should not be deterred by the bulkiness of Dr. Wilson's book, as this is due partly to the use of lavishly open print and partly to the author's effort, specially noticeable in the first half, of making the subject easily intelligible by supplying numerous illustrations and applications.

In his address on "Multiple Algebra," before the section of mathematics and astronomy of the American Association for the advancement of science in 1886 (Proceedings, Vol. 30, pp. 37–66), Professor Gibbs has set forth briefly the relation of his system to Grassmann's Ausdehnungslehre, Hamilton's quaternions, Cayley and Sylvester's theory of matrices, and Benjamin Peirce's linear associative algebra. This paper explains and illustrates very happily the simplicity and naturalness of his method as far as fundamental principles are concerned. But the value of vector methods is still so little appreciated that it may not be superfluous to call attention once more to their importance by briefly reciting some of the principal features of the system presented in Dr. Wilson's book.

Mathematical physics, in its broader aspect, has to deal with *fields* of *point functions*. By somewhat generalizing Lamé's definition it can be said that a point function is any quantity, or aggregate of quantities, whose value depends on the position of a point.* The simplest point function is a *scalar* whose value at any point of the field is given by a single (real) number. Next in importance comes the *vector*,

* Professor Gibbs uses for "point function" the expression "function of position in space" which is somewhat unwieldy. The present writer would also suggest that Dr. Wilson might have done well to speak of a point function, that is a vector quantity, simply as *vector*, and not as *vector function*, so as to reserve the latter term for a function of a vector, as in the expression linear vector function.

a point function having not merely numerical magnitude but also direction (and sense); its value at any point is represented geometrically by a rectilinear segment and given analytically by its components, *i. e.*, three scalars, each associated with a fixed direction. As examples of point functions of a higher order occurring in physics it will suffice to mention *pure strain*, with its six components (or three principal extensions in definite directions), and *homogeneous strain*, depending upon nine scalars.

The theory of scalar fields is nothing but an immediate generalization of the theory of the potential, a generalization on which Professor Gibbs very properly lays particular stress in his treatment of the potential (pp. 205–248) which forms one of the most interesting parts of Dr. Wilson's work.

Every scalar field gives rise to a vector field, *viz.*, the field of the vector that represents the *gradient* or *space derivative* ∇V of the scalar V. Thus, in the theory of the ordinary potential, the space derivative of the potential is the corresponding force. But the vector that results in this way from a scalar is a particular kind of vector: a vector whose *curl* is zero; the field of such a vector, and the vector itself, is said to be *potential*, or *irrotational*. It is therefore necessary to study vectors by themselves.

The idea of vectors and their geometrical addition is now familiar to everybody from the elements of mechanics. Dr. Wilson follows Mr. Heaviside in using heavy-faced (Clarendon) type to denote vectors. This departure from Professor Gibbs's original notation by small Greek letters is commendable as it leaves the Greek alphabet free for other uses. A vector $\mathbf{A}$ whose components along three mutually rectangular axes are A_1, A_2, A_3 is then expressed in the form

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors along the axes. This means that a vector is an extensive magnitude depending on three independent units.

The *scalar product* of two vectors $\mathbf{A}$, $\mathbf{B}$ is defined as the product of their lengths A, B into the cosine of the included angle θ:

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta;$$

this *dot* product is a scalar, *i. e.*, an ordinary algebraical quantity (real number). The *vector product* is similarly defined as

$$\mathbf{A}\times\mathbf{B} = AB\sin\theta;$$

this *cross* product, being the area of the parallelogram of the two vectors, can be represented by a vector at right angles to the plane of the two vectors. Both of these quantities are familiar from their frequent occurrence in mechanics, the scalar product as the work of a force, the vector product as the moment of a couple. Both products obey the ordinary rules of algebra except that the vector product is not commutative ($\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$) and that the vanishing of either product does not necessarily mean that one of the factors vanishes. For the unit vectors **i**, **j**, **k**, we have evidently:

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \quad \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{j} = 0,$$

and

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0, \quad \mathbf{j}\times\mathbf{k} = -\mathbf{k}\times\mathbf{j} = \mathbf{i},$$
$$\mathbf{k}\times\mathbf{i} = -\mathbf{i}\times\mathbf{k} = \mathbf{j}, \quad \mathbf{i}\times\mathbf{j} = -\mathbf{j}\times\mathbf{i} = \mathbf{k}.$$

It follows that if $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ we have

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3,$$

$$\mathbf{A}\times\mathbf{B} = (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k}.$$

The development of these simple ideas constitutes the first two chapters (pp. 1–114) of Dr. Wilson's book; they correspond to the first thirteen pages of Professor Gibbs's pamphlet. The algebra so obtained, especially when combined with the equally simple differentiation of a vector with respect to a scalar variable as given in the first twenty pages of the next chapter, suffices for the treatment of a large part of mechanics and admits of interesting applications to geometry.

Chapters III and IV (pp. 115–259), corresponding to pp. 14–39 of the pamphlet, are devoted to the differential and integral calculus of vectors. The central idea in these chapters is Hamilton's operator ∇ for which the pronunciation *del* is here proposed instead of *nabla* (p. 138). This operator is defined as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

so that, applied to a scalar, $V(x, y, z)$, it produces the vector that represents the *slope* or *gradient* of the point function V. The direction of ∇V is normal to the level-surface

$V(x, y, z) = \text{const.}$; its magnitude is the rate at which V changes as we pass from one level surface to the next along the normal. This vector ∇V, being independent of the coördinate axes, can be regarded as the *space derivative* of the scalar V. It is easy to see that, if $\mathbf{r}$ be the radius vector drawn from a fixed origin to the point (x, y, z), we have

$$d\mathbf{r} \cdot \nabla V = dV.$$

The space derivative of a vector $\mathbf{V}$ can be defined in precisely the same way. But as its interpretation involves the idea of a new kind of product, its consideration is reserved for the latter part of the book (p. 404). In the place of this space derivative $\nabla \mathbf{V}$, two other quantities, of the greatest importance in mathematical physics, are introduced, the *divergence* and the *curl* of the vector $\mathbf{V}$. These are most readily obtained by regarding the operator ∇ as an actual vector and forming its scalar and vector products with the vector $\mathbf{V} = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$, viz.:

$$\nabla \cdot \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = \text{div } \mathbf{V},$$

$$\nabla \times \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k}$$

$$= \text{curl } \mathbf{V}.$$

The meaning of these quantities is well explained by Dr. Wilson; the formulæ obtained by applying the operators ∇, $\nabla \cdot$, $\nabla \times$ to the sum or product of two or more scalars or vectors and by repeating these operations are fully developed.

In the integral calculus of vectors we are led naturally and with remarkable simplicity to line, surface, and volume integrals and to the theorems of Gauss, Green, and Stokes which appear here in a form which interprets itself and expresses most simply and directly their actual meaning.

Among the applications here discussed Professor Gibbs's original treatment of so-well worn a subject as the potential is specially noteworthy. According to the ordinary definition, the integral

$$\int \frac{V}{r}\,d\tau = \int\int\int \frac{V(x,\,y,\,z)\,dx\,dy\,dz}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}$$

represents the potential at the point (x_1, y_1, z_1) of a mass distributed with density V throughout the region τ; it is a scalar function of the variables x_1, y_1, z_1. An integral of this form, whatever may be the meaning of the scalar function V, is called by Professor Gibbs the *potential* of V; in symbols:

$$\text{Pot } V = \int \frac{V}{r}\,d\tau .$$

It can be proved, by means of Gauss's method of giving to the region τ a slight displacement in each of the coördinate directions, that

$$\nabla \text{ Pot } V = \text{Pot } \nabla V.$$

This point function which in the case when V is the density of a mass represents the Newtonian attraction due to this mass at the point (x_1, y_1, z_1) is called the *Newtonian* of V and denoted by the symbol New V:

$$\text{New } V = \nabla \text{ Pot } V = \int \frac{\mathbf{r}\,V}{r^3}\,d\tau.$$

If in the integral defining Pot V we replace the scalar V by a vector $\mathbf{V}$ we obtain a vector

$$\text{Pot } \mathbf{V} = \int \frac{\mathbf{V}}{r}\,d\tau.$$

Operating on this vector by means of $\nabla \times$ and $\nabla \cdot$, two more functions are obtained called the *Laplacian* and *Maxwellian* of $\mathbf{V}$ and denoted as Lap $\mathbf{V}$ and Max $\mathbf{V}$. These functions, as well as some of those arising from their combination, play a most important part in mathematical physics.

It should be noticed that this whole theory rests on but three simple fundamental operations: the scalar and vector products of two vectors and the space derivative of a scalar. With characteristic skill Professor Gibbs has selected, out of the broad field of multiple algebra and space analysis, just those operations which are most useful on account of their application in physics.

Dr. Wilson's presentation of these subjects is on the whole very clear and sufficiently exact. The idea of scalar and vector fields and their varieties might perhaps have been made more prominent, somewhat as Professor Föppl does in his "Geometrie der Wirbelfelder" (1897). As the book is distinctly designated as a text-book, it would have been better to make shorter chapters and sections, introduce more section headings and distinguish, externally, between what is essential and fundamental and what is mere illustration and application. The very large and heavy integral signs used here as in so many other books are neither handsome nor always necessary. Nor is there any necessity for three integral signs in a volume integral in which the volume element dv is not broken up into its linear factors; the three signs become desirable only when dv is replaced by $dx\,dy\,dz$ or $r^2 \sin\theta\, dr\, d\theta\, d\varphi$. A similar remark applies to surface integrals.

The second half of Dr. Wilson's book (Chapter V: linear vector functions, pp. 260–331; Chapter VI: rotations and strains, pp. 332–371; Chapter VII: miscellaneous applications, pp. 372–436) is almost entirely devoted to the linear vector function and its applications. The beginner will probably find it more difficult reading than the first half; the originality of the treatment will make it so much more interesting to the more advanced student.

It is here impossible to do more than indicate the fundamental idea underlying Professor Gibbs's treatment of the linear vector function.

If a vector be given in the usual form $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, any other vector $\mathbf{r}'$ can be written in the form

$$\begin{aligned}\mathbf{r}' &= c_1 x\mathbf{i} + c_2 y\mathbf{j} + c_3 z\mathbf{k}\\ &= c_1\mathbf{ii}\cdot\mathbf{r} + c_2\mathbf{jj}\cdot\mathbf{r} + c_3\mathbf{kk}\cdot\mathbf{r},\end{aligned}$$

since $x = \mathbf{i}\cdot\mathbf{r}$, $y = \mathbf{j}\cdot\mathbf{r}$, $z = \mathbf{k}\cdot\mathbf{r}$. For the sake of brevity this may be written

$$\mathbf{r}' = (c_1\mathbf{ii} + c_2\mathbf{jj} + c_3\mathbf{kk})\cdot\mathbf{r},$$

the expression in parenthesis being regarded as an operator that transforms $\mathbf{r}$ into $\mathbf{r}'$. More generally, any expression of the form

$$\Phi = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots,$$

in which $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{b}_1, \mathbf{b}_2, \cdots$ are vectors, can be regarded as an operator transforming a vector into another vector ; for the expression

$$\Phi \cdot \mathbf{r} = (\mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots) \cdot \mathbf{r} = \mathbf{a}_1\mathbf{b}_1 \cdot \mathbf{r} + \mathbf{a}_2\mathbf{b}_2 \cdot \mathbf{r} + \cdots$$

always represents a definite vector since $\mathbf{b}_1 \cdot \mathbf{r}$, $\mathbf{b}_2 \cdot \mathbf{r}$, $\cdots$ are mere scalars. The nature of the *indeterminate products* $\mathbf{a}_1\mathbf{b}_1$, $\mathbf{a}_2\mathbf{b}_2$, $\cdots$ or *dyads*, as Professor Gibbs calls them, results from the convention that any two such products are to be equal if, and only if, acting as operators on the same vector $\mathbf{r}$, they produce the same vector $\mathbf{r}'$.

The sum of any number of dyads,

$$\Phi = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots,$$

is called a *dyadic*. Thus a dyad as well as a dyadic has no direct geometrical meaning; it is a mere operator whose close relation to quaternions and matrices is obvious.

The linear vector function being defined by the property that $f(\mathbf{r}_1 + \mathbf{r}_2) = f(\mathbf{r}_1) + f(\mathbf{r}_2)$, from which the property $f(m\mathbf{r}) = mf(\mathbf{r})$ readily follows for any scalar m if f be assumed continuous, it can be shown that a linear vector function can always be represented by a dyadic. By expressing the vectors forming the dyadic in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the dyadic can be reduced to the *nonion* form :

$$\begin{aligned}\Phi = {} & a_{11}\mathbf{ii} + a_{12}\mathbf{ij} + a_{13}\mathbf{ik} \\ & + a_{21}\mathbf{ji} + a_{22}\mathbf{jj} + a_{23}\mathbf{jk} \\ & + a_{31}\mathbf{ki} + a_{32}\mathbf{kj} + a_{33}\mathbf{kk},\end{aligned}$$

or more generally to the form

$$\begin{aligned}\Phi = {} & b_{11}\mathbf{aA} + b_{12}\mathbf{aB} + b_{13}\mathbf{aC} \\ & + b_{21}\mathbf{bA} + b_{22}\mathbf{bB} + b_{23}\mathbf{bC} \\ & + b_{31}\mathbf{cA} + b_{32}\mathbf{cB} + b_{33}\mathbf{cC},\end{aligned}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as well as $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are any three non-coplanar vectors. Putting

$$b_{11}\mathbf{A} + b_{12}\mathbf{B} + b_{13}\mathbf{C} = \mathbf{l}, \quad b_{21}\mathbf{A} + b_{22}\mathbf{B} + b_{23}\mathbf{C} = \mathbf{m},$$
$$b_{31}\mathbf{A} + b_{32}\mathbf{B} + b_{33}\mathbf{C} = \mathbf{n},$$

it appears that any dyadic can be written as a sum of (at most) three terms :

$$\Phi = \mathbf{al} + \mathbf{bm} + \mathbf{cn},$$

where either $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ or $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ can be selected arbitrarily (but not coplanar). If after choosing $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ arbitrarily, the vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ result non-coplanar, the dyadic is *complete;* such a dyadic can produce *any* vector $\mathbf{r}'$ by operating on a properly chosen vector **r**. If **l**, **m**, **n** result coplanar, the dyadic can be reduced to the sum of two dyads and is called *planar;* it transforms any vector **r** into a vector lying in the plane of the first factors of the two dyads. If **l**, **m**, **n** result collinear, the dyadic can be reduced to a single dyad and is called *linear;* it transforms every vector into a vector collinear with the first factor of the single dyad.

The discussion on pp. 271–275 of the nature of the indeterminate product **ab** and its relation to the scalar product $\mathbf{a} \cdot \mathbf{b}$ and the vector product $\mathbf{a} \times \mathbf{b}$ is interesting theoretically. For all further developments the reader must be referred to the book itself. It must here suffice to mention the introduction of the "double products" of dyads and dyadics (pp. 306–315), which are not to be found in the pamphlet of 1881–84.

The applications treated in Chapters VI and VII give ample opportunity for practice in the theory developed in the earlier part of the work; they relate to rotations and strains, quadric surfaces, the propagation of light in crystals, and the curvature of surfaces. The brief section on light waves in non-isotropic bodies will arouse in every reader the wish that Professor Gibbs may soon find it possible to publish his lectures on the electro-magnetic theory of light.

The last section, on the curvature of surfaces, is preceded by a short discussion of variable dyadics; and here we find the interpretation of the space derivative of a vector, $\nabla \mathbf{V}$, as a dyadic.

In a work departing from familiar mathematical methods as much as does the present it is but natural that a large number of new terms and notations are introduced. These have been selected with unusual skill and good sense; indeed, this is one of the strong points of Professor Gibbs's system. Perhaps the expressions degree of nullity (of a dyadic) and idemfactor, a remnant of Sylvester's always ingenious, but somewhat extravagant, terminology, make an exception. The former of these terms seems hardly necessary, the latter might perhaps be replaced by unit dyadic.

It is very much to be hoped that Dr. Wilson's work will become a widely used text-book, especially as a general introduction to mathematical physics. Its usefulness for this purpose would have been much enhanced by a good index, by better figures, and by the use of all those external aids now so common in carefully printed mathematical textbooks.

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