where ρ_1 and ρ_2 denote the principal radii of S. For the second of the conditions (5) this gives $\rho_1 = \rho_2$, that is, S is a sphere, when real. Combining these results we have that *minimal surfaces and spheres are the only real surfaces for which the spherical representation of lines of length zero is the system of rectilinear generatrices of the sphere.*

In order to determine what surfaces have the lines of length zero on the sphere for spherical representation of its asymptotic lines we note that for the parameters used previously the equation of the asymptotic directions is*

$$rdu^2 + 2(s + z)dudv + tdv^2 = 0.$$

Hence, in order that the u and v lines may be asymptotic for the surface, the condition is

$$r = t = 0,$$

which as we have seen characterizes the sphere. Again, in order that the parametric lines may form a conjugate system we get from the above equation the condition

$$s + z = 0,$$

that is minimal surfaces. Recalling the above theorem we have the following:

In order that the asymptotic lines or a conjugate system on a surface may be represented upon the sphere by its imaginary generatrices, they must be lines of length zero on the surface.

PRINCETON,
November, 1901.

SOME PROPERTIES OF POTENTIAL SURFACES.

BY DR. EDWARD KASNER.

(Read before the American Mathematical Society, April 27, 1901.)

IN a previous paper, published in the BULLETIN, volume 7, pages 392–399, the author studied the algebraic curves $\varphi(x\ y) = 0$ defined by the condition $\varphi_{xx} + \varphi_{yy} \equiv 0$. Two classes of characteristic properties were obtained, the first translating directly the differential equation and the second arising from the well-known connection between harmonic

* Darboux, Leçons, I, p. 246.

functions and functions of the complex variable. The present paper extends some of the results to the corresponding type of surfaces,* *i. e.*, to the surfaces expressed in rectangular coördinates by an equation $\varphi(x, y, z) = 0$ where φ is a rational integral solution of the potential equation

$$(1) \qquad \Delta \varphi \equiv \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0.$$

The results obtained in § 3, however, apply not merely to algebraic but to all analytic potential surfaces.

§ 1. *Apolar Properties.*

The algebraic potential surfaces are characterized most simply by their relation to the imaginary circle at infinity.† Considering the latter as a degenerate class quadric, its equation in rectangular plane coördinates may be written

$$(2) \qquad Q \equiv u^2 + v^2 + w^2 = 0.$$

In homogeneous coördinates the condition that a quadric

$$u_P^2 \equiv \sum P_{ik} u_i u_k = 0 \qquad (i, k = 1, 2, 3, 4)$$

is apolar to a surface

$$\Phi \equiv a_x^n = 0$$

is expressed by the identical vanishing of the covariant

$$S \equiv a_P^2 a_x^{n-2} \equiv \sum P_{ik} \frac{\partial^2 \Phi}{\partial x_i \partial x_k} \cdot$$

Applying this to rectangular coördinates and assuming the quadric to be Q, this covariant takes the form

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \cdot$$

Comparing this with the defining equation (1) we have

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* The focal properties of the curves do not seem capable of direct generalization. The n-dimensional extensions of the results obtained in this paper are in general obvious.

† That there must be such a relation is seen a priori from the fact that the defining differential equation is invariant under the conform group, which contains the group of euclidean geometry as a subgroup.

I. *All potential surfaces are apolar to the imaginary circle at infinity considered as a degenerate quadric; conversely, any surface apolar to this circle is a potential surface.*

The imaginary element which enters into this characterization may be eliminated by considering the polar quadrics of the surface, which, from the definition of the apolar relation, must themselves be apolar to Q. If a quadric surface is apolar to Q it must intersect the plane at infinity in a conic which, as a curve of second order, is apolar to the imaginary circle considered as a curve of second class, and therefore contains triples of points mutually conjugate with respect to the circle. It follows that the asymptotic cone of the quadric contains among its generators triples of mutually orthogonal lines. The potential quadrics are thus identical with what may be termed the rectangular or equilateral hyperboloids* from their analogy to the rectangular hyperbolas. Theorem I may now be restated

II. *The polar quadric of any point with respect to a potential surface is a rectangular hyperboloid; conversely, if all the polar quadrics of a surface are of this species, it is a potential surface.*

Since the polar quadrics of a polar surface coincide with the polar quadrics of the original surface, we have

III. *The polar surfaces of a potential surface are also potential surfaces.*

§2. *Special Potential Surfaces.*

A relation between the potential curves and surfaces presents itself when the surface considered is cylindrical. The equation of such a surface may be assumed in the form $\varphi(x, y) = 0$, so that $\Delta\varphi$ reduces to $\varphi_{xx} + \varphi_{yy}$. Since the vanishing of this expression is the condition for a potential curve, the result may be stated

IV. *A cylindrical surface belongs to the class of potential surfaces when, and only when, its right section is a potential curve.*

This relation may be generalized by projecting the imaginary circle Q into an arbitrary conic and referring to theorem IV of the article cited above on potential curves.

IV′. *If a conical surface with its vertex V in the plane of a conic C is apolar to the latter (considered as a degenerate class quadric), then any plane passing through the polar line of V with respect to C intersects the conical surface in a curve apolar to the point pair cut out upon C; conversely, etc.*

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* An equation of the second degree in x, y, z represents such a quadric when the sum of the coefficients of x^2, y^2, and z^2 equals zero, as may be verified directly from the condition $\Delta\phi = 0$. For a complete discussion of these quadrics cf. H. Vogt, "Ueber ein besonderes Hyperboloid." *Crelle*, vol. 86 (1879), pp. 297-316.

If a potential cylindrical surface degenerates into a set of planes, the right section will consist of a set of lines forming a potential curve. Applying the theorem* concerning the asymptotes of potential curves, it follows that the lines must be concurrent and disposed symmetrical about their common point.

V. *A set of planes having a common perpendicular plane constitute a potential surface when, and only when, the planes intersect in a common line and are disposed symmetrically about it.*

Another type† of degenerate potential surfaces formed of a number of planes is obtained by adding to the above set the common perpendicular plane. For if $\varphi(x, y)=0$ represents the original set of planes, the new set may be represented in the form $z\varphi(x, y) = 0$, and the Laplacian of the first member is $z(\varphi_{xx} + \varphi_{yy})$.

Among the potential functions, those which are homogeneous play an important rôle. The corresponding surfaces are cones whose characteristic property may be deduced from the following

LEMMA. *If a cone S_n of nth order is apolar to a plane curve C_k of kth class, considered as a (degenerate) surface C_k' of kth class, then S_n intersects the plane of C_k in a curve S_n' which is apolar to C_k; the converse also holds.*

In the proof any system of homogeneous coördinates may be assumed since the relations considered are projective. Take then the vertex of the cone as one vertex of the fundamental tetrahedron and let the other three be taken in the plane of C_k. The equation of S_n is then of the form $S(x_1, x_2, x_3) = 0$, and that of C_k' is $C(u_1, u_2, u_3) = 0$. The equations of the related curves S_n' and C_k are the same as those of the surfaces, since x_1, x_2, x_3 and u_1, u_2, u_3 may be interpreted, independently of x_4 and u_4, as trilinear coördinates in the plane $u_4 = 0$ which is the plane of C_k. The condition of apolarity between the surfaces S_n, C_k' is thus identical with the condition of apolarity between the curves S_n', C_k. This is the result expressed in the lemma.

Applying this to the case when C_k becomes Q, the imaginary circle at infinity, we have‡

XI. *A conical surface is a potential surface when, and only*

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* Briot and Bouquet, Théorie des fonctions elliptiques, p. 227.

† Cf. the types referred to in Thomson and Tait's Natural Philosophy, 2d ed., § 781.

‡ This is equivalent to Clifford's theorem concerning nodal curves, "On the canonical form of spherical harmonics," Works, p. 234. Cf. W. Thomson, "On the general canonical form of a spherical harmonic of *n*th order," *Report 41st Meeting Brit. Assoc. Adv. Science*, 1871, p. 10.

when, it intersects the plane at infinity in a curve apolar to the imaginary circle.

§ 3. *Nodes and Multiple Curves.*

The preceding results relating to algebraic potential surfaces may be applied to the more general class of analytic potential surfaces. Referred to a system of rectangular coordinates with the origin situated on the surface, the equation an analytic surface is of the form

$$\varphi \equiv v_1 + v_2 + \cdots + v_k + \cdots = 0,$$

where v_k is a homogeneous function of degree k in x, y, z. Since

$$\Delta\varphi = \Delta v_1 + \Delta v_2 + \cdots + \Delta v_k + \cdots$$

and since no two terms in the second member are of the same degree, it follows that for a potential surface Δv_k must vanish identically for every value of k.

If now the origin is a node of order $k - 1$,

$$v_1 \equiv 0, \ v_2 \equiv 0, \ \cdots, \ v_{k-1} \equiv 0,$$

and $v_k = 0$ is the equation of the tangent cone. But from the above $\Delta v_k \equiv 0$, *i. e.*, the cone must be a potential surface.

VII. *If a potential surface has a node of order $k - 1$, the tangent cone is a potential cone of kth order. When the node is simple the cone is of the second order and contains triples of mutually orthogonal generators.*

If the analytic surface considered has a multiple curve of order $k - 1$, and the origin is taken anywhere on the curve, $v_k = 0$ represents the k planes tangent to the different sheets of the surface. Since these planes all pass through the line tangent to the multiple curve, theorem IV may be applied; therefore*

VIII. *If a potential surface has a multiple curve of order $k - 1$, then the k sheets passing through the curve intersect each other at angles equal to $2\pi/k$; when $k = 2$, the two sheets are orthogonal.*

The decomposable surface $\varphi\psi = 0$, formed by two potential surfaces $\varphi = 0$, $\psi = 0$ is not in general a potential surface. The Laplacian of a product may be developed

* For the corresponding theorem concerning the multiple points of potential curves, see Briot and Bouquet, *l. c.*, p. 225; Thomson and Tait, Natural philosophy, 2d ed., § 780, where reference is made to an earlier paper by Rankine, "Summary of the properties of certain stream lines," *Phil. Mag.*, 1864.

$$\Delta(\varphi\psi) = \varphi\Delta\psi + \psi\Delta\varphi + 2\Omega,$$

where

$$\Omega = \frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z}.$$

As is well known, Ω vanishes only when the two families $\varphi = \text{const}$. $\psi = \text{const}$, are orthogonal. Therefore

IX. *If a pair of potential surfaces* $\varphi = 0$, $\psi = 0$ *combined form a potential surface, the families* $\varphi =$ *const.,* $\psi =$ *const. are orthogonal.*

The projective generalization of this result is

IX′. *If a surface apolar to a conic decomposes into two surfaces apolar to the same conic, then the tangent planes to the latter surfaces at any point in their intersection cut the plane of the conic in a pair of lines conjugate with respect to the conic.*

A MODERN ENGLISH CALCULUS.

An Elementary Treatise on the Calculus, with Illustrations from Geometry, Mechanics and Physics. By George A. Gibson, M.A., F.R.S.E., Professor of Mathematics in the Glasgow and West of Scotland Technical College. London, Macmillan & Co., 1901. 12mo, pp. xix + 459.

In the year 1891 Harnack's Elements of the Differential and Integral Calculus, which appeared in Leipzig in 1881, was translated into English. This book gave the first systematic presentation in the English language of the leading principles of modern analysis in their relation to the foundations of the infinitesimal calculus. While not wholly free from errors, and sometimes difficult to read, owing to inadequate exposition of details, the book is nevertheless conceived in the spirit of modern mathematics and it lays stress on those principles of analysis which are essential for a rigorous development of the calculus.

The first book of English origin to treat the calculus from a modern standpoint was Lamb's Infinitesimal Calculus,* published in 1897. This is an excellent treatise and any later work on the calculus, of modern tendencies, must have many points of contact with it.

* A notice of this book by the present writer appeared in *Science*, new series, vol. 7 (1898), No. 176, p. 678.